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Research Article

Distributed Optimization of Multiagent Systems in Directed Networks with Time-Varying Delay

Junxiu Yan and Hui Yu

College of Science, China Three Gorges University, Yichang 443002, China

Correspondence should be addressed to Hui Yu; yuhui@ctgu.edu.cn

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This paper addresses a distributed consensus optimization problem of a first-order multiagent system with time-varying delay. A continuous-time distributed optimization algorithm is proposed. Different from most ways of solving distributed optimization problem, the Lyapunov-Razumikhin theorem is applied to the convergence analysis instead of the Lyapunov-Krasovskii functionals with LMI conditions. A sufficient condition for the control parameters is obtained to make all the agents converge to the optimal solution of the system. Finally, an example is given to validate the effectiveness of our theoretical result.

1. Introduction

In recent years, the distributed optimization problem of multiagent systems has been investigated by many researchers; researches on distributed optimization and control theorem have been developing rapidly and have been applied to various fields of industry and defense, like smart grid [1, 2], sensor networks [3], social networks [4], and so on. The objective of distributed optimization problem is to solve an optimization problem cooperatively in a distributed way, where the objective function is formed by a sum of local objective functions, and each agent can only know one local objective function. The ultimate goal is to make the states of all the agents converge to optimal solution of the optimization problem via local coordination. Compared with the consensus problem of multiagent systems, which makes all agents achieve a common state [5–8], not only does the optimization problem make all agents achieve the same state, but also at the same time the achieved state minimizes the optimization problem.

The current literatures about distributed optimization problems are more focused on discrete-time algorithms (see [9–12] and references therein). In both papers [9, 11], discrete-time subgradient algorithms are proposed for unconstrained, separable, convex optimization problem and each agent communicates with the other agents over a time-varying network topology. A projected consensus subgradient algorithm is proposed for constrained optimization problem in [10], and, in [12], the authors devise two distributed primal-dual subgradient algorithms over networks with dynamically changing topologies but satisfying a standard connectivity property. But, recently, some continuous-time methods have also been successfully used to solve distributed optimization problem. Based on the gradient algorithm and integral feedback, auxiliary-variables are introduced in continuous-time dynamical system [13–15]. From the control system viewpoint, a continuous-time multiagent system dynamic is proposed with undirected communication topology [13]; the algorithm is further investigated over a strongly connected and weight balanced directed graph [16], and even a modified system is proposed in [14] with auxiliary-variables no longer needing to exchange information. In [17], the authors present a second-order multiagent system for distributed optimization network under bound constraints, and, in [18], a distributed protocol design for the high-order agent-network under a connected communication topology is proposed. In order to avoid using auxiliary-variables, a family of Zero-Gradient-Sum algorithms are proposed over fixed communication topology in [19].

On the other hand, it is common that time-delay exists in practical systems because of the finite speeds of information transmission and spreading as well as traffic congestions. Therefore, time-delay should be taken into account in algorithm design of multiagent systems. For time-delay systems modelled by delayed differential equations, an effective way to deal with their convergence and stability analysis is based on the Lyapunov-Krasovskii functionals or Lyapunov-Razumikhin functions. Most of the existing works concentrate on Lyapunov functions combining with Linear Matrix Inequality (LMI) techniques to deal with the consensus problem of multiagent systems with time-delay [20, 21]. The methods based on Lyapunov-Krasovskii functionals can be applied to a wide variety of problems and may provide necessary and sufficient conditions of convergence and stability, but it often leads to computational complexity and poor scalability. When the number of the agents is large, it would be difficult to verify the solvability of the LMI conditions. However, based on the Lyapunov-Razumikhin theorem, the authors propose a neighbor-based distributed controller [7, 8] enabling the agents to achieve consensus along with interconnection delays, which can avoid verifying the LMI condition and reducing computational burden. In [15], distributed consensus optimization algorithms are proposed for continuous-time multiagent systems with time-delay, and some sufficiency conditions based on LMI are obtained.

Motivated by the above observations, the distributed consensus optimization problem of continuous-time multiagent systems with time-varying delay is considered. The interconnected graph is assumed to be directed, strongly connected, and weighted-balanced. The Lyapunov-Razumikhin function is used in the stability analysis. The convergence of the proposed algorithm is guaranteed with the model parameters satisfying some conditions. Meanwhile, the conditions can also give an estimate of the upper bound of the time-delay, which can avoid verifying and calculating the complicated LMI conditions. From the results, we can also see clearly the relationship among the parameters in the system.

The outline of this paper is organized as follows. Some basic knowledge on the algebraic graph theory and useful lemmas are presented in Section 2. The convergence results of the algorithm are established under the given communication condition on network topology by applying Lyapunov-Razumikhin Theorem in Section 3. An example is provided to illustrate the result in this paper in Section 4. Finally, the concluding remarks are given in Section 5.

Notations. $\mathscr{R}$ and $\mathscr{R}^n$ represent the set of real numbers and the set of $n \times 1$ real vectors, respectively; $I_n \in \mathscr{R}^{n\times n}$ is the $n \times n$ identity matrix; $\mathbf{1}_n$ (or $\mathbf{0}_n$) denotes an n dimensional column vector whose all entries being 1 (or 0); A^T represents the transpose of a matrix A; for vectors $x_1, x_2, \ldots, x_n$, $\mathrm{col}(x_1, x_2, \ldots, x_n) = [x_1^T, x_2^T, \ldots, x_n^T]^T$; for a vector w, then $\|w\| = \sqrt{w^T w}$ represents the standard Euclidean norm.

2. Preliminaries and Problem Statement

2.1. Preliminaries. Consider a multiagent system consisting of N agents, if each agent is regarded as a node, the communication topology among these agents can be described by a weighted digraph $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$ with the finite set of nodes $\mathscr{V} = \{1, 2, \ldots, N\}$ and edge set $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$. An edge starts from i and ends on j, which means that agent i can send information to agent j. The weighted adjacency matrix $\mathscr{A} = [a_{ij}] \in R^{N\times N}$ is defined as $a_{ij} > 0$ if $(i, j) \in \mathscr{E}$ and $a_{ij} = 0$ otherwise. If $\sum_{j=1}^{N} a_{ij} = \sum_{j=1}^{N} a_{ji}$ for all $i \in \mathscr{V}$, the digraph $\mathscr{G}$ is called weighted-balanced. A path is a sequence of connected edges in a graph. If there is a path between any two nodes of a digraph $\mathscr{G}$, then digraph $\mathscr{G}$ is said to be strongly connected, otherwise disconnected. The degree matrix $\mathscr{D} = \mathrm{diag}\{d_1, d_2, \ldots, d_N\} \in \mathscr{R}^{N\times N}$ of graph $\mathscr{G}$ is a diagonal matrix with the ith diagonal element being $d_i = \sum_{j=1}^{N} a_{ij}$ for $i \in \mathscr{V}$. The Laplacian of graph $\mathscr{G}$ is defined as $L = \mathscr{D} - \mathscr{A}$.

