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Research Article

Construction of a Class of Real Array Rank Distance Codes

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Rank distance codes are known to be applicable in various applications such as distributed data storage, cryptography, space time coding, and mainly in network coding. Rank distance codes defined over finite fields have attracted considerable attention in recent years. However, in some scenarios where codes over finite fields are not sufficient, it is demonstrated that codes defined over the real number field are preferred. In this paper, we proposed a new class of rank distance codes over the real number field $\mathbb{R}$. The real array rank distance (RARD) codes we constructed here can be used for all the applications mentioned above whenever the code alphabet is the real field $\mathbb{R}$. From the class of RARD codes, we extract a subclass of equidistant constant rank codes which is applicable in network coding. Also, we determined an upper bound for the dimension of RARD codes leading the way to obtain some optimal RARD codes. Moreover, we established examples of some RARD codes and optimal RARD codes.

1. Introduction

Coding theory investigates the properties of codes and their suitability for various applications. Codes are used for a variety of purposes, including data compression, cryptography, error correction, and, more recently, network coding. Various scientific disciplines, such as information theory, electrical engineering, mathematics, and computer science deal with codes in order to design efficient and reliable data transmission methods [1–7].

The "rank distance" measures the distance between two matrices by the rank of their difference. Codes of matrices with rank distance are called rank distance (rank metric) codes. The applicability of rank distance codes [8–12] defined over finite fields in a wide variety of applications such as distributed data storage, cryptography, and network coding attracted attention in recent years. However, codes defined over the real number field [13–15] are advantageous in some applications, where small perturbations or noise in the received data need to be handled gracefully. Real rank metric codes are well suited for applications involving analog signals or continuous data such as audio and image transmission when the received signal may not be perfectly quantized and may contain analog imperfections [16–20]. Also, in scenarios with interference from multiple sources, real rank distance codes can be advantageous for separating and decoding signals as they can exploit the continuous nature of the received data to distinguish between different sources more effectively.

In this work, we propose a new class of linear rank distance codes over the real number field $\mathbb{R}$. We make use of the concept of a rank distance code [6, 21, 22] and construct the special class of real array rank distance (RARD) codes. RARD codes are a versatile and powerful class of codes with applications ranging from communications and network coding to data storage and cryptography.

We begin by discussing the concept of a real array code, which provides a systematic framework for efficiently representing and transmitting some specific real-valued data. Real array codes go beyond traditional coding schemes by allowing the encoding and decoding of an array of real numbers.

Our main focus is on the construction of the class of RARD codes [23–26] and establishing the various properties of these codes. In this process, we obtain a subclass of equidistant constant rank codes also [27].

We look into the dimension properties of real array rank distance codes. This analysis sheds light on the trade-offs between code complexity and error correction capabilities by providing valuable insights into the relationship between rank distance and code dimensionality. We obtain an upper bound for the dimension of the real array rank distance codes, paving the way for defining an optimal RARD code.

We present illustrative examples of real array codes that achieve the upper bound on code dimension. This example demonstrates the feasibility and efficiency of real array codes in practical applications, reinforcing their real-world implementation potential.

The paper is organized as follows. In Section 2, we give some definitions and results as preliminaries. Section 3 introduces the construction of the proposed class of real array rank distance codes and studies some properties of these codes belonging to this class. Here, we obtain a subclass of equidistant constant weight codes as a special case. Finally, we conclude the paper by giving an upper bound for the RARD codes we proposed and give some examples.

