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*Research Article*

# Some New Generalized Retarded Gronwall-Like Inequalities and Their Applications in Nonlinear Systems

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The Gronwall inequalities are of significance in mathematics and engineering. This paper generalizes the Gronwall-like inequalities from different perspectives. Using the proposed inequalities, the difficulties to discuss the controllability of integrodifferential systems of mixed type can be solved. Meanwhile, two examples as their applications are also given to show the effectiveness of our main results.

## 1. Introduction

Integral inequalities provide a powerful and important tool in the study of qualitative properties of solutions of nonlinear differential, integral, and integrodifferential equations, as well as in the modeling of science and engineering problems (see [1]). One of the most famous inequalities of this type is known as "Gronwall's inequality," "Bellman's inequality," or "Gronwall-Bellman's inequality" (see [2, 3]). Recently, the celebrated Gronwall inequality and its generalizations play increasingly important roles in the qualitative analysis of differential, integral, and integrodifferential equations. Based on the different purposes, many researchers put their efforts in exploring new inequalities and their applications in many fields, and many useful Gronwall-like integral inequalities have been established in various problems (see [4–17]).

Lipovan [18] proved a Gronwall-like inequality, and in order to show its applications, Lipovan applied his main results to the qualitative analysis of solutions to certain integral equations, functional differential equations, and retarded differential equations. Ye et al. [19] gave a generalized Gronwall inequality with singularity which can be applied to weakly singular Volterra integral equations and fractional integral and integrodifferential equations. Liu [20] proved a comparison result, which is widely known later and always used to provide explicit bounds on solutions and estimate on noncompactness. In addition, some existence theorems of solutions and iterative approximation of the unique solution for the nonlinear integrodifferential equations of mixed type are obtained. However, it is worth mentioning that it is difficult to deal with integrodifferential systems which include a Fredholm operator in nonlinearity unless powerful integral inequalities are established.

In this paper, we prove a generalization of the Gronwall inequality. As an application, we show that the inequality can be applied to the controllability analysis of abstract control system and existence analysis. Sufficient conditions ensuring the controllability of certain impulsive integrodifferential system of mixed type are obtained. The main difficulties from the Fredholm operator can be overcome.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas and prove some generalized Gronwall-like inequalities. In Section 3, we discuss the controllability of impulsive integrodifferential systems of mixed type in a Banach space as an application. In Section 4, we give another example to illustrate the application of our main results vividly. Finally, conclusions are given in Section 5.

## 2. Integral Inequalities

In this section, we present a generalization of the Gronwall-like inequality which can be called a comparison result

in many literatures (see [20]). Unless otherwise stated, we denote $\mathbb{R}_+ = (0, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ in this paper.

**Lemma 1.** *Suppose that $u$, $f$, $g \in C([t_0, T), \mathbb{R}^+)$. Let $w \in C(\mathbb{R}^+, \mathbb{R}^+)$ be nondecreasing with $w(u) > 0$ for $u > 0$ and let $\alpha$, $\beta \in C^1([t_0, T), [t_0, T))$ be nondecreasing with $\alpha(t)$, $\beta(t) \le t$ on $[t_0, T)$. If*

$$u(t) \le m_1(t) + m_2(t) \int_{\alpha(t_0)}^{\alpha(t)} f(s) w(u(s)) ds + m_3(t) \int_{\beta(t_0)}^{\beta(t)} g(s) w(u(s)) ds, \tag{1}$$

*where $m_i(t)$ is a nonnegative, continuous function defined on $t_0 \le t < T$, and there exist nonnegative constants $M_i$ such that $m_i(t) \le M_i$ $(i = 1, 2, 3)$, then, for $t_0 \le t < t_1$, one has*

$$u(t) \le G^- \left( G(M_1) + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) ds \right), \tag{2}$$

*where $G(r) = \int_1^r (1/w(s)) ds$, $r > 0$, $t_1 \in (t_0, T)$ is chosen so that*

$$G(M_1) + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) ds \in \mathrm{Dom}(G^-), \tag{3}$$

*for all $t$ lying in the interval $[t_0, t_1)$.*

*Proof.* Noting the conditions we imposed, we have

$$u(t) \le M_1 + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) w(u(s)) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) w(u(s)) ds. \tag{4}$$

Let us denote

$$U(t) = M_1 + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) w(u(s)) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) w(u(s)) ds. \tag{5}$$

