Hindawi Publishing Corporation
Journal of Engineering
Volume 2016, Article ID 3279047, 7 pages
http://dx.doi.org/10.1155/2016/3279047

Research Article

Constitutive Equations of Yield Stress Sensitivity to Strain Rate of Metals: A Comparative Study

Ammar A. Al Salahi and Ramzi Othman

Mechanical Engineering Department, Faculty of Engineering, King Abdul-Aziz University, P.O. Box 80248, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to Ramzi Othman; rothman1@kau.edu.sa

Received 2 March 2016; Accepted 15 May 2016

Academic Editor: Yuanxin Zhou

Several constitutive equations have proposed to model the strain rate sensitivity of metals to strain rate. This paper presents a comparative of six equations reported in the open literature. All equations are used to fit the yield stress of three copper materials and one steel material at two different temperatures. A specific cost function and an optimization problem are defined. The authors recommend the use of the Cowper-Symonds equation or a modified-Eyring equation as both of them fit well the experimental data while using only three material constants. A modified flow stress Johnson-Cook equation is then proposed for metallic materials.

1. Introduction

Metallic materials are largely used in several industrial fields, for example, aeronautical, naval, automobile, and military industries. In these applications, transportation vehicles have to be designed against impact loads. Thus, the characterization and modeling of metals' sensitivity to strain rate are highly important. The split Hopkinson bar is largely used to characterize materials in the high strain rate range [1, 2] while the direct-impact Hopkinson bar is used at the very high strain rate [3, 4]. In terms of constitutive equations, the Johnson-Cook law [5, 6] has been widely used to model the behavior of metallic materials including temperature or (and) strain rate effect(s) [3, 7–9]. This constitutive equation separates the hardening, temperature, and strain rates. Namely, they are written in a multiplicative form. The strain rate effect is considered as varying linearly in terms of the logarithm of strain rate.

Several studies have showed that the linear variation of yield or flow stress in terms of strain rate is only valid in the quasi-static and intermediate strain rate ranges [10–13]. However, there is a sharp increase in the strain rate sensitivity at high strain rates. This increase cannot be considered by the classical Johnson-Cook equation. Some modified Johnson-Cook equations have then been proposed [14]. Huh and Kang [15] proposed a quadratic for the strain rate sensitivity. Tuazon et al. [16] expressed the dependence on the logarithm of strain rate as a power-law. Couque [17] proposed a modified Johnson-Cook equation where the strain rate sensitivity is written in terms of a four-constant equation. El-Qoubaa and Othman [18, 19] have proposed a modified-Eyring equation for polymers yield stress sensitivity to strain rates. This model was successfully applied to several metallic materials in [20].

The aim of this work is to compare and discuss the above constitutive equations: the standard Johnson-Cook, Huh-Kang, Tuazon et al., Couque, and modified-Eyring equations. The pioneering Cowper-Symonds equation is also considered [21]. In terms of strain rate dependence, it gives similar relation as The Zerilli-Armstrong equation [22].

2. Methodology

2.1. Experimental Data. In this work, we are interested in evaluating several constitutive equations. Thus, they are used here to fit the strain rate sensitivity of the yield stress of two metallic materials: steel and copper. These two materials are extensively characterized in the literature. In terms of the copper yield stress, we rely upon the experimental data of Couque [17]. For the steel yield stress, we will rely upon the experimental data of Clarke et al. [11].

Table 1: Constitutive equations.

Names	Equations (σ_y is the yield stress and $\dot{\varepsilon}$ is the strain rate)	Constants
Standard Johnson-Cook [5]	$\sigma_y = A\left(1 + C\log\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\right)$	A and C are two material constants, and $\dot{\varepsilon}_0 = 1\,\mathrm{s}^{-1}$ is a reference strain rate
Huh-Kang [15]	$\sigma_y = A\left(1 + C_1\log\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right) + C_2\log^2\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\right)$	A, C_1,and C_2 are three material constants, and $\dot{\varepsilon}_0 = 1\,\mathrm{s}^{-1}$ is a reference strain rate
Tuazon et al. [16]	$\sigma_y = A\left(1 + C\log^p\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\right)$	A, C, and p are three material constants, and $\dot{\varepsilon}_0 = 1\,\mathrm{s}^{-1}$ is a reference strain rate
Couque [17]	$\sigma_y = A\left(1 + C\log\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right) + D\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_1}\right)^k\right)$	A, C, D, and k are four material constants, and $\dot{\varepsilon}_0 = 1\,\mathrm{s}^{-1}$ and $\dot{\varepsilon}_1 = 10^3\,\mathrm{s}^{-1}$ are two reference strain rates
Cowper-Symonds [21] or Zerilli-Armstrong [22]	$\sigma_y = A\left(1 + D\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_1}\right)^k\right)$	A, D, and k are three material constants, and $\dot{\varepsilon}_1 = 1\,\mathrm{s}^{-1}$ is a reference strain rate
Modified-Eyring [18–20]	$\sigma_y = \sigma_0 + \frac{k_B T}{V_0}\exp\left(\sqrt{\frac{\dot{\varepsilon}}{\dot{\varepsilon}_c}}\right)\log\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)$	σ_0, V_0, and $\dot{\varepsilon}_c$ are three material constants, and k_B is the Boltzmann constant, T is the absolute temperature, and $\dot{\varepsilon}_0 = 1\,\mathrm{s}^{-1}$ is a reference strain rate