2. Preliminaries

Let $V = \mathbb{R}^N$ denote the N-dimensional vector space over the real field $\mathbb{R}$. Consider the set

$$V^n = \left(\mathbb{R}^N\right)^n = \left\{\left(\mathbf{X_1^T}, \mathbf{X_2^T}, \ldots, \mathbf{X_n^T}\right) \middle| \mathbf{X_i} \in V\right\}. \quad (1)$$

Each element $X \in V^n$ is expressed in the form of an n-tuple $X = [\mathbf{X_1^T}, \mathbf{X_2^T}, \mathbf{X_3^T}, \ldots \mathbf{X_n^T}]$, where $X_i \in \mathbb{R}^N$. Let $M_{N\times n}(\mathbb{R})$ denote the collection of all $N \times n$ matrices over $\mathbb{R}$. We can define a bijection $T\colon V^n \longrightarrow M_{N\times n}(\mathbb{R})$ such that for any $X = [\mathbf{X_1^T}, \mathbf{X_2^T}, \ldots \mathbf{X_n^T}] \in V^n$, the associated matrix is denoted by

$$T(X) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{N1} & \alpha_{N2} & \cdots & \alpha_{Nn} \end{pmatrix}, \quad (2)$$

where the i^{th} column represents the i^{th} coordinate X_i of X over $\mathbb{R}$ (w.r.t. the standard basis of $\mathbb{R}^N$).

Definition 1. The *rank* of an element $X \in V^n$ is the rank of the associated matrix $T(X)$ over $\mathbb{R}$ denoted by $r(X)$.

Definition 2. The *rank distance* denoted by d_R on V^n is defined as follows:

$$d_R(X, Y) = r(X - Y), \forall X, Y \in V^n, \quad (3)$$

where $r(X)$ denotes the rank of X over $\mathbb{R}$.

By the basic properties of the rank of a matrix, it can be proved that d_R defines a metric on V^n. The space V^n over $\mathbb{R}$ equipped with the rank metric d_R is defined as a "(*Real*) *rank distance space*."

3. Real Array Rank Distance (RARD) Codes

Let N, n be positive integers, where $N \le n$. For our purpose, we consider the rank distance space V^n as a vector space of dimension $N \times n$ over $\mathbb{R}$.

Definition 3. A real array rank distance code of length n over $\mathbb{R}$ is a subset of the rank distance space V^n over $\mathbb{R}$. A linear RARD code is a linear subspace of dimension k in the rank distance space V^n. By $C[N, n, k]$, we denote a linear $[N, n, k]$ RARD code.

Note that the dimension k of the code $C[N, n, K]$ is atmost Nn.

Definition 4. Let C be a linear RARD code. The minimum rank distance d of C is defined as $d = \min\{d_R(X, Y) \mid X, Y \in C, X \ne Y\}$. In other words, $d = \min\{r(X - Y) \mid X, Y \in C, X \ne Y\}$, i.e. $d = \min\{r(X) \mid X \in C \text{ and } X \ne 0\}$.

3.1. Construction of a Class of RARD Code. For any given positive integers N, n, and r with $r \le N \le n$, we construct a class of linear RARD codes and prove that, for this class of codes, the dimension k is given by $(n - r + 1)(N - r + 1)$ and minimum distance r.

We start by constructing three sets $B_1, B_2, B_3 \subseteq V^n$. The union of these three sets forms a basis B for any code C belonging to this class.

(i) Stage-I (construction of the set B_1)

(1) Let $s = n - N + 1$ and $t = N - r + 1$

(2) Choose an element $P = [\mathbf{X_1^T}, \mathbf{X_2^T}, \ldots, \mathbf{X_t^T}] \in V^t$, where $\mathbf{X_i} \in \mathbb{R}^N$ for $1 \le i \le t$, such that every $t \times t$ submatrix of $T(P)$ is nonsingular.

(3) Let $T(P) = [a_{ij}]_{N\times t}$

(4) Let

$$\begin{aligned} B_1 &= \left\{M_{ij} : M_{ij} \in V^n; 1 \le i \le s \;\&\; 1 \le j \le t\right\}, \text{with} \\ M_{ij} &= \left[\mathbf{O_1^T}, \mathbf{O_2^T}, \ldots, \mathbf{O_{i-1}^T}, a_{1j}\mathbf{E_1^T}, a_{2j}\mathbf{E_2^T}, \ldots, a_{Nj}\mathbf{E_N^T}, \mathbf{O_i^T}, \mathbf{O_{i+1}^T} \ldots, \mathbf{O_{n-N}^T}\right], \end{aligned} \quad (4)$$

where $[a_{1j}, a_{2j}, a_{3j}, \ldots, a_{Nj}]^T$ is the j^{th} coordinate of P, $\mathbf{O_i}$ & $\mathbf{E_j} \in \mathbb{R}^N$ ($\mathbf{O_i}$ denotes the zero vector of length "N," $1 \le i \le n - N$ and $\mathbf{E_j} = [0, 0,$

$\ldots, 0, 1, 0, \ldots, 0]$, 1 is at the j^{th} position, for $1 \le j \le N$).