Obviously, we have $U(t_0) = M_1$ and

$$U'(t) = M_2 f(\alpha(t)) w(u(\alpha(t))) \alpha'(t) + M_3 g(\beta(t)) w(u(\beta(t))) \beta'(t). \tag{6}$$

Since $\alpha(t) \le t$ and $\beta(t) \le t$ on $[t_0, T)$, then

$$U'(t) \le M_2 f(\alpha(t)) w(U(t)) \alpha'(t) + M_3 g(\beta(t)) w(U(t)) \beta'(t). \tag{7}$$

By the definitions of $G$, we obtain that

$$\frac{d}{dt} G(U(t)) \le M_2 f(\alpha(t)) \alpha'(t) + M_3 g(\beta(t)) \beta'(t) \tag{8}$$

integrate both sides, and we conclude that

$$G(U(t)) \le G(M_1) + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) ds. \tag{9}$$

Because $G^-$ is increasing on $\mathrm{Dom}(G^-)$, we get

$$u(t) \le G^- \left( G(M_1) + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) ds \right). \tag{10}$$

□

*Remark 2.* Next, we shall show that Lemma 1 generalizes some existing results:

(1) For $m_1(t) \equiv k$, $m_2(t) \equiv 1$, and $m_3(t) \equiv 0$, we obtain theorem in [18]. Further supposing that $\alpha(t) \equiv t$, we get the celebrated Bihari's inequality.

(2) Set $w(u) \equiv u$. Note that $G(u) = \int_1^\infty (1/s) ds = \infty$; then the previous result (2) holds.

(3) Compared with Theorem 1 in [19], this lemma has a different range of applications.

**Corollary 3.** *Suppose that $u$, $f$, $g \in C([t_0, T), \mathbb{R}^+)$. Let $\alpha$, $\beta \in C^1([t_0, T), [t_0, T))$ be nondecreasing with $\alpha(t)$, $\beta(t) \le t$ on $[t_0, T)$. If*

$$u(t) \le M_1 + M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) u(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) u(s) ds, \tag{11}$$

*where $M_i$ $(i = 1, 2, 3)$ is nonnegative constants, then, for $t_0 \le t < T$, one has*

$$u(t) \le M_1 \exp\left( M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) ds + M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) ds \right). \tag{12}$$

*Remark 4.* It is easy to get the following results.

(1) Assume that $M_2 = 1$ and $M_3 = 0$; we know that corollary in [18] is valid.

(2) With $M_3 = 0$ and $\alpha(t) = t$, we obtain the celebrated Gronwall-Bellman inequality.

**Theorem 5.** *Suppose that $u, f, g \in C([t_0, T), \mathbb{R}^+)$. Let $w \in C(\mathbb{R}^+, \mathbb{R}^+)$ be nondecreasing with $w(u) > 0$ for $u > 0$ and let $\alpha, \beta \in C^1([t_0, T), [t_0, T))$ be nondecreasing with $\alpha(t), \beta(t) \le t$ on $[t_0, T)$. If*

$$\begin{aligned} u(t) \le M_1 &+ M_2 \int_{\alpha(t_0)}^{\alpha(t)} f(s) w(u(s)) ds \\ &+ M_3 \int_{\beta(t_0)}^{\beta(t)} g(s) w(u(s)) ds \\ &+ M_4 \int_{t_0}^{T} [u(s)]^{\lambda} ds, \end{aligned} \tag{13}$$

*where $M_i$ $(i = 1, 2, 3, 4)$ are nonnegative constants, $0 \le \lambda < 1$, and $\int_1^{\infty} (1/w(s)) ds = \infty$, then, for $t_0 \le t < T$, one has*

$$u(t) \le m^{-}(0), \tag{14}$$

*where $m(s) = (2s - MM_1)^{1-\lambda} - s^{1-\lambda} - (1-\lambda) MM_4 T$, $M$ is a positive constant, and $m^{-}(\cdot)$ represents the inverse of $m(\cdot)$.*

*Proof.* By Lemma 1 and Remark 2 (2), for $t_0 \le t < T$, there exists a constant $M > 0$ such that

$$u(t) \le M \left( M_1 + M_4 \int_{t_0}^{T} [u(s)]^{\lambda} ds \right). \tag{15}$$