2.2. Constitutive Equations. The constitutive equations studied in this work are dressed in Table 1. Here, we are only interested in the strain rate sensitivity. We should notice that the equations are sometimes changed from their original form for identification purposed.

2.3. Identification Procedure. In this section, we depict the methodology followed in order to identify the material constants of each of the constitutive equations that are dressed in Table 1. Actually, each constitutive equation depends on a set of two, three, or four material constants. The standard Johnson-Cook model depends on only two material constants: A and C. The Huh-Kang, Tuazon et al., Cowper-Symonds, and modified-Eyring equations depend each on three material constants. Finally, the Couque equation depends on four material constants. The main idea of this section is to find for each constitutive equation a set of material constants that reduces the difference between the experimental yield stresses and the yield stresses obtained by that equation.

Let $\dot{E} = (\dot{\varepsilon}_i)$ be a vector that collects the experimental strain rate values which are collected from the literature as explained in Section 2, where $\dot{\varepsilon}_i$ denotes the strain rate obtained for a test i. Similarly, let $\widehat{\Sigma} = (\widehat{\sigma}_i)$ be a vector which collects the yield stresses $\widehat{\sigma}_i$ measured at strain rates $\dot{\varepsilon}_i$. Using a constitutive equation χ from Table 1, it is possible to build a vector $\Sigma_\chi = (\sigma_{\chi i})$ which collects the yield stresses $\sigma_{\chi i}$ that are calculated at strain rates $\dot{\varepsilon}_i$.

In order to obtain the best material constants for each constitutive equation, we need to optimize a cost function f_χ. This cost function is built in terms of the difference between the experimental and calculated yield stresses. Let $\|\,\|_2$ and $\|\,\|_\infty$ be the Euclidean norm and the maximum norm, respectively. It is possible to define an error using the Euclidean norm.

More precisely,

$$\mathrm{EucErr}_\chi = \frac{\left\|\Sigma_\chi - \widehat{\Sigma}\right\|_2}{\left\|\widehat{\Sigma}\right\|_2}. \quad (1)$$

Likewise, it is possible to define an error using the maximum norm. Namely,

$$\mathrm{MaxErr}_\chi = \frac{\left\|\Sigma_\chi - \widehat{\Sigma}\right\|_\infty}{\left\|\widehat{\Sigma}\right\|_\infty}. \quad (2)$$

The Euclidean norm-based error gives a measurement of the average difference between the experimental yield stress and the yield stress predicted by the considered constitutive equation. It can be considered as a global error measurement. On the opposite, the max norm-based error focuses on the tests where the maximum difference is encountered. It can then be considered as a local error measurement.

In this study the cost function is defined as the average between the Euclidean norm-based error and the maximum norm-based error:

$$f_\chi\left(\kappa_1^\chi, \kappa_2^\chi, \ldots\right) = \frac{\mathrm{EucErr}_\chi + \mathrm{MaxErr}_\chi}{2}, \quad (3)$$

where $\kappa_1^\chi, \kappa_2^\chi, \ldots$ are the material constants of the constitutive equation χ. The best material constants are then obtained by minimizing the cost function f_χ:

$$\kappa_1^\chi, \kappa_2^\chi, \ldots = \operatorname{argmin} f_\chi. \quad (4)$$

3. Results and Discussion

3.1. Standard Johnson-Cook Model. The standard Johnson-Cook equation is used to fit the compression yield stress of three copper materials (Figure 1(a)) and for steel at two temperatures (Figure 2(b)). The material constants, the errors, and the correlation coefficient are calculated and depicted in Table 2. It is clear that this equation cannot model the increase in strain rate sensitivity at high strain rates. This is observed in the five situations studied here. The error is important. The minimum error is obtained for the case of steel at 273 K. It is equal to 13.4%. It can increase up to 48.2% which is obtained with the case of copper 105.