(ii) Stage-II (construction of the sets B_2 and B_3)

(1) Choose an element $P_h = [\mathbf{X}_1^T, \mathbf{X}_2^T, \ldots, \mathbf{X}_{(N-h)-r+1}^T] \in V^{(N-h)-r+1}$, $h = 1$ to $N - r$, where $\mathbf{X}_i \in \mathbb{R}^{(N-h)}$ for $1 \le i \le (N-h) - r + 1$, such that every $[(N-h) - r + 1] \times [(N-h) - r + 1]$ submatrix of $T(P_h)$ is nonsingular.

(2) Let $T(P_h) = [a_{ij}]_{(N-h)\times(N-h-r+1)}$

(3) Let

$$B_2 = \left\{U_{ij} : U_{ij} \in V^n; 1 \le i \le N - r \,\&\, 1 \le j \le (N-i) - r + 1\right\}, \text{with}$$
$$U_{ij} = \left[\mathbf{O}_1^T, \mathbf{O}_2^T, \ldots, \mathbf{O}_{s+i-1}^T, a_{1j}\mathbf{E}_1^T, a_{2j}\mathbf{E}_2^T, a_{3j}\mathbf{E}_3^T, \ldots, a_{(N-i)j}\mathbf{E}_{N-i}^T\right], \tag{5}$$

where $[a_{1j}, a_{2j}, a_{3j}, \ldots, a_{(N-i)j}]^T$ be the j^{th} coordinate of P_i, $i = 1$ to $N - r$, $\mathbf{O}_i \& \mathbf{E}_j \in \mathbb{R}^N$ ($\mathbf{O}_i$ denotes the zero vector of length "N," $1 \le i \le n - r$ and $\mathbf{E}_j = [0, 0, \ldots, 0, 1, 0, \ldots, 0]$, 1 is at the j^{th} position, for $1 \le j \le N - i$).

(4) Let

$$B_3 = \left\{L_{ij} : L_{ij} \in V^n; 1 \le i \le N - r \,\&\, 1 \le j \le (N-i) - r + 1\right\}, \text{with}$$
$$L_{ij} = \left[a_{1j}\mathbf{E}_{i+1}^T, a_{2j}\mathbf{E}_{i+2}^T, a_{3j}\mathbf{E}_{i+3}^T, \ldots, a_{(N-i)j}\mathbf{E}_N^T, \mathbf{O}_1^T, \mathbf{O}_2^T, \ldots, \mathbf{O}_{s+i-1}^T\right], \tag{6}$$

where $[a_{1j}, a_{2j}, a_{3j}, \ldots, a_{(N-i)j}]^T$ be the j^{th} coordinate of P_i, $i = 1$ to $N - r$, $\mathbf{O}_i \& \mathbf{E}_j \in \mathbb{R}^N$ ($\mathbf{O}_i$ denotes the zero vector of length "N," $1 \le i \le n - r$ and $\mathbf{E}_j = [0, 0, \ldots, 0, 1, 0, \ldots, 0]$, 1 is at the j^{th}position, for $1 \le j \le N - i$).

(iii) Now, consider the set $B = B_1 \cup B_2 \cup B_3$.

The following results aid in the construction of the code C.