Define

$$\begin{aligned} &p(t) \\ &= M \left( M_1 + M_4 \int_{t_0}^{t} [u(s)]^{\lambda} ds + M_4 \int_{t_0}^{T} [u(s)]^{\lambda} ds \right); \end{aligned} \tag{16}$$

we have

$$\begin{aligned} p(t_0) &= MM_1 + MM_4 \int_{t_0}^{T} [u(s)]^{\lambda} ds, \\ p'(t) &\le MM_4 [p(t)]^{\lambda}. \end{aligned} \tag{17}$$

Integrating from $t_0$ to $t$, we get

$$[p(t)]^{1-\lambda} - [p(t_0)]^{1-\lambda} \le (1-\lambda) MM_4 (t - t_0); \tag{18}$$

then

$$p(t) \le \left\{ [p(t_0)]^{1-\lambda} + (1-\lambda) MM_4 (t - t_0) \right\}^{1/(1-\lambda)}. \tag{19}$$

Since

$$\begin{aligned} &2p(t_0) - MM_1 = p(T) \\ &\quad \le \left\{ [p(t_0)]^{1-\lambda} + (1-\lambda) MM_4 (T - t_0) \right\}^{1/(1-\lambda)}, \end{aligned} \tag{20}$$

we can deduce that

$$(2p(t_0) - MM_1)^{1-\lambda} - [p(t_0)]^{1-\lambda} \le (1-\lambda) MM_4 T. \tag{21}$$

Let

$$m(s) = (2s - MM_1)^{1-\lambda} - s^{1-\lambda} - (1-\lambda) MM_4 T. \tag{22}$$

Observe that $m \in C([MM_1/2, \infty], \mathbb{R}^+)$ and

$$\begin{aligned} &m\left(\frac{MM_1}{2}\right) = -\left(\frac{MM_1}{2}\right)^{1-\lambda} - (1-\lambda) MM_4 T < 0, \\ &\lim_{s \to 0} \frac{m(s)}{s^{1-\lambda}} = 2^{1-\lambda} - 1 > 0. \end{aligned} \tag{23}$$

It is easy to get that there exists a $s_0$ such that $m(s_0) = 0$; then $p(t_0) \le s_0$. Therefore

$$u(t) \le p(t_0) \le m^{-}(0), \quad t \in [t_0, T). \tag{24}$$

The proof is completed. □

*Remark 6.* (i) If $M_1 = M_4 = 0$, we have $m^{-}(t_0) = 0$; that is, $u(t) \equiv 0$.

(ii) Generally speaking, the spectral radius of Fredholm operators should not be less than one. However, there is no doubt that here the above inequality is satisfied as a particular case.

## 3. Controllability of Differential Systems of Mixed Type

In this section, we shall give an application to show that the proposed inequalities are useful in investigating the existence of mild solutions and controllability of differential systems of mixed type. Unfortunately, since the spectral radius of Fredholm operators should not be less than one, the inequality used in previous paper may be not suitable (see [21–26]). Therefore, more powerful integral inequalities should be established to solve the problem. In order to illustrate this problem, we consider the following impulsive integrodifferential system in a Banach space:

$$\begin{aligned} x'(t) &= A(t) x(t) + f(t, x(t), (Sx)(t), (Tx)(t)) \\ &\quad + (Bu)(t), \quad t \in I = [0, b], \\ \Delta x(t_i) &= I_i(x(t_i)) = x(t_i^+) - x(t_i^-), \\ &\qquad i = 1, 2, \ldots, s, \\ x(0) &= x_0, \end{aligned} \tag{25}$$

where operators $S$ and $T$ are defined as follows:

$$\begin{aligned} (Sx)(t) &= \int_0^t k(t, s, x(s)) ds, \\ (Tx)(t) &= \int_0^b h(t, s, x(s)) ds. \end{aligned} \tag{26}$$