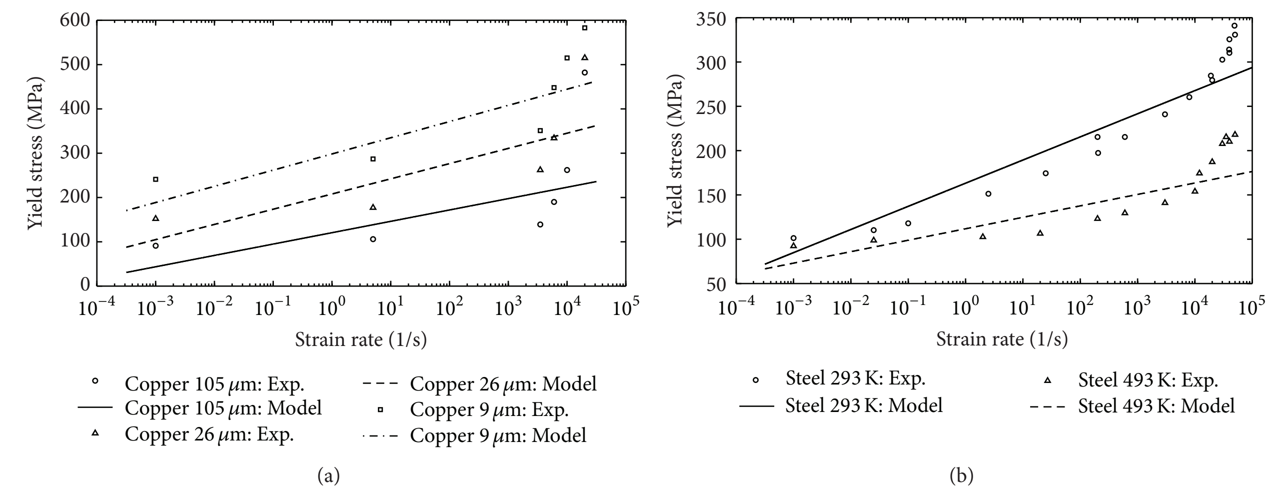

Figure 1: Yield stress fitting by the standard Johnson-Cook model: (a) copper and (b) steel.

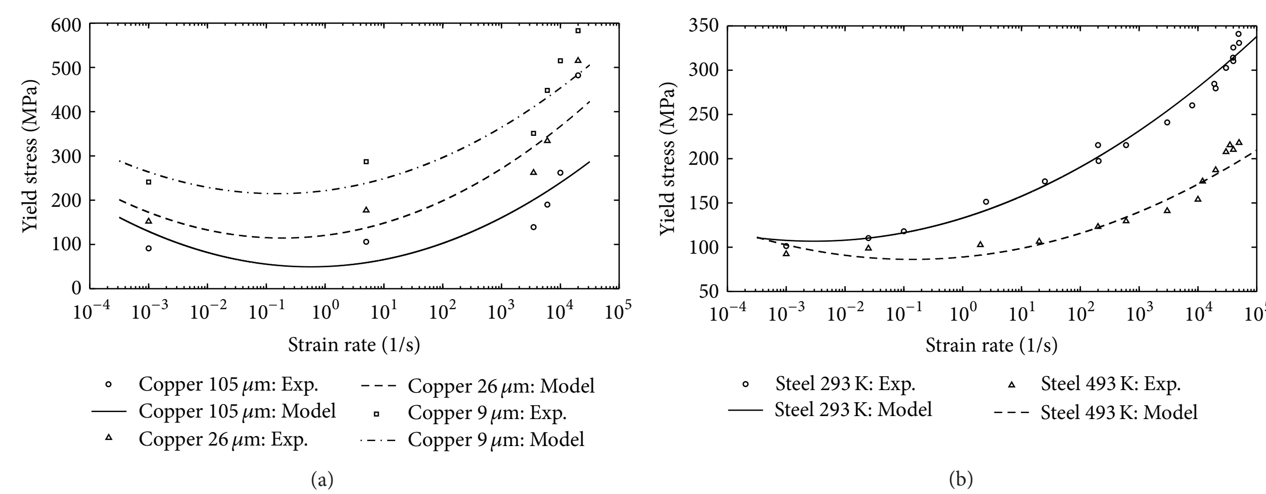

Figure 2: Yield stress fitting by the Huh-Kang model: (a) copper and (b) steel.

Table 2: Material constants and errors obtained with the standard Johnson-Cook equation.