The next lemmas related to the important properties of Laplace L and provide useful mathematical tools.

Lemma 1 (see [22]). *Laplace matrix L has at least one zero eigenvalue with $\mathbf{1}_N = [1, 1, \ldots, 1] \in \mathscr{R}^N$ as its eigenvector, and all the nonzero eigenvalues of L have positive real parts. Laplacian L has a simple zero eigenvalue if and only if $\mathscr{G}$ is strongly connected.*

Lemma 2. *For matrices A, B, C and D with appropriate dimensions, the Kronecker product $\otimes$ satisfies* (1) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$; (2) $(A \otimes B)^T = A^T \otimes B^T$; (3) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

Lemma 3 (see [23]). *For a given real matrix $S = \begin{pmatrix} X & Y \\ Y^T & Z \end{pmatrix}$ with $X^T = X$ and $Z^T = Z$, then the following conditions are equivalent:*

(1) $S > 0$;

(2) $X > 0,\ Z - Y^T X^{-1} Y > 0$;

(3) $Z > 0,\ X - Y X^{-1} Y^T > 0$.

2.2. Problem Statement. We consider a multiagent system consisting of N agents. The dynamics of the ith agent, $i \in \mathscr{V}$, is described by

$$\dot{x}_i(t) = u_i(t), \tag{1}$$

where $x_i \in \mathscr{R}^m$ denotes the state of agent i and $u_i \in \mathscr{R}^m$ is the control input.

Consider the multiagent optimization problem, in which the goal is to minimize the sum of local cost functions associated with the individual agent. More specially, it can be expressed as

$$\text{minimize} \quad f(x) = \sum_{i=1}^{N} f_i(x), \quad x \in \mathscr{R}^m. \tag{2}$$

Let $\mathbf{x} = \mathrm{col}(x_1, x_2, \ldots, x_N) \in \mathscr{R}^{Nm}$. Next, we provide an alternative formulation of (2), that is,

$$\begin{aligned} &\text{minimize} \quad f(\mathbf{x}) = \sum_{i=1}^{N} f_i(x_i), \quad x_i \in \mathscr{R}^m, \\ &\text{subject to} \quad (L \otimes I_m)\mathbf{x} = 0_{Nm}. \end{aligned} \tag{3}$$

We can see that the problem (2) on $\mathscr{R}^m$ is equivalent to the problem (3) on $\mathscr{R}^{Nm}$.

In this paper, our goal is to design a distributed controller for each agent such that the states of all the agents converge to the optimal solution of the optimization problem (2) via local communication.

Before proceeding, we give the following assumption on the local cost function f_i based on convex analysis [24].

Assumption 4. (a) For each $i \in \mathscr{V}$, f_i is differentiable and its gradient is Lipschitz with constant $\rho_i > 0$ in $\mathscr{R}^m$:

$$\left\|\nabla f_i(x) - \nabla f_i(y)\right\| \leq \rho_i \left\|x - y\right\|, \quad \forall x, y \in \mathscr{R}^m. \tag{4}$$

(b) for $i \in \mathscr{V}$, f_i is m_i-strongly convex with constant $m_i > 0$:

$$(x - y)^T \left(\nabla f_i(x) - \nabla f_i(y)\right) \geq m_i \left\|x - y\right\|^2, \quad \forall x, y \in \mathscr{R}^m. \tag{5}$$

Remark 5. Under Assumption 4(b), we can note that f is strictly convex; then the problem (3) has an unique optimal solution.

Assumption 6. The digraph $\mathscr{G}$ is weighted-balanced and strongly connected.

From Lemma 1 and Assumption 6, there exists a matrix $Q \in \mathscr{R}^{N\times(N-1)}$ with

$$\begin{aligned} \mathbf{1}_N^T Q &= 0, \\ Q^T Q &= I_{N-1}, \\ QQ^T &= I_N - \frac{1}{N}\mathbf{1}_N\mathbf{1}_N^T, \end{aligned} \tag{6}$$

such that the matrix $Q^T L Q = H$, where the real parts of all the eigenvalues of H are positive, and $H + H^T$ is positive definite.

When considering the presence of time-varying communication delay among the information transmission, the continuous-time distributed optimization protocol is proposed for agent i $(i \in \mathscr{V})$ as follows:

$$\begin{aligned} u_i(t) &= -k\sum_{j=1}^{N} a_{ij}\left[x_i(t-\tau(t)) - x_j(t-\tau(t))\right] - w_i(t) \\ &\quad - \gamma\nabla f_i(x_i(t)), \\ \dot{w}_i(t) &= \alpha\sum_{j=1}^{N} a_{ij}\left[x_i(t-\tau(t)) - x_j(t-\tau(t))\right], \\ &\qquad w_i(0) = 0, \end{aligned} \tag{7}$$

where $w_i(t)$ is an auxiliary state of agent i and $\tau(t)$ is a continuously differentiable function satisfying $\tau(t) \in [0, \tau]$ with $\tau > 0$ for all $t > 0$ and $k,\ \alpha,\ \gamma$ are the scalar tuning positive parameters; $-\gamma\nabla f_i(x_i(t))$ is the gradient term to guide the agents for optimization; $-k\sum_{j=1}^{N} a_{ij}[x_i(t-\tau(t)) - x_j(t-\tau(t))]$ is the consensus term with time-delay to make all the agents converge to the same point; $-w_i(t)$ is an integral term to correct the error caused by the consensus term.

Let

$$\begin{aligned} \mathbf{w}(t) &= \operatorname{col}\left(w_1(t), w_2(t), \ldots, w_N(t)\right), \\ \nabla\overline{f}(x(t)) & \\ &= \operatorname{col}\left(\nabla f_1(x_1(t)), \nabla f_2(x_2(t)), \ldots, \nabla f_N(x_N(t))\right). \end{aligned} \tag{8}$$

Then the closed-loop system of (1) and (7) can be expressed as a compact form:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= -k\left(L \otimes I_m\right)\mathbf{x}(t-\tau(t)) - \mathbf{w}(t) - \gamma\nabla\overline{f}(\mathbf{x}(t)), \\ \dot{\mathbf{w}}(t) &= \alpha\left(L \otimes I_m\right)\mathbf{x}(t-\tau(t)). \end{aligned} \tag{9}$$

Let the right-side of closed-loop system (9) be equal to 0; then we can get the equilibrium point $(\mathbf{x}^*, \mathbf{w}^*)$, that is,

$$\begin{aligned} -k\left(L \otimes I_m\right)\mathbf{x}^* - \mathbf{w}^* - \gamma\nabla\overline{f}\left(\mathbf{x}^*\right) &= 0, \\ \alpha\left(L \otimes I_m\right)\mathbf{x}^* &= 0. \end{aligned} \tag{10}$$