Lemma 5. *The set $B = B_1 \cup B_2 \cup B_3$, where B_i's are the sets constructed as above, is linearly independent over $\mathbb{R}$.*

Proof. Suppose

$$\sum_{j=1}^{t}\sum_{i=1}^{s} m_{ij}M_{ij} + \sum_{j=1}^{(N-i)-r+1}\sum_{i=1}^{N-r} u_{ij}U_{ij} + \sum_{j=1}^{(N-i)-r+1}\sum_{i=1}^{N-r} l_{ij}L_{ij} = \left[\mathbf{O}_1^T, \mathbf{O}_2^T, \ldots, \mathbf{O}_n^T\right], \tag{7}$$

where $M_{ij} \in B_1$, $U_{ij} \in B_2$, $L_{ij} \in B_3$, $m_{ij}, u_{ij}, l_{ij} \in \mathbb{R}$, and $\mathbf{O}_i$ denotes the zero vector of length "N."

Let $LHS = X = [\mathbf{X}_1^T, \mathbf{X}_2^T, \ldots, \mathbf{X}_n^T]$. Equating the coordinates, we get $\mathbf{X}_i^T = \mathbf{O}_i^T$, $\forall i = 1$ to n. For $i = 1$ to N, comparing the i^{th} coordinate of $\mathbf{X}_i^T$ with $\mathbf{O}_i^T$, we get

$$\begin{aligned} m_{11}a_{11} + m_{12}a_{12} + \cdots + m_{1t}a_{1t} &= 0, \\ m_{11}a_{21} + m_{12}a_{22} + \cdots + m_{1t}a_{2t} &= 0, \\ &\vdots \\ m_{11}a_{N1} + m_{12}a_{N2} + \cdots + m_{1t}a_{Nt} &= 0. \end{aligned} \tag{8}$$

Equation (8) is a homogeneous system of N linear equations with $t = N - r + 1$ variables $m_{11}, m_{12}, \ldots, m_{1t}$. This system has only a trivial solution as the coefficient matrix of (8) is exactly $T(P)$, i.e. $m_{1j} = 0$ for $j = 1$ to t.

In the same way, we can prove that $m_{ij} = 0$ for $2 \le i \le s \,\&\, 1 \le j \le t$, $u_{ij} = o$, and $l_{ij} = 0$ for $1 \le i \le N - r \,\&\, 1 \le j \le (N-i) - r + 1$. Hence, B is linearly independent.

Now, let C be the span of B over $\mathbb{R}$. Clearly, C is a vector space with basis B over $\mathbb{R}$. Equipping C with the rank distance, we obtain the proposed RARD code.

The following results give the various properties of the RARD code C. □

Theorem 6. *The dimension of the linear RARD code C constructed above is $(n - r + 1)(N - r + 1)$.*

Proof. The dimension of the linear RARD code is the cardinality of B.

$$\begin{aligned} |B| &= \left|B_1 \cup B_2 \cup B_3\right| \\ &= \left|B_1\right| + \left|B_2\right| + \left|B_3\right|, \text{as } B_i'\text{s are pairwise disjoint.} \end{aligned} \tag{9}$$

(i) The set B_1 contains $st = (n - N + 1)(N - r + 1)$ elements.

(ii) The number of elements in the set B_2 is

$$\begin{aligned} |B_2| &= |U_{1j}| + |U_{2j}| + \ldots + |U_{(N-r)j}|, 1 \le i \le N-r \,\&\, 1 \le j \le (N-i)-r+1 \\ &= (N-r) + (N-r-1) + \ldots + 1 \\ &= \frac{(N-r)(N-r+1)}{2}. \end{aligned} \tag{10}$$

(iii) The number of elements in the set B_3 is

$$\begin{aligned} |B_3| &= |L_{1j}| + |L_{2j}| + \ldots + |L_{(N-r)j}|, 1 \le i \le N-r \,\&\, 1 \le j \le (N-i)-r+1 \\ &= (N-r) + (N-r-1) + \ldots + 1 \\ &= \frac{(N-r)(N-r+1)}{2}. \end{aligned} \tag{11}$$

Hence,

$$\begin{aligned} |B| &= (n-N+1)(N-r+1) + \frac{(N-r)(N-r+1)}{2} + \frac{(N-r)(N-r+1)}{2} \\ &= (n-r+1)(N-r+1). \end{aligned} \tag{12}$$

□

Theorem 7. *Each nonzero element of the RARD code C has rank at least r over* $\mathbb{R}$.