Metal	A (MPa)	C	Error (%)
Copper 105	120.5	0.0924	48.2
Copper 26	207.9	0.0717	28.8
Copper 9	298.3	0.0532	19.5
Steel 293 K	163.2	0.0695	13.4
Steel 493 K	111.8	0.0502	18.6

Table 3: Material constants and errors obtained by the Huh-Kang equation.

Metal	A (MPa)	C_1	C_2	Error (%)
Copper 105	50.0	0.0450	0.0398	41.4
Copper 26	120.5	0.0594	0.0177	20.9
Copper 9	221.6	0.0330	0.0088	15.3
Steel 293 K	132.9	0.0675	0.0058	5.3
Steel 493 K	88.9	0.0303	0.0076	9.8

3.2. Huh-Kang Model. The Huh-Kang equation is a modified form of the standard Johnson-Cook model with a quadratic relation between yield stress and the logarithm of the strain rate. The fit of this model to the experimental data of copper and steel is shown in Figure 2. The material constants and the errors are reported in Table 3. The model follows roughly the experimental data. The fit is highly better with steel than copper. The errors are ranging from 5.3% for steel 293 K to 41.4% for copper 105. It can catch the sharp increase in the yield strain at high strain rate. However, it predicts an increase of the yield stress in the quasi-static strain rate range because of the quadratic form of the constitutive equation. This increase at low strain rates was never reported in the open literature, to the best knowledge of the authors. Thus, it is considered here nonphysical.

3.3. Tuazon et al. Model. Tuazon et al. [16] modified the Johnson-Cook equation by adding an exponent p to

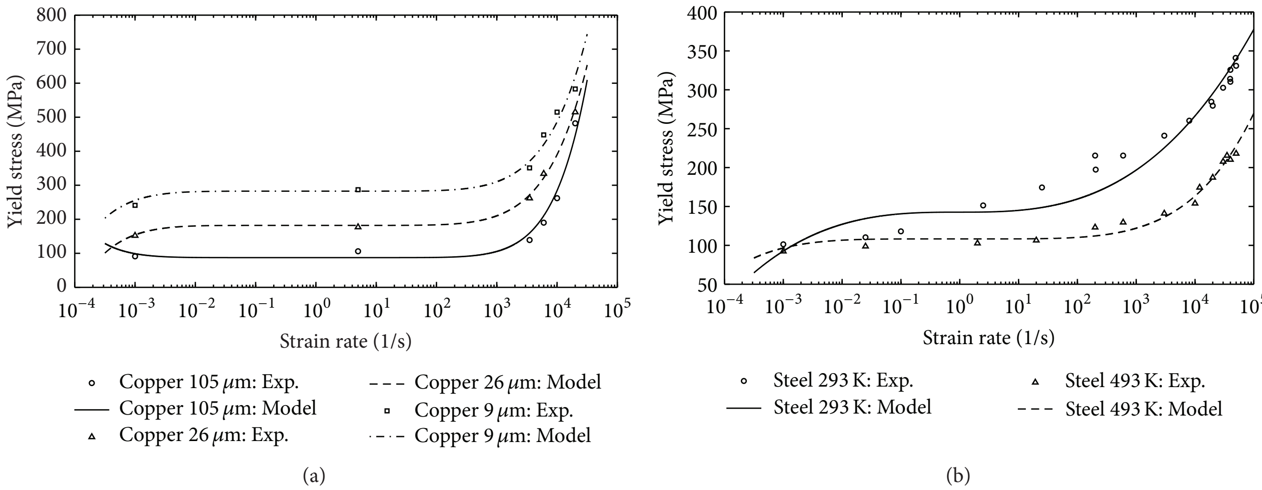

FIGURE 3: Yield stress fitting by the Tuazon et al. model: (a) copper and (b) steel.