According to the properties of Laplacian matrix and from (10), one can obtain

$$\begin{aligned} \mathbf{x}^* &= \mathbf{1}_N \otimes \pi, \quad \pi \in \mathscr{R}^m, \\ \mathbf{w}^* &= -\gamma\nabla\overline{f}\left(\mathbf{x}^*\right). \end{aligned} \tag{11}$$

Under Assumption 6, we have $\mathbf{1}_N^T L = 0$. Left multiplying the second equation of (9) by $\mathbf{1}_N^T \otimes I_m$ and using initial conditions $w_i(0) = 0$, we obtain $\sum_{j=1}^{N} \dot{w}_j(t) = 0$; then

$$\sum_{j=1}^{N} w_j(t) = \sum_{j=1}^{N} w_j(0) = 0, \quad \forall t \geq 0. \tag{12}$$

Left multiplying the second equation of (11) by $\mathbf{1}_N^T \otimes I_m$ again results in

$$\begin{aligned} 0 = \sum_{j=1}^{N} w_j^* &= -\gamma\left(\mathbf{1}_N^T \otimes I_m\right)\nabla\overline{f}\left(\mathbf{x}^*\right) = -\gamma\sum_{j=1}^{N}\nabla f_i(\pi) \\ &= -\gamma\nabla f\left(\mathbf{x}^*\right). \end{aligned} \tag{13}$$

Thus, the optimal condition $\nabla f(\mathbf{x}^*) = 0$ is satisfied, which means $\mathbf{x}^* = \mathbf{1}_N \otimes x^*,\ x^* \in \mathscr{R}^m$ is the optimal solution of the optimization problem (3).

Using the transformation

$$\begin{aligned} \overline{\mathbf{x}}(t) &= \mathbf{x}(t) - \mathbf{x}^*, \\ \overline{\mathbf{w}}(t) &= \mathbf{w}(t) - \mathbf{w}^*, \end{aligned} \tag{14}$$

one can shift the equilibrium point into the origin; then the system (9) can be transformed into the following form:

$$\begin{aligned} \dot{\overline{\mathbf{x}}}(t) &= -k\left(L \otimes I_m\right)\overline{\mathbf{x}}(t-\tau(t)) - \overline{\mathbf{w}}(t) - \gamma\Psi\left(\overline{\mathbf{x}}(t)\right), \\ \dot{\overline{\mathbf{w}}}(t) &= \alpha\left(L \otimes I_m\right)\overline{\mathbf{x}}(t-\tau(t)), \end{aligned} \tag{15}$$

where $\Psi(\overline{\mathbf{x}}(t)) = \nabla\overline{f}(\mathbf{x}(t)) - \nabla\overline{f}(\mathbf{x}^*)$.

3. Main Results

Before analyzing the consensus and optimization problem (9), we introduce the stability of time-delay systems. Consider the following time-delay system:

$$\dot{x} = f(t, x_t), \quad t > t_0,$$
$$x(\theta) = \varphi(\theta), \quad \theta \in [-\tau, t_0], \tag{16}$$

where $x_t(\theta) = x(t+\theta)$, $\forall \theta \in [-\tau, t_0]$ and $f(t, 0) = 0$. In the sequel, suppose that $t_0 = 0$. Let $C([-\tau, 0], \mathscr{R}^n)$ be a Banach space of continuous function defined on an interval $[-\tau, 0]$, taking values in $\mathscr{R}^n$ with topology of uniform convergence, and with a norm $\|\varphi\|_c = \max_{\theta\in[-\tau,t_0]}\|\varphi(\theta)\|$.

The definition of the stability of the solution $x = 0$ is given as follows in terms of the solution of the delayed equation (16).

Lemma 7 (see [25]). *Let ϕ_1, ϕ_2, and ϕ_3 be continuous, nonnegative, nondecreasing function with $\phi_1(s) > 0$, $\phi_2(s) > 0$, $\phi_3(s) > 0$ for $s > 0$ and $\phi_1(0) = \phi_2(0) = 0$. For system (16), suppose that the function $f : \mathscr{R} \times C([-\tau, 0], \mathscr{R}^n) \to \mathscr{R}$ takes bounded sets of $C([-\tau, 0], \mathscr{R}^n)$ in bounded sets of R^n. There is a continuous function $V(t, x)$ such that*

$$\phi_1(\|x\|) \le V(t, x) \le \phi_2(\|x\|), \quad t \in \mathscr{R}, \ x \in \mathscr{R}^n. \tag{17}$$

In addition, there exists a continuous nondecreasing function $\phi(s)$ with $\phi(s) > s$, $s > 0$ such that

$$\dot{V}(t, x)\big|_{(16)} \le -\phi_3(s). \tag{18}$$

If

$$V(t+\theta, x(t+\theta)) < \phi(V(t, x)), \quad \theta \in [-\tau, 0], \tag{19}$$

then the solution $x = 0$ of system (16) is uniformly asymptotically stable.

Usually, $V(t, x)$ is called Lyapunov-Razumikhin function if it satisfies (17) and (18) in Lemma 7.

Then the main results can be obtained as follows.

Theorem 8. *Suppose Assumptions 4 and 6 hold, satisfy*

$$2\gamma\underline{m} \ge \gamma^2\overline{\rho}^2 + \underline{\lambda}_2 \tag{20}$$

and take

$$k > k^* = \frac{2}{\underline{\lambda}_1} + \alpha \tag{21}$$

and assume that

$$\tau < \tau^* = \frac{\underline{\lambda}_2}{(k-\alpha)\left[(k+\alpha)\overline{\lambda}_1 + k^3\overline{\lambda}_2\right] + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right)(\overline{\mu}+1)}, \tag{22}$$

where $q > 1$, $\overline{\mu} = 1 + k/\alpha$, and $\delta = \lambda_{\max}(P_1)/\lambda_{\min}(P)$, and, respectively,

$$\underline{\lambda}_1 = \lambda_{\min}\left(H^T + H\right),$$
$$\underline{\lambda}_2 = \lambda_{\min}\left(R - I_{2N-2}\right);$$
$$\overline{\lambda}_1 = \lambda_{\max}\left(HH^T\right),$$
$$\overline{\lambda}_2 = \lambda_{\max}\left(H^2\left(H^2\right)^T\right), \tag{23}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and the largest nonzero eigenvalue of positive semidefinite matrix, respectively.