$A(t)$ is a family of linear operators which generates an evolution operator

$$\begin{aligned} G : \Delta &= \{(t, s) \in [0, b] \times [0, b] : 0 \le s \le t \le b\} \\ &\longrightarrow L(\mathbb{X}), \end{aligned} \tag{27}$$

where $L(\mathbb{X})$ is the space of all bounded linear operators in $\mathbb{X}$ and $\mathbb{X}$ is a Banach space. Assume that $k \in C[\Delta \times \mathbb{X}, \mathbb{X}]$ and $h \in C[I \times I \times \mathbb{X}, \mathbb{X}]$. $f$ is continuous. $0 = t_0 < t_1 < t_2 < \cdots < t_s < t_{s+1} = b$. $I_i \in C[\mathbb{X}, \mathbb{X}]$, $(i = 1, 2, \ldots, s)$ are impulsive functions, and $x(t_i^+)$ and $x(t_i^-)$ represent the right and the left limits of $x(t)$ at $t = t_i$, respectively. $B \in L[U, \mathbb{X}]$ is a bounded linear operator and the control function $u(\cdot)$ is given in $L^2[I, U]$ and $U$ is a Banach space. Set $T_r = \{x \in \mathbb{X} \mid \|x\| \le r\}$ and $B_r = \{x \in \mathrm{PC}[I, \mathbb{X}] \mid \|x\| \le r\}$. $\mathrm{PC}[I, \mathbb{X}] = \{x \in C[I, \mathbb{X}] \mid x$ is continuous on $(t_i, t_{i+1})$, $i = 0, 1, \ldots, s$, $x(t_i^-) = x(t_i)$, and the right limit $x(t_i^+)$ exists, $i = 1, 2, \ldots, s\}$. Obviously, $\mathrm{PC}[I, \mathbb{X}]$ is a Banach space with the norm $\|x\|_{\mathrm{PC}} = \sup_{t \in I}\{\|x(t)\|\}$.

Suppose that the following hypotheses are satisfied.

$(H_1)$ $A(t) : D(A) \to \mathbb{X}$ is a family of linear operators, generating an equicontinuous evolution system $\{G(t, s) : (t, s) \in \Delta\}$; that is, $(t, s) \to \{G(t, s)x : x \in B_r\}$ is equicontinuous for $t > 0$ and for all bounded subsets $B_r$.

$(H_2)$ For any $r > 0$, $f$ is uniformly continuous on $I \times T_r \times T_r \times T_r$ and $I_i$ $(i = 1, 2, \ldots, s)$ are bounded on $T_r$. There exist functions $b_p \in C[I, \mathbb{R}^+]$ $(p = 1, 2, 3, 4)$ and $k^*$, $h^* > 0$ such that

$$\begin{aligned} \|f(t, x, y, z)\| &\le b_1(t) + b_2(t)\|x\|^\lambda + b_3(t)\|y\| \\ &\quad + b_4(t)\|z\|, \\ \|k(t, s, x)\| &\le k^* \|x\|^\lambda, \\ \|h(t, s, x)\| &\le h^* \|x\|^\lambda, \end{aligned} \tag{28}$$

where $0 \le \lambda < 1$. Define $b_p = \max\{b_p(t) \mid t \in I\}$.

$(H_3)$ The linear operator $W : L^2[I, U] \to \mathbb{X}$ is defined by

$$Wu = \int_0^b G(b, s) Bu(s)\, ds. \tag{29}$$

(i) $W$ has an invertible operator $W^{-1}$ which takes values in $L^2[I, U]/\mathrm{Ker}W$ and there exist positive constants $L_B$ and $L_W$ such that $\|B\| \le L_B$ and $\|W^{-1}\| \le L_W$;

(ii) there exists $K_W \in C[I, \mathbb{R}^+]$ such that, for any bounded set $H \subset \mathbb{X}$,

$$\alpha\left(\left(W^{-1}H\right)(t)\right) \le K_W(t)\, \alpha(H). \tag{30}$$

Define $K_W = \max\{K_W(t) \mid t \in I\}$; $\alpha(\cdot)$ represents the Kuratowski noncompactness measure.

$(H_4)$ There exist $l_q \in C[I, \mathbb{R}^+]$ $(q = 1, 2)$ such that, for any equicontinuous set, $D \subset B_r$, such that

$$\begin{aligned} &\alpha\left(f(t, D(s), (TD)(s), (SD)(s))\right) \\ &\quad \le l_1(t)\left[\alpha(D(s))\right]^\lambda + l_2(t)\, \alpha((TD)(s)). \end{aligned} \tag{31}$$

Define $l_q = \max\{l_q(t) \mid t \in I\}$.