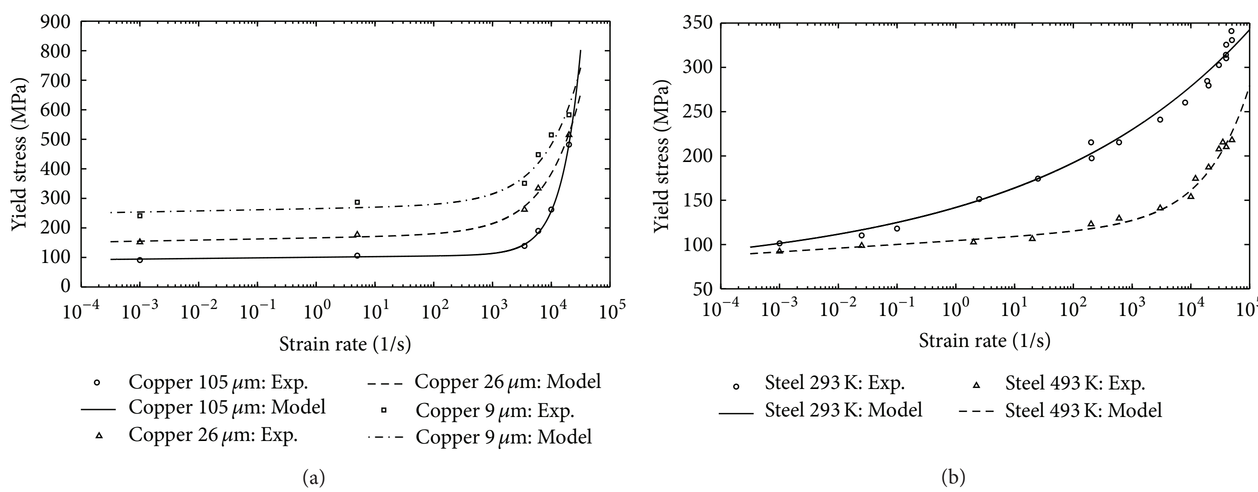

FIGURE 4: Yield stress fitting by the Couque model: (a) copper and (b) steel.

the logarithm of strain rate. More precisely, the expression $\log(\dot{\varepsilon}/\dot{\varepsilon}_0)$ in the standard Johnson-Cook equation is substituted by the expression $\log^p(\dot{\varepsilon}/\dot{\varepsilon}_0)$. The Tuazon et al. model fits quite well the experimental data of steel and copper (Figure 3). The error ranges between 3.4% (obtained with copper 26) and 15.2% (obtained with steel 293) (see Table 4). This equation can catch the increase in the strain rate sensitivity at high strain rates. However, it predicts a sharp drop in yield stress at low strain rates. This behavior was never reported in the literature, to the best of the authors knowledge, and thus considered here nonphysical. Hence, we rather recommend using the Tuazon et al. equation for strain rates higher than $10^{-2}\ \mathrm{s}^{-1}$.

3.4. Couque Model. Couque [17] modified the standard Johnson-Cook equation by adding a third term which is written as a power of the strain rate and not the logarithm of strain rate, that is, $(\dot{\varepsilon}/\dot{\varepsilon}_1)^k$. This equation fits well the experimental data of copper and steel (Figure 4) over the total strain range, that is, between $10^{-3}\ \mathrm{s}^{-1}$ and $5 \times 10^4\ \mathrm{s}^{-1}$. The model describes well the sharp increase in the yield stress which is recorded at high strain rate. The error is low and ranges between 2.8% for copper 105 and 6.2% for copper 9 (Table 5).

TABLE 4: Material constants and errors obtained by Tuazon et al. model.

Metal	A (MPa)	C	p	Error (%)
Copper 105	95.4	2.26×10^{-9}	9.2544	9.1
Copper 26	182.0	2.24×10^{-7}	6.9553	3.4
Copper 9	282.7	1.46×10^{-7}	6.9458	6.1
Steel 293 K	142.6	0.0015	2.8758	15.2
Steel 493 K	108.2	1.14×10^{-5}	4.8235	6.9

3.5. Cowper-Symonds Model. The Cowper-Symonds equation uses a power equation of the strain rate. It fits well the experimental data for the copper materials and for the steel at