Then, the optimization problem (3) for multiagent system (1) can be solved by the optimization control (7), where

$$R = \begin{pmatrix} (k-\alpha)\left(H^T + H\right) & I_{N-1} \\ I_{N-1} & 2I_{N-1} \end{pmatrix} \otimes I_m. \tag{24}$$

Proof. Let

$$e(t) = \left(T^T \otimes I_m\right)\overline{\mathbf{x}}(t),$$
$$\vartheta(t) = \left(T^T \otimes I_m\right)\overline{\mathbf{w}}(t),$$
$$T = \left[\frac{\mathbf{1}_N}{\sqrt{N}} Q\right]. \tag{25}$$

Denote $e = \mathrm{col}(e_1, e_2)$, and $\vartheta = \mathrm{col}(\vartheta_1, \vartheta_2)$ with $e_1, \vartheta_1 \in \mathscr{R}^m$, and $e_2, \vartheta_2 \in \mathscr{R}^{m(N-1)}$. By the structure of T and (6), we can know that T is an orthogonal matrix. Then the system (15) can be rewritten as

$$\dot{e}_1(t) = -\gamma\left(\frac{\mathbf{1}_N^T}{\sqrt{N}} \otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)),$$
$$\dot{e}_2(t) = -k\left(H \otimes I_m\right)e_2(t-\tau(t)) - \vartheta_2(t) - \gamma\left(Q^T \otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)),$$
$$\dot{\vartheta}_1(t) = 0,$$
$$\dot{\vartheta}_2(t) = \alpha\left(H \otimes I_m\right)e_2(t-\tau(t)). \tag{26}$$

Let $\boldsymbol{\varepsilon}(t) = \mathrm{col}(e(t), \vartheta(t)) = \mathrm{col}(e_1(t), e_2(t), \vartheta_1(t), \vartheta_2(t))$, and construct the Lyapunov-Razumikhin function as

$$V(\boldsymbol{\varepsilon}(t)) = \boldsymbol{\varepsilon}^T(t)P\boldsymbol{\varepsilon}(t) \tag{27}$$

with

$$P = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & I_{N-1} & 0 & I_{N-1} \\ 1 & 0 & \frac{k}{\alpha} & 0 \\ 0 & I_{N-1} & 0 & \frac{k}{\alpha}I_{N-1} \end{pmatrix} \otimes I_m. \tag{28}$$

We can have the fact that $P = \begin{pmatrix} I_N & I_N \\ I_N & (k/\alpha)I_N \end{pmatrix} \otimes I_m$ is positive definite since $k > \alpha$.

The derivation of V along the system (26) is given by

$$\begin{aligned}
\dot{V}(\boldsymbol{\varepsilon}(t)) &= 2\boldsymbol{\varepsilon}^T(t) P\dot{\boldsymbol{\varepsilon}}(t) \\
&= 2\left[e_1(t)+\vartheta_1(t)\right]^T \dot{e}_1(t) \\
&\quad + 2\left[e_2(t)+\vartheta_2(t)\right]^T \dot{e}_2(t) \\
&\quad + 2\left[e_1(t)+\frac{k}{\alpha}\vartheta_1(t)\right]^T \dot{\vartheta}_1(t) \\
&\quad + 2\left[e_2(t)+\frac{k}{\alpha}\vartheta_2(t)\right]^T \dot{\vartheta}_2(t).
\end{aligned} \tag{29}$$

Combining the third equation of (26) and (12) gives $\vartheta_1(t) = 0$, $\forall t \geq 0$; then

$$\begin{aligned}
\dot{V}(\boldsymbol{\varepsilon}(t)) &= 2e_1^T(t)\dot{e}_1(t) + 2\left[e_2(t)+\vartheta_2(t)\right]^T \dot{e}_2(t) \\
&\quad + 2\left[e_2(t)+\frac{k}{\alpha}\vartheta_2(t)\right]^T \dot{\vartheta}_2(t) \\
&= 2e_1^T(t)\dot{e}_1(t) + 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1\dot{\boldsymbol{\varepsilon}}_{2:N}(t),
\end{aligned} \tag{30}$$

where $\boldsymbol{\varepsilon}_{2:N}(t) = \mathrm{col}(e_2^T(t), \vartheta_2^T(t))$, $P_1 = \begin{pmatrix} I_{N-1} & I_{N-1} \\ I_{N-1} & (k/\alpha)I_{N-1} \end{pmatrix} \otimes I_m$.

For the second and fourth equalities of system (26), we have a compact form

$$\dot{\boldsymbol{\varepsilon}}_{2:N}(t) = C\boldsymbol{\varepsilon}_{2:N}(t) + E\boldsymbol{\varepsilon}_{2:N}(t-\tau(t)) + F \tag{31}$$

with $C = \begin{pmatrix} 0 & -I_{N-1} \\ 0 & 0 \end{pmatrix} \otimes I_m$, $E = \begin{pmatrix} -kH & 0 \\ \alpha H & 0 \end{pmatrix} \otimes I_m$, and $F = \begin{pmatrix} -\gamma(Q^T \otimes I_m)\Psi(\overline{\mathbf{x}}(t)) \\ 0 \end{pmatrix}$.

By the Leibniz-Newton formula

$$\begin{aligned}
\boldsymbol{\varepsilon}_{2:N}(t-\tau(t)) &= \boldsymbol{\varepsilon}_{2:N}(t) - \int_{t-\tau(t)}^{t} \dot{\boldsymbol{\varepsilon}}_{2:N}(s)\,ds \\
&= \boldsymbol{\varepsilon}_{2:N}(t) - C\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - E\int_{-2\tau(t)}^{-\tau(t)} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - \int_{-\tau(t)}^{0} F(t+s)\,ds.
\end{aligned} \tag{32}$$

Therefore, the system (31) can be rewritten as

$$\begin{aligned}
\dot{\boldsymbol{\varepsilon}}_{2:N}(t) &= \overline{F}\boldsymbol{\varepsilon}_{2:N}(t) - EC\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - E^2\int_{-2\tau(t)}^{-\tau(t)} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - E\int_{-\tau(t)}^{0} F(t+s)\,ds + F,
\end{aligned} \tag{33}$$

where $\overline{F} = E + C$.