Proof. Let X be an arbitrary nonzero element of C. X can be expressed in terms of elements of B.

$$\text{i.e. } X = \sum_{j=1}^{t}\sum_{i=1}^{s} m_{ij}M_{ij} + \sum_{j=1}^{(N-i)-r+1}\sum_{i=1}^{N-r} u_{ij}U_{ij} + \sum_{j=1}^{(N-i)-r+1}\sum_{i=1}^{N-r} l_{ij}L_{ij}, \tag{13}$$

where $M_{ij} \in B_1$, $U_{ij} \in B_2$, $L_{ij} \in B_3$, and $m_{ij}, u_{ij}, l_{ij} \in \mathbb{R}$. The rank of X is the rank of the corresponding matrix $T(X)$. $T(X)$ can be expressed as follows:

$$\begin{pmatrix} \sum_{j=1}^{t} a_{1j}m_{1j} & \sum_{j=1}^{t} a_{1j}m_{2j} & \cdots \sum_{j=1}^{t} a_{1j}m_{sj} & \sum_{j=1}^{N-r} a_{1j}u_{1j} & \cdots u_{(N-r)1} & 0_1 & \cdots 0_{r-1} \\ \sum_{j=1}^{N-r} a_{1j}l_{1j} & \sum_{j=1}^{t} a_{2j}m_{1j} & \cdots \sum_{j=1}^{t} a_{2j}m_{(s-1)j} & \sum_{j=1}^{t} a_{2j}m_{sj} & \cdots \sum_{j=1}^{2} a_{2j}u_{1j} & u_{(N-r)1} & \cdots 0_{r-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{j=1}^{N-2r+3} a_{1j}l_{(r-2)j} & \vdots & \vdots & \vdots & \vdots & \vdots & 0_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_1 & \vdots & \vdots & \vdots & \vdots & \vdots & \sum_{j=1}^{N-2r+3} a_{(N-r+2)j}u_{(r-2)j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{r-1} & 0_{r-2} & \cdots\cdots 0_1 & \cdots & \cdots & \cdots & \sum_{j=1}^{t} a_{Nj}m_{sj} \end{pmatrix}. \quad (14)$$

Consider the lowest nonzero diagonal of $T(X)$. To prove each nonzero element of the code C has rank at least r over $\mathbb{R}$, it is enough to prove that this diagonal contains at least r nonzero elements. We may assume that this diagonal is the main diagonal. Consider the $N \times N$ submatrix of $T(X)$ containing the main diagonal, which is denoted by $T_N(X)$. Let

$$\lambda_i = \sum_{j=1}^{t} a_{ij}m_{1j}, \; i = 1 \text{ to } N, \quad (15)$$

be the main diagonal elements of $T_N(X)$ and we assume that not all of the m_{1j} are zero for $j = 1$ to t. We have to prove $\operatorname{rank}(T_N(X)) \geq r$.

On the contrary, we assume that $\operatorname{rank}(T_N(X)) < r$. Then, at least $N - r + 1$ of λ_i must be zero (if $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the diagonal elements of a square matrix and if every product of r distinct λ_i vanishes, then at least $N - r + 1$ of λ_i must be zero [28]). Without loss of generality, we may assume that $\lambda_1, \lambda_2, \ldots, \lambda_{N-r+1}$ are nonzero. Hence,

$$\sum_{j=1}^{t} a_{ij}m_{1j} = o, \quad (16)$$

for $i = 1$ to $N - r + 1$. Equation (16) is a homogeneous system of $N - r + 1$ equations in $N - r + 1$ variables. The coefficient matrix of the above homogeneous system is a $(N - r + 1) \times (N - r + 1)$ submatrix of $T(P)$. By the nature of $T(P)$, the system has only a trivial solution. So, $m_{1j} = 0$ for $j = 1$ to $N - r + 1 = t$, which contradicts our assumption. Hence, $\operatorname{rank}(T(X)) \geq r$. In the same way, we can prove the result in the case of other diagonals. □