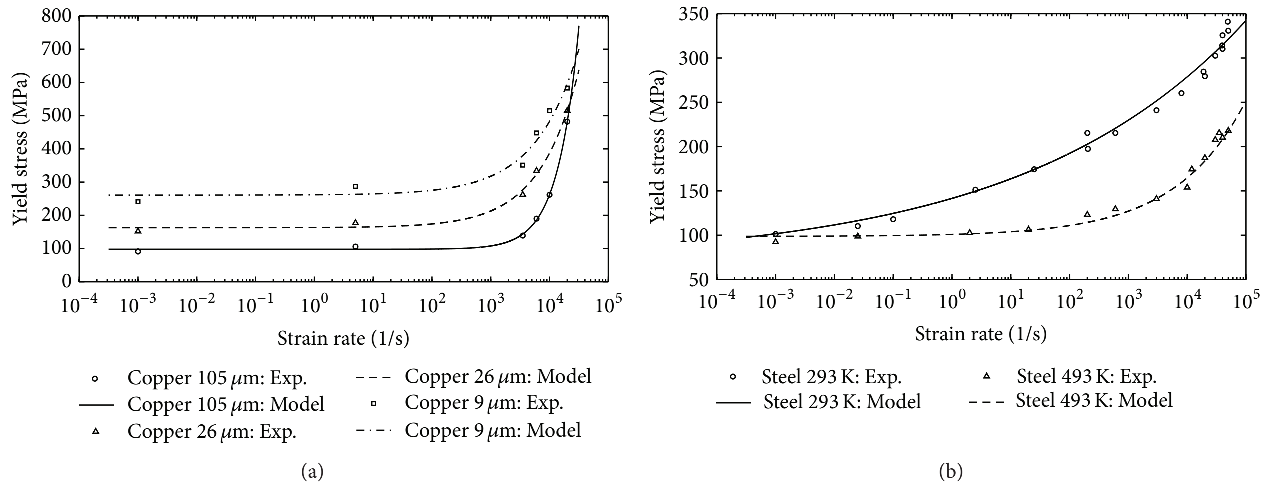

Figure 5: Yield stress fitting by the Cowper-Symonds model: (a) copper and (b) steel.

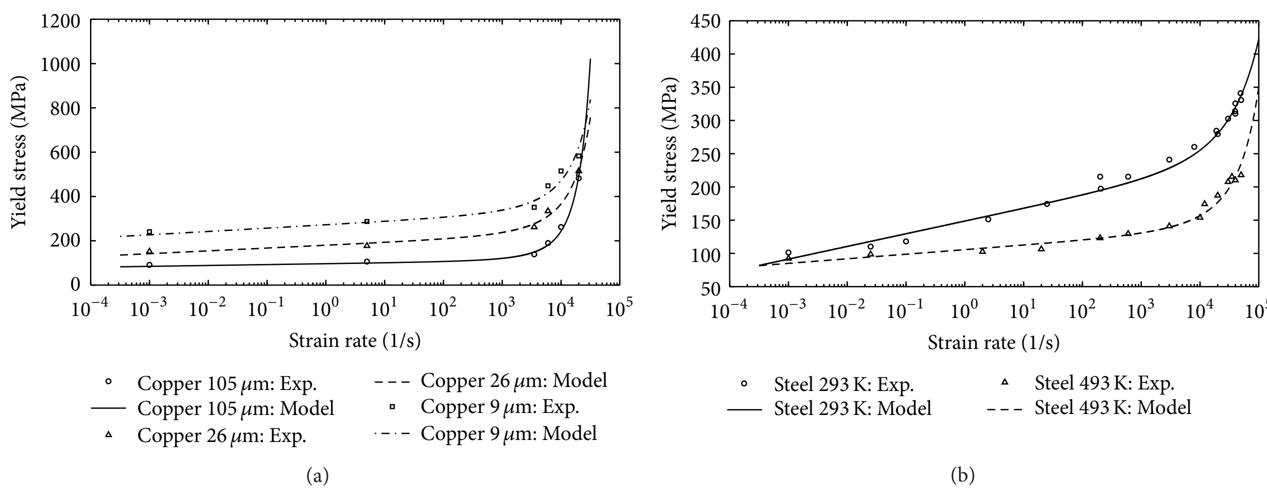

Figure 6: Yield stress fitting by the modified-Eyring model: (a) copper and (b) steel.

Table 5: Material constants and errors obtained by Couque equation.

Metal	A (MPa)	C	D (Pa)	k	Error (%)
Copper 105	100.6	0.0090	0.0669	1.3417	2.8
Copper 26	166.1	0.0096	0.2265	0.7327	3.6
Copper 9	265.1	0.0062	0.1462	0.7167	6.2
Steel 293 K	77.5	0.0077	1.9118	0.1206	4.9
Steel 493 K	104.2	0.0174	0.1011	0.5829	4.2

Table 6: Material constants and errors obtained by Cowper-Symonds equation.

Metal	A (MPa)	D (Pa)	k	Error (%)
Copper 105	97.9	$2.1616e-05$	1.2227	4.8
Copper 26	162.8	0.0035	0.6504	5.3
Copper 9	260.9	0.0036	0.5934	7.0
Steel 293 K	687.8	1.058	0.1150	4.9
Steel 493 K	98.6	0.0232	0.3650	5.5

different temperatures (Figure 5). Particularly, it catches well the sharp increase in the yield stress at high strain rates. The error is quite low (Table 6). It ranges between 4.8% (obtained for copper 105) and 7% (obtained for copper 9).

3.6. Modified-Eyring Model. The modified-Eyring model is developed by El-Qoubaa and Othman [18–20] based on the original work of Eyring [23] except that they used an activation volume decreasing with an increasing strain rate. The model works well here with copper and steel over the studied strain rate range (Figure 6). It fits well the increase in yield stress and the increase in the strain rate sensitivity of the yield stress that is observed at high strain rate. The error is low and ranges between 5.8 obtained with copper 26 and 7.8% obtained with copper 9 (Table 7).