Thus, we can get

$$\begin{aligned}
&2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1\dot{\boldsymbol{\varepsilon}}_{2:N}(t) \\
&\quad = \boldsymbol{\varepsilon}_{2:N}^T(t)\left(\overline{F}^T P_1 + P_1\overline{F}\right)\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 EC\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 E^2\int_{-2\tau(t)}^{-\tau(t)} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 E\int_{-\tau(t)}^{0} F(t+s)\,ds \\
&\quad\quad + 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 F.
\end{aligned} \tag{34}$$

Combining (30) and (34) gives

$$\begin{aligned}
\dot{V}(\boldsymbol{\varepsilon}(t)) &= \boldsymbol{\varepsilon}_{2:N}^T(t)\left(\overline{F}^T P_1 + P_1\overline{F}\right)\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 EC\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 E^2\int_{-2\tau(t)}^{-\tau(t)} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad - 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 E\int_{-\tau(t)}^{0} F(t+s)\,ds \\
&\quad + 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 F + 2e_1^T(t)\dot{e}_1(t).
\end{aligned} \tag{35}$$

Note that $2a^Tb \leq a^T\Phi a + b^T\Phi^{-1}b$ holds for any appropriate positive definite matrix Φ; then let $a^T = -\boldsymbol{\varepsilon}_{2:N}^T(t)P_1EC$, $b = \boldsymbol{\varepsilon}_{2:N}(t+s)$, and $\Phi = P_1^{-1}$; one can obtain

$$\begin{aligned}
&- 2\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 EC\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad = \int_{-\tau(t)}^{0} 2\left(-\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 EC\right)\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad \leq \tau(t)\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 ECP_1^{-1}\left(P_1EC\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad\quad + \int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}^T(t+s) P_1\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\quad \leq \tau\boldsymbol{\varepsilon}_{2:N}^T(t) P_1 ECP_1^{-1}\left(P_1EC\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad\quad + \int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}^T(t+s) P_1\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds.
\end{aligned} \tag{36}$$

Similarly, let $a^T = -\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E^2$, $b = \boldsymbol{\varepsilon}_{2:N}(t+s)$, and $\Phi = P_1^{-1}$; we have

$$\begin{aligned}
&-2\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E^2\int_{-2\tau(t)}^{-\tau(t)}\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&= \int_{-2\tau(t)}^{-\tau(t)} 2\left(-\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E^2\right)\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\le \tau(t)\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E^2P_1^{-1}\left(P_1E^2\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad + \int_{-2\tau(t)}^{-\tau(t)}\boldsymbol{\varepsilon}_{2:N}^T(t+s)P_1\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds \\
&\le \tau\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E^2P_1^{-1}\left(P_1E^2\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad + \int_{-2\tau(t)}^{-\tau(t)}\boldsymbol{\varepsilon}_{2:N}^T(t+s)P_1\boldsymbol{\varepsilon}_{2:N}(t+s)\,ds,
\end{aligned} \tag{37}$$

and let $a^T = -\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E$, $b = F(t+s)$, and $\Phi = P_1^{-1}$; there is

$$\begin{aligned}
&-2\boldsymbol{\varepsilon}_{2:N}^T(t)P_1E\int_{-\tau(t)}^{0}F(t+s)\,ds \\
&\le \tau(t)\boldsymbol{\varepsilon}_{2:N}^T(t)P_1EP_1^{-1}\left(P_1E\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad + \int_{-\tau(t)}^{0}F^T(t+s)P_1F(t+s)\,ds \\
&\le \tau\boldsymbol{\varepsilon}_{2:N}^T(t)P_1EP_1^{-1}\left(P_1E\right)^T\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad + \int_{-\tau(t)}^{0}F^T(t+s)P_1F(t+s)\,ds.
\end{aligned} \tag{38}$$

Due to

$$F(t) = \begin{pmatrix} -\gamma\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)) \\ 0 \end{pmatrix}, \tag{39}$$

then

$$\begin{aligned}
\|F(t)\|^2 &= \gamma^2\left\|\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t))\right\|^2 \le \gamma^2\overline{\rho}^2\|\overline{\mathbf{x}}(t)\|^2 \\
&= \gamma^2\overline{\rho}^2\|e(t)\|^2 \le \gamma^2\overline{\rho}^2\|\boldsymbol{\varepsilon}(t)\|^2,
\end{aligned} \tag{40}$$

with the transformation $e(t) = (T^T\otimes I_m)\overline{\mathbf{x}}(t)$; we have

$$\begin{aligned}
2e_1^T(t)\dot{e}_1(t) &= -2\gamma e_1^T(t)\left(\frac{\mathbf{1}_N^T}{\sqrt{N}}\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)) \\
&= -2\gamma\overline{\mathbf{x}}^T(t)\Psi(\overline{\mathbf{x}}(t)) \\
&\quad + 2\gamma e_2^T(t)\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t));
\end{aligned} \tag{41}$$

then, from Assumption 4, it follows that

$$\begin{aligned}
&2\boldsymbol{\varepsilon}_{2:N}^T(t)P_1F + 2e_1^T(t)\dot{e}_1(t) \\
&= -2\gamma\left(e_2^T(t)+\vartheta_2^T(t)\right)\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)) \\
&\quad - 2\gamma\overline{\mathbf{x}}^T(t)\Psi(\overline{\mathbf{x}}(t)) \\
&\quad + 2\gamma e_2^T(t)\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)) \\
&= -2\gamma\vartheta_2^T(t)\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t)) \\
&\quad - 2\gamma\overline{\mathbf{x}}^T(t)\Psi(\overline{\mathbf{x}}(t)) \\
&\le \vartheta_2^T(t)\vartheta_2(t) + \gamma^2\left\|\left(Q^T\otimes I_m\right)\Psi(\overline{\mathbf{x}}(t))\right\|^2 \\
&\quad - 2\gamma\underline{m}\|\overline{\mathbf{x}}(t)\|^2 \\
&\le \vartheta_2^T(t)\vartheta_2(t) + \gamma^2\overline{\rho}^2\|\overline{\mathbf{x}}(t)\|^2 - 2\gamma\underline{m}\|\overline{\mathbf{x}}(t)\|^2 \\
&\le \boldsymbol{\varepsilon}_{2:N}^T(t)\boldsymbol{\varepsilon}_{2:N}(t) - \left(2\gamma\underline{m}-\gamma^2\overline{\rho}^2\right)\|\overline{\mathbf{x}}(t)\|^2,
\end{aligned} \tag{42}$$

where $\underline{m} = \min\{m_1, m_2, \ldots, m_N\}$ and $\overline{\rho} = \max\{\rho_1, \rho_2, \ldots, \rho_N\}$.