Theorem 8. *Each element of the basis B has rank r over $\mathbb{R}$.*

Proof. The result follows from the fact that $B = B_1 \cup B_2 \cup B_3$ and if E is an arbitrary element of B and $T(E)$ is its corresponding matrix representation, then $T(E)$ has exactly "r" nonzero columns, which are, in fact, scalar multiples of unit vectors. □

Remark 9. Theorems 6, 7, and 8 imply that C is a linear RARD code with minimum distance r and dimension $(N - r + 1)(n - r + 1)$.

Example 1. An RARD code "C" with minimum distance $r = 2$ over the space $V^4 = (\mathbb{R}^3)^4$.

Here, $s = n - N + 1 = 2$ and $t = N - r + 1 = 2$.

(i) Stage-I (construction of set B_1)

We choose an element $P = [\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}] \in V^2$ over $\mathbb{R}^3$ such that $\mathbf{X}_1 = [1, 1, 1]$, $\mathbf{X}_2 = [1, 2, 3]$. $T(P)$ is the 3×2 matrix, in which every 2×2 submatrix is nonsingular.

$$\begin{aligned} B_1 &= \left\{M_{ij} : M_{ij} \in V^n;\ 1 \leq i \leq s \;\&\; 1 \leq j \leq t\right\} \\ &= \left\{M_{ij} : M_{ij} \in V^5;\ 1 \leq i \leq 2 \;\&\; 1 \leq j \leq 2\right\} \\ &= \{M_{11}, M_{12}, M_{21}, M_{22}\}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} M_{11} &= \left[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}, \mathbf{X}_3^{\mathrm{T}}, \mathbf{X}_4^{\mathrm{T}}\right] \\ &= \left[\mathbf{E}_1^{\mathrm{T}}, \mathbf{E}_2^{\mathrm{T}}, \mathbf{E}_3^{\mathrm{T}}, \mathbf{O}_1^{\mathrm{T}}\right], \\ M_{12} &= \left[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}, \mathbf{X}_3^{\mathrm{T}}, \mathbf{X}_4^{\mathrm{T}}\right] \\ &= \left[\mathbf{E}_1^{\mathrm{T}}, 2\mathbf{E}_2^{\mathrm{T}}, 3\mathbf{E}_3^{\mathrm{T}}, \mathbf{O}_1^{\mathrm{T}}\right], \\ M_{21} &= \left[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}, \mathbf{X}_3^{\mathrm{T}}, \mathbf{X}_4^{\mathrm{T}}\right] \\ &= \left[\mathbf{O}_1^{\mathrm{T}}, \mathbf{E}_1^{\mathrm{T}}, \mathbf{E}_2^{\mathrm{T}}, \mathbf{E}_3^{\mathrm{T}}\right], \\ M_{22} &= \left[\mathbf{X}_1^{\mathrm{T}}, \mathbf{X}_2^{\mathrm{T}}, \mathbf{X}_3^{\mathrm{T}}, \mathbf{X}_4^{\mathrm{T}}\right] \\ &= \left[\mathbf{O}_1^{\mathrm{T}}, \mathbf{E}_1^{\mathrm{T}}, 2\mathbf{E}_2^{\mathrm{T}}, 3\mathbf{E}_3^{\mathrm{T}}\right]. \end{aligned} \quad (18)$$

(ii) Stage-II (construction of sets B_2 and B_3)

(1) $h = 1$ *to* $N - r = 1$

(2) We choose the element $P_1 = [X_1^T] \in V^{(N-h)-r+1} = V^1$ over $\mathbb{R}^{N-h} = \mathbb{R}^2$ such that $\mathbf{X}_1 = [1, 1]$.