3.7. Comparison. In the previous sections, six equations are used to fit the yield stress of three copper materials and one steel material at two temperatures. The standard Johnson-Cook equation gives the biggest error as mainly it cannot

Table 7: Material constants and errors obtained by the modified-Eyring equation.

Metal	σ_0 (MPa)	V_0 (Å^3)	$\dot{\varepsilon}_c$ (s^{-1})	Error (%)
Copper 105	96.5	2346	2036	6.0
Copper 26	179.8	737.5	5919	5.8
Copper 9	272.3	620.2	7049	7.8
Steel 293 K	148.6	491.3	89815	6.7
Steel 493 K	105.6	1365	28578	6.6

fit the behavior at high strain rates. The Huh-Kang equation gives only slightly better fit. Moreover, it predicts a sharp increase in the yield stress at quasi-static strain rates and this is a nonphysical behavior. The Tuazon et al. model yield an acceptable fit in the intermediate and high strain rate ranges. However, it predicts a sharp decrease in the yield stress at very low strain rates which is also a nonphysical behavior. Couque, Cowper-Symonds, and the modified-Eyring equations fit well the experimental data. Couque model yields the least value of error, then Cowper-Symonds, and finally the modified-Eyring equation. However, Couque equation uses four material constants while the modified-Eyring and Cowper-Symonds equations use only three each. It is then recommended to use the following modified Johnson-Cook equation for modeling of metallic materials including hardening, strain rate, and temperature effect:

$$\sigma_y = \left(\sigma_0 + \frac{k_B T_r}{V_0} \exp\left(\sqrt{\frac{\dot{\varepsilon}}{\dot{\varepsilon}_c}}\right) \log\left(\frac{\dot{\varepsilon}}{\dot{\varepsilon}_0}\right)\right)\left(1 + B' {\varepsilon_p}^n\right) \cdot \left(1 - \left(\frac{T - T_r}{T_m - T_r}\right)^m\right), \quad (5)$$

where σ_0, B', V_0, $\dot{\varepsilon}_c$, n, and m are six material constants and ε_p, T, T_r, T_m, $\dot{\varepsilon}$, $\dot{\varepsilon}_0$, and k_B are the plastic strain, absolute temperature, the room temperature, the melting temperature, the strain rate, a reference strain rate, and the Boltzmann constant, respectively. In this equation, the standard Johnson-Cook flow stress equation is modified using the strain rate sensitivity as predicted for room temperature by the modified-Eyring equation, while keeping the original hardening and temperature effects as first written by the standard equation.

4. Conclusion

In this work, we have compared six constitutive equations that predict the strain rate sensitivity of metallic materials. They are mainly used to fit the strain rate sensitivity of the yield stress of three copper materials and a steel alloy at two different temperatures. It is recommended to use either the Cowper-Symonds equation or the modified-Eyring equation as both of them give a close fit to the experimental data while using only three material constants each. A modified Johson-Cook equation is then proposed by including the strain rate sensitivity predicted by the modified-Eyring equation in the standard modified Johnson-Cook flow stress equation.

Competing Interests

The authors declare no conflict of interests regarding the publication of this paper.

References

[1] D. Y. Hu, K. P. Meng, H. L. Jiang, J. Xu, and R. R. Liu, "Strain rate dependent constitutive behavior investigation of AerMet 100 steel," *Materials and Design*, vol. 87, pp. 759–772, 2015.

[2] J. Kajberg and K. G. Sundin, "High-temperature split-hopkinson pressure bar with a momentum trap for obtaining flow stress behaviour and dynamic recrystallisation," *Strain*, vol. 50, no. 6, pp. 547–554, 2014.

[3] X. Guo, T. Heuzé, R. Othman, and G. Racineux, "Inverse identification at very high strain rate of the johnson-cook constitutive model on the Ti-6Al-4V alloy with a specially designed direct-impact Kolsky bar device," *Strain*, vol. 50, no. 6, pp. 527–538, 2014.

[4] C. K. H. Dharan and F. E. Hauser, "Determination of stress-strain characteristics at very high strain rates," *Experimental Mechanics*, vol. 10, no. 9, pp. 370–376, 1970.

[5] G. R. Johnson and W. H. Cook, "Fracture characteristics of three metals subjected to various strains, strain rates, temperatures and pressures," *Engineering Fracture Mechanics*, vol. 21, no. 1, pp. 31–48, 1985.