**Theorem 7.** *Assume that conditions $(H_1)$–$(H_4)$ hold. Then the system (25) is controllable.*

*Proof.* Using $(H_3)$ (i), for every $x \in \mathrm{PC}[I, X]$, without loss of generality, define the control

$$\begin{aligned} u_{0x}(t) &= W^{-1}\Bigg[x_1 - G(b, 0)x_0 \\ &\quad - \int_0^b G(b, s) f(s, x(s), (Tx)(s), (Sx)(s))\, ds\Bigg](t), \\ u_{jx}(t) &= W^{-1}\Bigg[x_1 - G(b, 0)x_0 \\ &\quad - \int_0^b G(b, s) f(s, x(s), (Tx)(s), (Sx)(s))\, ds \\ &\quad - \sum_{i=1}^{j} G(t, t_i) I_i(x(t_i))\Bigg](t), \end{aligned} \tag{32}$$

where $j = 1, \ldots, s$. Define operator $Q$ as follows:

$$\begin{aligned} (Qx)(t) &= G(t, 0)x_0 + \int_0^t G(t, s)\left(f + Bu_{sx}\right)(s)\, ds \\ &\quad + \sum_{0 < t_i < t} G(t, t_i) I_i(x(t_i)); \end{aligned} \tag{33}$$

clearly, using the control $u_{sx}(t)$, the fixed point of operator $Q$ is a solution of the system (25), and $x_1 = (Qx)(b)$; that is, system (25) is controllable. From the conditions we imposed, it is easy to get that operator $Q$ is continuous.

Set $\Omega_0 = \{x \in \mathrm{PC}[I, \mathbb{X}] \mid x = \lambda Qx,\ 0 \le \lambda \le 1\}$. Assume that there exists $\lambda_0 \in [0, 1]$ such that $\overline{x}(t) = \lambda_0(Q\overline{x})(t)$. Next, we shall use the method of piecewise discussion.

(i) When $t \in [0, t_1]$,

$$\overline{x}(t) = \lambda_0 G(t, 0)x_0 + \lambda_0 \int_0^t G(t, s)\left(f + Bu_{0\overline{x}}\right)(s)\, ds. \tag{34}$$

From Ji et al. [23], we know that there exists $L_G > 0$ such that $\|G(t, s)\| \le L_G$ for any $(t, s) \in I \times I$. Thus

$$\begin{aligned} \|\overline{x}(t)\| &\le L_G\|x_0\| + L_G \int_0^t \left\|\left(f + Bu_{0\overline{x}}\right)(s)\right\| ds \\ &\le L_G\|x_0\| + L_G \int_0^t \|f\|\, ds \\ &\quad + L_G L_B \int_0^t \|u_{0\overline{x}}(s)\|\, ds, \end{aligned} \tag{35}$$

where

$$\begin{aligned}
&\int_0^t \|f\|\,ds \\
&\quad\le \int_0^b b_1(s)\,ds + b_2\int_0^t \|\overline{x}(s)\|^\lambda\,ds \\
&\qquad + b_3k^*t_1\int_0^t \|\overline{x}(s)\|^\lambda\,ds + b_4h^*t_1\int_0^b \|\overline{x}(s)\|^\lambda\,ds, \\
&\|u_{0\overline{x}}\|_{L^2} \\
&\quad= \left\|W^{-1}\left[x_1 - G(b,0)\,x_0 - \int_0^b G(b,s)\,f\,ds\right]\right\|_{L^2} \\
&\quad\le L_W\left[\|x_1\| + L_G\|x_0\| + L_G\int_0^b \|f\|\,ds\right].
\end{aligned} \tag{36}$$

Let $u(t) = \|\overline{x}(t)\|$; then $u(t) \in C[[0,t_1],\mathbb{R}^+]$; we have

$$\begin{aligned}
u(t) &\le L_G\|x_0\| + L_GL_Bb\left[L_W\|x_1\| + L_WL_G\|x_0\|\right] \\
&\quad + L_G\left[\int_0^b b_1(s)\,ds + b_2\int_0^t [u(s)]^\lambda\,ds\right. \\
&\quad \left. + b_3k^*t_1\int_0^t [u(s)]^\lambda\,ds + b_4h^*t_1\int_0^b [u(s)]^\lambda\,ds\right] \\
&\quad + L_G^2L_BL_Wb\left[\int_0^b b_1(s)\,ds + b_2\int_0^b [u(s)]^\lambda\,ds\right. \\
&\quad \left. + b_3k^*t_1\int_0^b [u(s)]^\lambda\,ds + b_4h^*t_1\int_0^b [u(s)]^\lambda\,ds\right].
\end{aligned} \tag{37}$$