According to the Lyapunov-Razumikhin Theorem, take $\phi(s) = qs$ for some constant $q > 1$. In case that

$$V(\boldsymbol{\varepsilon}(t+\theta)) < qV(\boldsymbol{\varepsilon}(t)), \quad \theta\in[-2\tau, 0], \tag{43}$$

then

$$\begin{aligned}
&\boldsymbol{\varepsilon}_{2:N}^T(t+s)P_1\boldsymbol{\varepsilon}_{2:N}(t+s) \\
&\le e_1^T(t+s)e_1(t+s) + \boldsymbol{\varepsilon}_{2:N}^T(t+s)P_1\boldsymbol{\varepsilon}_{2:N}(t+s) \\
&= \boldsymbol{\varepsilon}^T(t+s)P\boldsymbol{\varepsilon}(t+s) < q\boldsymbol{\varepsilon}^T(t)P\boldsymbol{\varepsilon}(t).
\end{aligned} \tag{44}$$

Next, considering the integral term in (38) and according to (40), we can obtain

$$\begin{aligned}
&\int_{-\tau(t)}^{0}F^T(t+s)P_1F(t+s)\,ds \\
&\le \lambda_{\max}(P_1)\int_{-\tau(t)}^{0}F^T(t+s)F(t+s)\,ds \\
&\le \lambda_{\max}(P_1)\gamma^2\overline{\rho}^2\int_{-\tau(t)}^{0}\boldsymbol{\varepsilon}^T(t+s)\boldsymbol{\varepsilon}(t+s)\,ds \\
&\le \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P)}\gamma^2\overline{\rho}^2\int_{-\tau(t)}^{0}\boldsymbol{\varepsilon}^T(t+s)P\boldsymbol{\varepsilon}(t+s)\,ds \\
&\le \delta\gamma^2\overline{\rho}^2q\int_{-\tau(t)}^{0}\boldsymbol{\varepsilon}^T(t)P\boldsymbol{\varepsilon}(t)\,ds \\
&\le \delta\gamma^2\overline{\rho}^2q\tau\left[e_1^T(t)e_1(t) + \boldsymbol{\varepsilon}_{2:N}^T(t)P_1\boldsymbol{\varepsilon}_{2:N}(t)\right] \\
&\le \delta\gamma^2\overline{\rho}^2q\tau\overline{\mathbf{x}}^T(t)\overline{\mathbf{x}}(t) + \delta\gamma^2\overline{\rho}^2q\tau\boldsymbol{\varepsilon}_{2:N}^T(t)P_1\boldsymbol{\varepsilon}_{2:N}(t) \\
&\le \delta\gamma^2\overline{\rho}^2q\tau\boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) + \delta\gamma^2\overline{\rho}^2q\tau\boldsymbol{\varepsilon}_{2:N}^T(t)P_1\boldsymbol{\varepsilon}_{2:N}(t),
\end{aligned} \tag{45}$$

and, substituting (44) into the integral term in (36), we can obtain

$$\begin{aligned}
&\int_{-\tau(t)}^{0} \boldsymbol{\varepsilon}_{2:N}^{T}(t+s) P_1 \boldsymbol{\varepsilon}_{2:N}(t+s)\, ds \\
&\quad < \int_{-\tau(t)}^{0} q\boldsymbol{\varepsilon}^{T}(t) P \boldsymbol{\varepsilon}(t)\, ds \\
&\quad < \int_{-\tau(t)}^{0} \left[q e_1^T(t) e_1(t) + q\boldsymbol{\varepsilon}_{2:N}^{T}(t) P_1 \boldsymbol{\varepsilon}_{2:N}(t)\right] ds \\
&\quad < q\tau e_1^T(t) e_1(t) + q\tau \boldsymbol{\varepsilon}_{2:N}^{T}(t) P_1 \boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad < q\tau \|\overline{\mathbf{x}}(t)\|^2 + q\tau \boldsymbol{\varepsilon}_{2:N}^{T}(t) P_1 \boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad \le q\tau \boldsymbol{\varepsilon}^{T}(t)\boldsymbol{\varepsilon}(t) + q\tau \boldsymbol{\varepsilon}_{2:N}^{T}(t) P_1 \boldsymbol{\varepsilon}_{2:N}(t).
\end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned}
&\int_{-2\tau(t)}^{-\tau(t)} \boldsymbol{\varepsilon}_{2:N}^{T}(t+s) P_1 \boldsymbol{\varepsilon}_{2:N}(t+s)\, ds \\
&\quad < q\tau \boldsymbol{\varepsilon}^{T}(t)\boldsymbol{\varepsilon}(t) + q\tau \boldsymbol{\varepsilon}_{2:N}^{T}(t) P_1 \boldsymbol{\varepsilon}_{2:N}(t).
\end{aligned} \tag{47}$$

Then from (35) and above inequalities, we have

$$\begin{aligned}
\dot{V}(\boldsymbol{\varepsilon}(t)) &\le -\boldsymbol{\varepsilon}_{2:N}^{T}(t)\left(R - I_{2N-2}\right)\boldsymbol{\varepsilon}_{2:N}(t) + \tau \boldsymbol{\varepsilon}_{2:N}^{T}(t) \\
&\cdot \left[P_1 E C P_1^{-1}\left(P_1 E C\right)^T + P_1 E^2 P_1^{-1}\left(P_1 E^2\right)^T \right. \\
&\left. + P_1 E P_1^{-1}\left(P_1 E\right)^T + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right) P_1\right]\boldsymbol{\varepsilon}_{2:N}(t) \\
&+ \left(2q\tau + \delta\gamma^2\overline{\rho}^2 q\tau\right)\boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) - \left(2\gamma\underline{m} - \gamma^2\overline{\rho}^2\right) \\
&\cdot \|\overline{\mathbf{x}}(t)\|^2,
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
R &= -\left(\overline{F}^T P_1 + P_1 \overline{F}\right) = \begin{pmatrix} (k-\alpha)\left(H^T + H\right) & I_{N-1} \\ I_{N-1} & 2I_{N-1} \end{pmatrix} \\
&\otimes I_m, \\
&P_1 E C P_1^{-1}\left(P_1 E C\right)^T + P_1 E^2 P_1^{-1}\left(P_1 E^2\right)^T \\
&+ P_1 E P_1^{-1}\left(P_1 E\right)^T \\
&= \begin{pmatrix} (k-\alpha)\left[(k+\alpha) H H^T + k^3 H^2 \left(H^2\right)^T\right] & 0 \\ 0 & 0 \end{pmatrix} \\
&\otimes I_m.
\end{aligned} \tag{49}$$

According to Lemma 3, if k satisfies condition (21), then $R - I_{2N-2}$ is positive definite; we have

$$\begin{aligned}
&\boldsymbol{\varepsilon}_{2:N}^{T}(t)\left(R - I_{2N-2}\right)\boldsymbol{\varepsilon}_{2:N}(t) \ge \underline{\lambda}_2 \boldsymbol{\varepsilon}_{2:N}^{T}(t)\boldsymbol{\varepsilon}_{2:N}(t) \\
&\quad = \underline{\lambda}_2 \boldsymbol{\varepsilon}_{2:N}^{T}(t)\boldsymbol{\varepsilon}_{2:N}(t) + \underline{\lambda}_2 e_1^T(t) e_1(t) \\
&\qquad - \underline{\lambda}_2 e_1^T(t) e_1(t) = \underline{\lambda}_2 \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) - \underline{\lambda}_2 e_1^T(t) e_1(t) \\
&\quad \ge \underline{\lambda}_2 \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) - \underline{\lambda}_2 \|\overline{\mathbf{x}}(t)\|^2,
\end{aligned} \tag{50}$$