[6] G. R. Johnson and W. H. Cook, "A constitutive model and data for metals subjected to large strains, high strain rates and high temperatures," in *Proceedings of the 7th International Symposium on Ballistics*, vol. 21, pp. 541–547, 1983.

[7] O. Pantalé and B. Gueye, "Influence of the constitutive flow law in FEM simulation of the radial forging process," *Journal of Engineering*, vol. 2013, Article ID 231847, 8 pages, 2013.

[8] Y. Guo and Y. Li, "A novel approach to testing the dynamic shear response of Ti-6Al-4V," *Acta Mechanica Solida Sinica*, vol. 25, no. 3, pp. 299–311, 2012.

[9] G. Ljustina, M. Fagerström, and R. Larsson, "Rate sensitive continuum damage models and mesh dependence in finite element analyses," *The Scientific World Journal*, vol. 2014, Article ID 260571, 8 pages, 2014.

[10] W. Moćko, J. A. Rodríguez-Martínez, Z. L. Kowalewski, and A. Rusinek, "Compressive viscoplastic response of 6082-T6 and 7075-T6 aluminium alloys under wide range of strain rate at room temperature: experiments and modelling," *Strain*, vol. 48, no. 6, pp. 498–509, 2012.

[11] K. D. Clarke, R. J. Comstock, M. C. Mataya, C. J. Tyne, and D. K. Matlock, "Effect of strain rate on the yield stress of ferritic stainless steels," *Metallurgical and Materials Transactions A: Physical Metallurgy and Materials Science*, vol. 39, no. 4, pp. 752–762, 2008.

[12] Y. Wang, D. Lin, Y. Zhou, Y. Xia, and C. C. Law, "Dynamic tensile properties of Ti-47Al-2Mn-2Nb alloy," *Journal of Materials Science*, vol. 34, no. 3, pp. 509–513, 1999.

[13] T. Suo, L. Ming, F. Zhao, Y. Li, and X. Fan, "Temperature and strain rate sensitivity of ultrafine-grained copper under uniaxial compression," *International Journal of Applied Mechanics*, vol. 5, no. 2, Article ID 1350016, 15 pages, 2013.

[14] V. V. Balandin, V. V. Balandin, A. M. Bragov, L. A. Igumnov, A. Y. Konstantinov, and A. K. Lomunov, "High-rate deformation and fracture of steel 09G2S," *Mechanics of Solids*, vol. 49, no. 6, pp. 666–672, 2014.

[15] H. Huh and W. J. Kang, "Crash-worthiness assessment of thin-walled structures with the high-strength steel sheet," *International Journal of Vehicle Design*, vol. 30, no. 1-2, pp. 1–21, 2002.

[16] B. J. Tuazon, K.-O. Bae, S.-H. Lee, and H.-S. Shin, "Integration of a new data acquisition/processing scheme in SHPB test and characterization of the dynamic material properties of high-strength steels using the optional form of Johnson-Cook model," *Journal of Mechanical Science and Technology*, vol. 28, no. 9, pp. 3561–3568, 2014.

[17] H. Couque, "The use of the direct impact Hopkinson pressure bar technique to describe thermally activated and viscous regimes of metallic materials," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 372, no. 2023, Article ID 20130218, 10 pages, 2014.

[18] Z. El-Qoubaa and R. Othman, "Characterization and modeling of the strain rate sensitivity of polyetheretherketone's compressive yield stress," *Materials and Design*, vol. 66, pp. 336–345, 2015.

[19] A. A. Al-Juaid and R. Othman, "Modeling of the strain rate dependency of polycarbonate's yield stress: evaluation of four constitutive equations," *Journal of Engineering*, vol. 2016, Article ID 6315421, 9 pages, 2016.

[20] R. Othman, "A modified eyring equation for modeling yield and flow stresses of metals at strain rates ranging from 10^{-5} to 5×10^4 s^{-1}," *Advances in Materials Science and Engineering*, vol. 2015, Article ID 539625, 6 pages, 2015.

[21] G. Cowper and P. Symonds, "Strain hardening and strain-rate effects in the impact loading of cantilever beams," Tech. Rep., Division of Applied Mathematics, Brown University, Providence, RI, USA, 1952.

[22] F. J. Zerilli and R. W. Armstrong, "Dislocation-mechanics-based constitutive relations for material dynamics calculations," *Journal of Applied Physics*, vol. 61, no. 5, pp. 1816–1825, 1987.

[23] H. Eyring, "Viscosity, plasticity, and diffusion as examples of absolute reaction rates," *The Journal of Chemical Physics*, vol. 4, no. 4, pp. 283–291, 1936.