Since $\int_1^\infty (1/s^\lambda)ds = \infty$, then by Theorem 5, there exists a constant $C_0$ such that $u(t) \le C_0$, $t \in [0,t_1]$; that is, there exists a constant $C_0$ independent of $u$ such that $u(t) \le C_0$, $t \in [0,t_1]$. The above inequality implies that $\|\overline{x}(t)\| \le C_0$. From $(H_2)$, there also exists a constant $M_0 > 0$ independent of $\overline{x}$ such that

$$\begin{aligned}
&\left\|f\left(t,\overline{x}(t),\overline{x}'(t),(T\overline{x})(t),(S\overline{x})(t)\right)\right\| \le M_0, \\
&\qquad \left\|I_1\left(\overline{x}(t_1)\right)\right\| \le M_0, \\
&\qquad \forall t \in [0,t_1],\quad \|\overline{x}(t)\| \le C_0.
\end{aligned} \tag{38}$$

Thus $\|\overline{x}(t_1^+)\| = \|\overline{x}(t_1) + I_1(\overline{x}(t_1))\| \le C_0 + M_0$.

(ii) When $t \in (t_1,t_2]$,

$$\overline{v}(t) = \begin{cases} \overline{x}(t), & t \in (t_1,t_2], \\ \overline{x}(t_1^+), & t = t_1. \end{cases} \tag{39}$$

Then $\overline{v}(t) \in C[[t_1,t_2],\mathbb{X}]$ and

$$\begin{aligned}
\overline{v}(t) &= \lambda_0G(t,0)\,x_0 + \lambda_0\int_0^{t_1} G(t,s)\left(f + Bu_{1\overline{v}}\right)(s)\,ds \\
&\quad + \lambda_0\int_{t_1}^t G(t,s)\left(f + Bu_{1\overline{v}}\right)(s)\,ds \\
&\quad + \lambda_0G(t,t_1)\,I_1\left(x(t_1)\right).
\end{aligned} \tag{40}$$

From results (38) and similar to the proof of (i), we can know that there exists $C_1 > 0$ that does not depend on $\overline{x}$ such that $\|\overline{v}(t)\| \le C_1$, $t \in [t_1,t_2]$. So $\|\overline{x}(t)\| \le C_1$, $t \in (t_1,t_2]$.

By the same method as above, we can prove that there exists a constant $C_s > 0$ that does not depend on $\overline{x}$ such that

$$\|\overline{x}(t)\| \le C_s, \quad t \in (t_s,b]. \tag{41}$$

Let $C = \max\{C_i \mid 0 \le i \le s\}$; then $\|\overline{x}(t)\| \le C$, $t \in I$. Thus $\Omega_0$ is a bounded set in $\mathrm{PC}[I,\mathbb{X}]$. Take $R > C$; let $\Omega = \{x \in \mathrm{PC}[I,\mathbb{X}] \mid \|x\|_{\mathrm{PC}} < R\}$; obviously $\Omega$ is a bounded open set in $\mathrm{PC}[I,\mathbb{X}]$ and $\theta \in \Omega$. From the choice of $R$, we know that if $x \in \partial\Omega$ and $\lambda \in [0,1]$, we have $x \ne \lambda Qx$.

Let $H \subset \overline{\Omega}$ be a countable set and $H \subset \overline{\mathrm{co}}(\{\theta\}\bigcup Q(H))$. By $(H_1)$ and $(H_2)$, it is easy to see that $Q(H)$ is equicontinuous on each $[t_i,t_{i+1}]$, $(i = 0,1,2,\ldots,s)$.

Next, we shall prove that $(QH)(t)$ is relatively compact for each $[t_i,t_{i+1}]$. In the same way, we discuss step by step as follows.

(i) When $t \in [t_0,t_1]$

$$\begin{aligned}
&\alpha(H(t)) \le \alpha((QH)(t)) \le \alpha\left(G(t,0)\,x_0 \mid x \in H\right) \\
&\quad + \alpha\left(\int_0^t G(t,s)\left(f + Bu_{0x}\right)(s)\,ds \mid x \in H\right) \\
&\quad \le 2L_Gl_1\int_0^t [\alpha(H(s))]^\lambda\,ds \\
&\quad + 2L_Gl_2k^*b\int_0^t \alpha(H(s))\,ds \\
&\quad + L_GL_B\int_0^t \alpha\left(u_{0x} \mid x \in H\right)ds, \\
&\alpha\left(u_{0x} \mid x \in H\right) \le K_W\left(2L_Gl_1\int_0^b [\alpha(H(s))]^\lambda\,ds\right. \\
&\quad \left. + 2L_Gl_2k^*b\int_0^t \alpha(H(s))\,ds\right).
\end{aligned} \tag{42}$$