$T(P)$ is the 2×1 matrix, whose every 1×1 submatrix is nonsingular.

$$\begin{aligned}
B_2 &= \left\{U_{ij}: U_{ij} \in V^5; 1 \le i \le N - r \ \&\ 1 \le j \le (N-i) - r + 1\right\} \\
&= \left\{U_{ij}: U_{ij} \in V^5; 1 \le i \le 1 \ \&\ 1 \le j \le 2 - i\right\} \\
&= \{U_{11}\}, \text{where} \\
U_{11} &= \left[\mathbf{X}_1^{\mathbf{T}}, \mathbf{X}_2^{\mathbf{T}}, \mathbf{X}_3^{\mathbf{T}}, \mathbf{X}_4^{\mathbf{T}}\right] \\
&= \left[\mathbf{O}_1^{\mathbf{T}}, \mathbf{O}_2^{\mathbf{T}}, \mathbf{E}_1^{\mathbf{T}}, \mathbf{E}_2^{\mathbf{T}}\right] \\
B_3 &= \left\{L_{ij}: L_{ij} \in V^5; 1 \le i \le N - r \ \&\ 1 \le j \le (N-i) - r + 1\right\} \\
&= \left\{L_{ij}: L_{ij} \in V^5; 1 \le i \le 1 \ \&\ 1 \le j \le 2 - i\right\} \\
&= \{L_{11}\}, \text{where} \\
L_{11} &= \left[\mathbf{X}_1^{\mathbf{T}}, \mathbf{X}_2^{\mathbf{T}}, \mathbf{X}_3^{\mathbf{T}}, \mathbf{X}_4^{\mathbf{T}}\right] \\
&= \left[\mathbf{E}_2^{\mathbf{T}}, \mathbf{E}_3^{\mathbf{T}}, \mathbf{O}_1^{\mathbf{T}}, \mathbf{O}_2^{\mathbf{T}}\right].
\end{aligned} \tag{19}$$

Then,

$$\begin{aligned}
B &= B_1 \cup B_2 \cup B_3 \\
&= \{M_{11}, M_{12}, M_{21}, M_{22}, U_{11}, L_{11}\} \\
|B| &= (N - r + 1)(n - r + 1) \\
&= (3 - 2 + 1)(4 - 2 + 1) \\
&= 6.
\end{aligned} \tag{20}$$

The RARD code C is given by

$$\text{i.e. } C = \left\langle \sum_{j=1}^{2}\sum_{i=1}^{2} m_{ij}M_{ij} + \sum_{j=1}^{2-i}\sum_{i=1}^{1} u_{ij}U_{ij} + \sum_{j=1}^{2-i}\sum_{i=1}^{1} l_{ij}L_{ij} \right\rangle. \tag{21}$$

The matrix representation of any element P of C is

$$T(P) = \begin{pmatrix} m_{11} + m_{12} & m_{21} + m_{22} & u_{11} & 0 \\ l_{11} & m_{11} + 2m_{12} & m_{21} + 2m_{22} & u_{11} \\ 0 & l_{11} & m_{11} + 3m_{12} & m_{11} + 3m_{12} \end{pmatrix}. \tag{22}$$

The nonzero elements of C have rank at least 2.

3.2. A Subclass of Equidistant Constant Rank Codes. Equidistant constant rank codes have many applications especially in network coding, ARQ systems, etc. The following result extracts a subclass of equidistant constant rank RARD code with distance "r" from the class of RARD codes.

Theorem 10. *If $N = r$, the RARD code "C" is an equidistant constant rank code with distance r.*

Proof. Since "C" is a linear rank distance space, it is enough to prove that every nonzero element of "C" has rank r. When $N = r$, by the construction $t = 1, B_2 = \phi$ and $B_3 = \phi$. We choose an element $P = [\mathbf{X}_1^{\mathbf{T}}]$, where $\mathbf{X}_1 = [a_{11}, a_{21}, \ldots, a_{N1}]$, from V such that every 1×1 submatrix of $T(P)$ is nonsingular, i.e. $a_{i1} \ne 0$, $\forall i = 1$ to N. Thus, the basis is