if condition (20) is satisfied and due to the fact that $\boldsymbol{\varepsilon}_{2:N}^T(t)\boldsymbol{\varepsilon}_{2:N}(t) \le \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t)$; then

$$\begin{aligned}
&\dot{V}(\boldsymbol{\varepsilon}(t)) \le -\underline{\lambda}_2 \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) + \underline{\lambda}_2 \|\overline{\mathbf{x}}(t)\|^2 + \tau \boldsymbol{\varepsilon}_{2:N}^T(t) \\
&\quad \cdot \left\{(k-\alpha)\left[(k+\alpha)\overline{\lambda}_1 + k^3\overline{\lambda}_2\right] + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right)\overline{\mu}\right\} \\
&\quad \cdot \boldsymbol{\varepsilon}_{2:N}(t) + 2\left(\delta\gamma^2\overline{\rho}^2 q\tau + q\tau\right)\boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) - \left(2\gamma\underline{m}\right. \\
&\quad \left. - \gamma^2\overline{\rho}^2\right)\|\overline{\mathbf{x}}(t)\|^2 \le -\underline{\lambda}_2 \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) + \tau \boldsymbol{\varepsilon}^T(t) \\
&\quad \cdot \left\{(k-\alpha)\left[(k+\alpha)\overline{\lambda}_1 + k^3\overline{\lambda}_2\right] + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right)\overline{\mu}\right. \\
&\quad \left. + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right)\right\}\boldsymbol{\varepsilon}(t) - \left(2\gamma\underline{m} - \gamma^2\overline{\rho}^2 - \underline{\lambda}_2\right) \\
&\quad \cdot \|\overline{\mathbf{x}}(t)\|^2 \le -\underline{\lambda}_2 \boldsymbol{\varepsilon}^T(t)\boldsymbol{\varepsilon}(t) + \tau \boldsymbol{\varepsilon}^T(t) \\
&\quad \cdot \left\{(k-\alpha)\left[(k+\alpha)\overline{\lambda}_1 + k^3\overline{\lambda}_2\right]\right. \\
&\quad \left. + \left(2q + \delta\gamma^2\overline{\rho}^2 q\right)\left(\overline{\mu} + 1\right)\right\}\boldsymbol{\varepsilon}(t)
\end{aligned} \tag{51}$$

and we take τ has the upper bound in (22); then $\dot{V}(\boldsymbol{\varepsilon}(t))$ is negative definite. Thus by the Lyapunov-Razumikhin theorem, we can conclude that $\boldsymbol{\varepsilon}(t) \to 0$; that is, $e(t) \to 0_{mN}$, $\vartheta(t) \to 0_{mN}$ as $t \to \infty$.

With the transformation $\overline{\mathbf{x}}(t) = (T \otimes I_m)e(t)$ and $\overline{\mathbf{w}}(t) = (T \otimes I_m)\vartheta(t)$ and T is a orthogonal matrix, we can obtain $\overline{\mathbf{x}}(t) \to 0_{m\mathrm{N}}$, $\overline{\mathbf{w}}(t) \to 0_{mN}$, which means $\mathbf{x}(t) \to \mathbf{x}^*$, $\mathbf{w}(t) \to \mathbf{w}^*$ as $t \to \infty$. As a result, this proof is completed. □

Remark 9. The continuous-time protocol considered in this paper is based on the algorithm proposed in [15], and under the same communication topology, but the conditions of convergence analysis needed by this paper are more relaxed. From (20) and (21), it is clearly shown that k^* is independent of parameters β and γ but dependent on α and communication topology, while τ^* is independent of constant m_i in this paper compared to [15]. We can know when the number of the agents is large, it would be difficult to verify the LMI condition, but, in this paper, it only needs the model parameters to meet some boundary conditions, and when considering the dynamic system with time-varying delay, the Lyapunov function with Razumikhin technique is also an effective method compared to Lyapunov-Krasovskii method.

4. Simulations

In this section, we give an example to validate our theoretical results. In the example, we consider a multiagent system

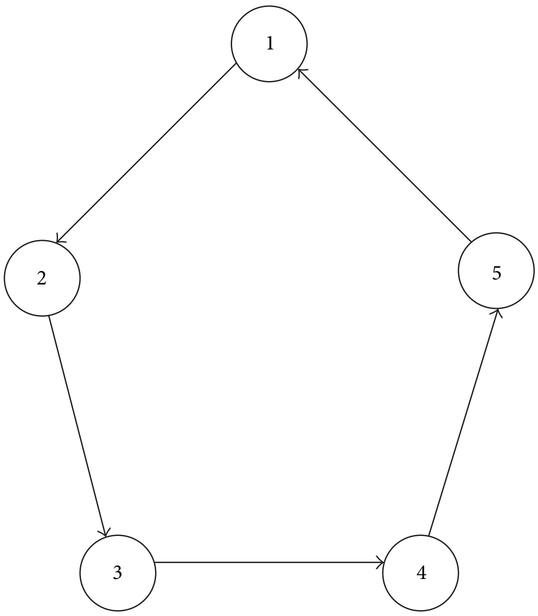

FIGURE 1: Connected graph.

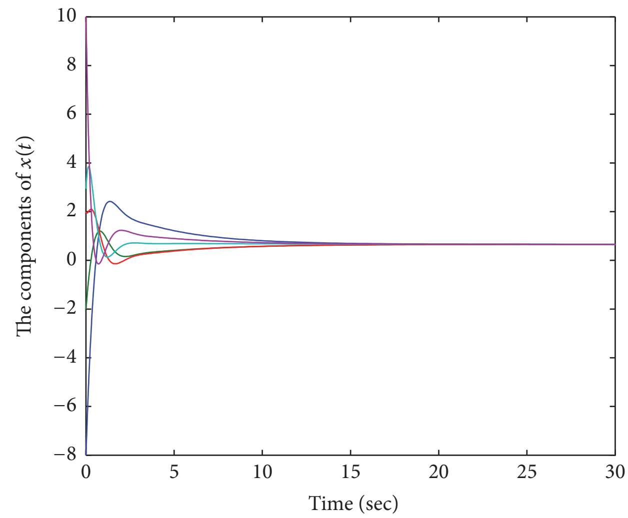

FIGURE 2: The trajectories of x_i.