Let $m(t) = \alpha(H(t))$, $t \in [0,t_1]$; then $m(t) \in C[[0,t_1],\mathbb{R}^+]$. Thus

$$\begin{aligned}
m(t) &\le 2L_Gl_1\int_0^t [\alpha(H(s))]^\lambda\,ds \\
&\quad + 2L_Gl_2k^*b\int_0^t \alpha(H(s))\,ds
\end{aligned}$$

$$+ 2K_W L_G l_1 b \int_0^t [\alpha(H(s))]^\lambda ds + 2K_W L_G l_2 b k^* t \int_0^t \alpha(H(s)) ds. \tag{43}$$

By Remark 6 (i), we have $m(t) \equiv 0, t \in [0, t_1]$. Thus $\alpha(H(t)) = 0$; that is, $H$ is a relatively compact set in $PC[[0, t_1], \mathbb{X}]$. Since $I_1 \in C[\mathbb{X}, \mathbb{X}]$, $\alpha(I_1(H(t_1))) = 0$, then $H(t_1)$ is a relatively compact set in $\mathbb{X}$.

(ii) For $t \in (t_1, t_2]$, we know

$$\begin{aligned} &\alpha(H(t)) \\ &\quad \le \alpha(G(t,0)x_0 \mid x \in H) \\ &\qquad + \alpha\left(\int_0^t G(t,s)(f + Bu_{1x})(s) ds \mid x \in H\right) \\ &\qquad + \alpha(I_1(x(t_1)) \mid x \in H) \\ &\quad \le 2L_G l_1 \int_0^t [\alpha(H(s))]^\lambda ds \\ &\qquad + 2L_G l_2 k^* b \int_0^t \alpha(H(s)) ds \\ &\qquad + L_G L_B \int_0^t \alpha(u_{1x} \mid x \in H) ds. \end{aligned} \tag{44}$$

Similar to the proof of (i), we can deduce that $m(t) \equiv 0$, $t \in [t_1, t_2]$. Therefore $\alpha(H(t)) = 0, t \in [t_1, t_2]$. In particular, $\alpha(H(t_2)) = 0$, so $H$ is a relatively compact set in $C[[t_1, t_2], \mathbb{X}]$. Similarly, we can show that $H$ is a relatively compact set in $C[[t_i, t_{i+1}], \mathbb{X}]$ $(i = 2, 3, \ldots, s)$. Thus $H$ is a relatively compact set in $PC[I, \mathbb{X}]$.

In conclusion, we deduce that $Q$ has at least one fixed point in $\Omega$ by the Mönch fixed point theorem; that is, system (25) has at least one mild solution in $PC[I, \mathbb{X}]$. Thus system (25) is controllable on $I$. □

## 4. The Uniqueness and Global Existence of Solutions

Consider the following integral equation:

$$u(t) = k(t) + \int_0^{\alpha(t)} f(s) w(u(s)) ds + \int_0^{\beta(t)} g(s) w(u(s)) ds, \tag{45}$$

where $k, f, g \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $w(0) = 0$ and $\alpha, \beta \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ are nondecreasing with $\alpha(t), \beta(t) \le t$ on $\mathbb{R}\mathbb{R}^+$. Assume that (45) has a solution $u \in C([0, T], \mathbb{R}^+)$ on some maximal interval of existence $[0, T]$. Moreover, if $T < \infty$,

$$\limsup_{t \to T} u(t) = \infty. \tag{46}$$

**Theorem 8.** *Assume that*

$$\|w(x) - w(y)\| \le Z(\|x - y\|) \tag{47}$$

*with $Z \in C(\mathbb{R}^+, \mathbb{R}^+)$ nondecreasing, $Z(x) > 0$ for $x > 0$. If*

$$\int_0^1 \frac{1}{Z(s)} ds = \int_1^\infty \frac{1}{Z(s)} ds = \infty, \tag{48}$$

*then (45) has a unique solution defined on $\mathbb{R}^+$. Moreover, if $k$ is bounded on $\mathbb{R}^+$ and if either $\alpha$, $\beta$ is bounded on $\mathbb{R}^+$ or $\int_0^\infty f(s)ds$, $\int_0^\infty g(s)ds < \infty$, then its solution is bounded on $\mathbb{R}^+$.*

*Proof.* Suppose that, on some interval $[0, t_0]$, (45) has two solutions $u_1, u_2 \in C([0, t_0], \mathbb{R}^+)$; we obtain

$$\begin{aligned} u_1(t) - u_2(t) &= \int_0^{\alpha(t)} f(s)[w(u_1(s)) - w(u_2(s))] ds \\ &\quad + \int_0^{\beta(t)} g(s)[w(u_1(s)) - w(u_2(s))] ds. \end{aligned} \tag{49}$$

Denote $u(t) = \|u_1(t) - u_2(t)\|$; we have

$$u(t) \le \int_0^{\alpha(t)} f(s) Z(u(s)) ds + \int_0^{\beta(t)} g(s) Z(u(s)) ds. \tag{50}$$

Set

$$G(r) = \int_1^r \frac{1}{Z(s)} ds, \quad r > 0. \tag{51}$$

Then $G(0) = -\infty$ and $G(\infty) = \infty$. There exists $\epsilon > 0$, where

$$u(t) \le \epsilon + \int_0^{\alpha(t)} f(s) Z(u(s)) ds + \int_0^{\beta(t)} g(s) Z(u(s)) ds, \quad 0 \le t \le t_0. \tag{52}$$

From Theorem 5, we know that

$$u(t) \le G^-\left(G(\epsilon) + \int_0^{\alpha(t)} f(s) Z(u(s)) ds + \int_0^{\beta(t)} g(s) Z(u(s)) ds\right), \quad 0 \le t \le t_0. \tag{53}$$

From Remark 6, $u(t) \to 0$, $\epsilon \to 0$; then $u_1(t) = u_2(t)$, and the uniqueness of the solution can be obtained.

Next, we will show that the solution is global; that is, $T = \infty$, where $T$ is the maximal time of existence. If $T < \infty$, we set $k_0 = \max_{0 \le t \le T}\{k(t)\}$, and we obtain that

$$u(t) \le k_0 + \int_0^{\alpha(t)} f(s) Z(u(s)) ds + \int_0^{\beta(t)} g(s) Z(u(s)) ds, \quad 0 \le t < T, \tag{54}$$

as $w(u(s)) = w(u(s)) - w(0) \leq Z(u(s))$ for $0 \leq t < T$. By Lemma 1 and $u \in C[[0,T],\mathbb{R}^+]$, we deduce that

$$u(t) \leq G^{-}\left(G(k_0) + \int_0^{\alpha(t)} f(s)\,ds + \int_0^{\beta(t)} g(s)\,ds\right), \quad 0 \leq t < T. \tag{55}$$

Since $k$ is bounded on $\mathbb{R}^+$ and either $\alpha, \beta$ is bounded on $\mathbb{R}^+$ or $\int_0^\infty f(s)ds, \int_0^\infty g(s)ds < \infty$ satisfies; then $u(t)$ is bounded on $\mathbb{R}^+$. Thus the previous inequality (55) contradicts (46). Thus the global existence is proved. □

## 5. Conclusions

This paper generalizes a more general Gronwall-like inequality with a Fredholm operator. Using the proposed inequality, we solve a difficult problem in the research of the controllability of integrodifferential systems of mixed type in Banach space. Meanwhile, we also prove the uniqueness and global existence of solutions for a class of integral equations. Therefore, the results we obtained are very important and powerful tools. However, it should be more useful than we can imagine in qualitative properties of many other nonlinear problems, such as existence, estimation of solutions, dependence of solutions on parameters in nonlinear analysis, and control.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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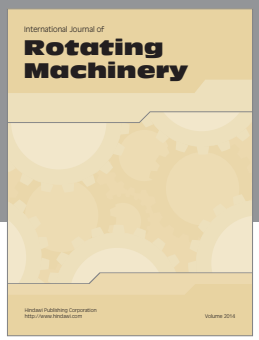

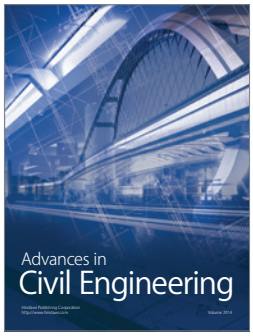


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