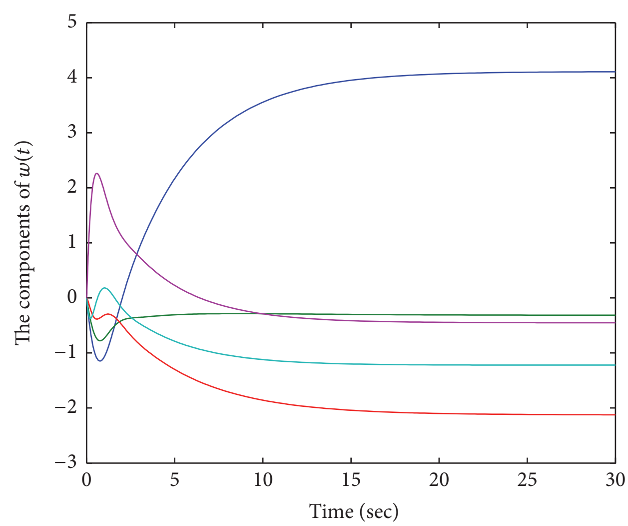

FIGURE 3: The trajectories of w_i.

consisting of five agents. Suppose that the interconnected topology is described as in Figure 1.

Consider the following optimization problem:

$$\text{minimize} \quad f(x) = \sum_{i=1}^{N} f_i(x), \quad x \in \mathscr{R}, \tag{52}$$

where the local objective function is given as follows:

$$\begin{aligned} f_1(x) &= 0.7(x-8)^2, \\ f_2(x) &= 0.6x^2 - 2, \\ f_3(x) &= (x+2)^2, \\ f_4(x) &= 0.8x^2 + 2x, \\ f_5(x) &= \sin\frac{x}{2} + \frac{x^2}{2}. \end{aligned} \tag{53}$$

Obviously, for $i = 1, 2, \ldots, 5$, f_i is differentiable and satisfies Assumption 4. Choosing $\alpha = 0.6$, $k = 2.2$, $\gamma = 0.4$, $q = 1.1$ and time-varying delay $\tau(t) = 0.01|\cos(t)|$, we can obtain $\overline{\rho} = 2$, $\underline{m} = 1$, $k^* = 2.0472$ and $\tau^* = 0.0004$.

Let the initial values $x(0) = [x_1(0), x_2(0), x_3(0), x_4(0), x_5(0)]^T = [-8, -2, 2, 3, 10]^T$, $w(0) = [w_1(0), w_2(0), w_3(0), w_4(0), w_5(0)]^T = [0, 0, 0, 0, 0]^T$. The simulation results are shown in Figures 2 and 3.

We can see that the trajectories x_i of each agent i converge to the global optimal solution $x^* = 0.6565$ of the objective function $f(x) = \sum_{i=1}^{N} f_i(x)$ and all the trajectories w_i converge to a constant, respectively, for $i = 1, 2, \ldots, 5$. The optimal value of $f(x)$ is 45.2602.

5. Conclusion

In this paper, the consensus optimization problem of multiagents with communication delays was considered. By a continuous-time algorithm, consensus and optimization under some parameter bound conditions are ensured. Graph theory is used to describe the interconnection topologies. Lyapunov-Razumikhin theory were employed for stability analysis. The connectivity assumption of directed graph plays a key role in the analysis of algorithm convergence. Numerical examples were given to illustrate the theoretical results.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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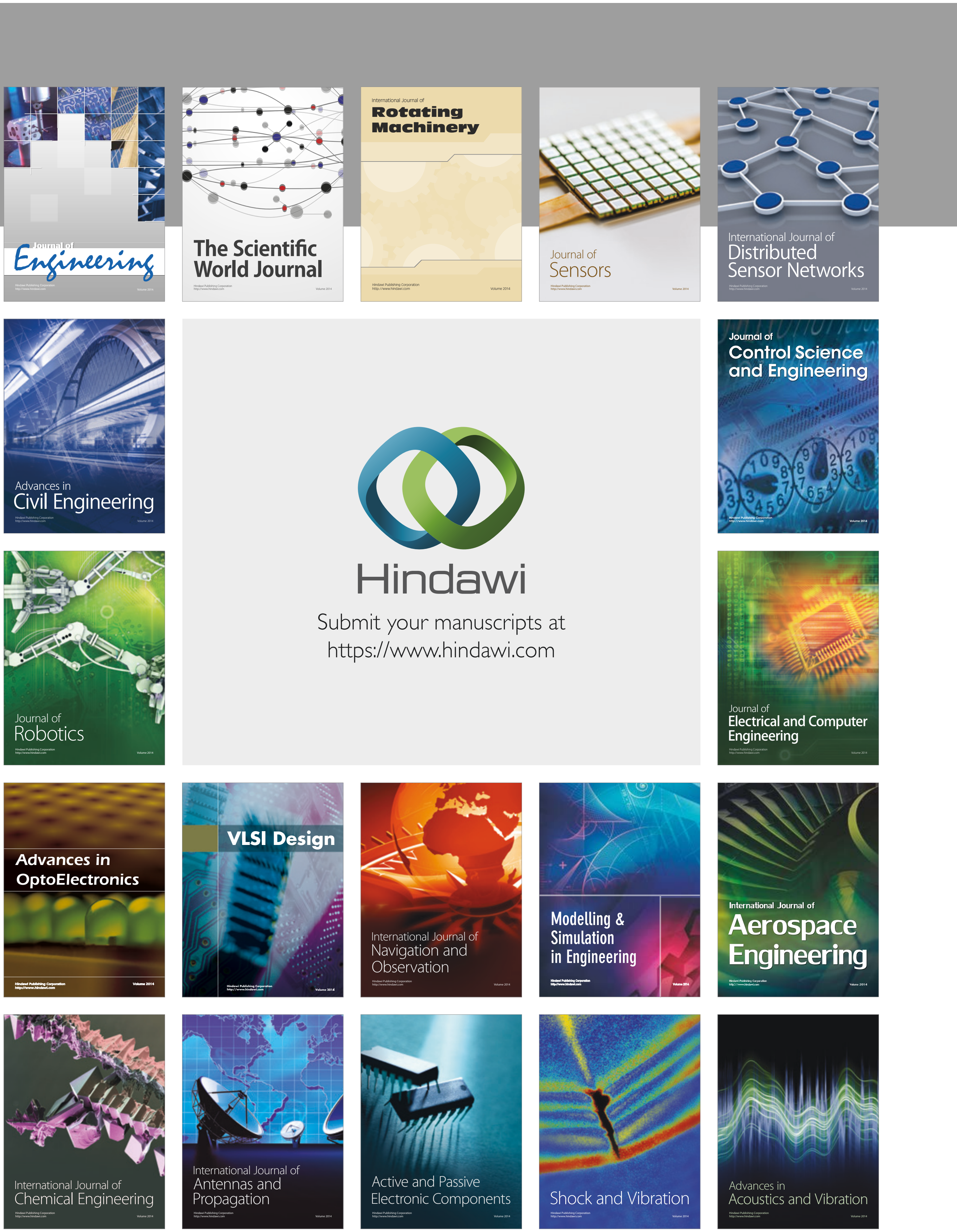
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