$$\begin{aligned}
B &= B_1 \\
&= \left\{M_{ij} : M_{ij} \in V^n; 1 \le i \le s \ \&\ 1 \le j \le t\right\} \\
M_{i1} &= \left[\mathbf{O}_1^{\mathbf{T}}, \mathbf{O}_2^{\mathbf{T}}, \ldots, \mathbf{O}_{\mathbf{i-1}}^{\mathbf{T}}, a_{11}\mathbf{E}_1^{\mathbf{T}}, a_{21}\mathbf{E}_2^{\mathbf{T}}, \ldots, a_{N1}\mathbf{E}_{\mathbf{N}}^{\mathbf{T}}, \mathbf{O}_{\mathbf{i}}^{\mathbf{T}}, \mathbf{O}_{\mathbf{i+1}}^{\mathbf{T}} \ldots, \mathbf{O}_{\mathbf{n-N}}^{\mathbf{T}}\right],
\end{aligned} \tag{23}$$

where $[a_{11}, a_{21}, a_{31}, \ldots, a_{N1}] = X_1$. Thus, $C = < \sum_{j=1}^{t} \sum_{i=1}^{s} m_{ij} M_{ij} >$. Let X be any arbitrary nonzero element of C. Then, $T(X)$ is

$$\begin{pmatrix} a_{11}m_{11} & a_{11}m_{21} & a_{11}m_{31} & \cdots & a_{11}m_{r1} & \cdots & a_{11}m_{s1} & 0 & 0 & \cdots 0 \\ 0 & a_{21}m_{11} & a_{21}m_{21} & \cdots & a_{21}m_{(r-1)1} & \cdots & a_{21}m_{(s-1)1} & a_{21}m_{s1} & 0 & \cdots 0 \\ 0 & 0 & a_{31}m_{11} & \cdots & a_{31}m_{(r-2)1} & \cdots & a_{31}m_{(s-2)1} & a_{31}m_{(s-1)1} & m_{s1} & \cdots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N1}m_{11} & \cdots & \cdots & \cdots & \cdots & \cdots a_{N1}m_{s1} \end{pmatrix}. \quad (24)$$

As the number of nonzero rows of $T(X)$ is $N = r$, the rank of $T(X)$ cannot exceed r. Also, by the construction, $T(X)$ has rank r for any choice of $m_{11}, m_{21}. \cdots, m_{s1}$ except when all of them are zero, in which case we have the zero matrix.

Thus, $T(X)$ has rank either r or 0. As $T(X)$ is nonzero, its rank must be r. That is, if $N = r$, the RARD code C is an equidistant constant rank code with distance r. □

3.3. An Upper Bound for the Dimension of RARD Codes. The following result gives an upper bound for the dimension of an RARD code for any given N, n, and r.

Theorem 11. *Let $C[N, n, k]$ be an RARD code with every nonzero vector having rank at least r. Then, the dimension of C is at most $n(N - r + 1)$.*

Proof. Let $V^n = (\mathbb{R}^N)^n$ be the rank distance space of dimension $N \times n$ over $\mathbb{R}$. Let U be a subspace of V^n such that every element of U has rank at most "r", and dim $U = nr$. If we replace r by $r - 1$, we get a subspace W of V^n with dimension dim $W = n(r - 1)$, whose elements have rank at most $r - 1$. Consider a new subspace C of V^n with dimension k consisting of all the nonzero vectors having rank at least r. Since $C \cap W = \{0\}$,

$$\begin{aligned} \dim(C \cap W) &= \dim(C) + \dim(W) - \dim(C + W) \\ \text{we have} \dim(C) &= \dim(C \cap W) - \dim(W) + \dim(C + W) \\ k &\leq 0 - n(r - 1) + nN \\ &= nN - n(r - 1) \\ &= n(N - r + 1). \end{aligned} \quad (25)$$

Thus, the dimension of $C[n, N, k]$ is at most $n(N - r + 1)$. □

Remark 12. We call an RARD code achieving the upper bound on its dimension as an optimal RARD code.

The following are examples of optimal RARD codes.

Example 2. Let $V^2 = (\mathbb{R}^2)^2$ be the rank distance space. We construct an RARD code with minimum distance $r = 2$. Let C be the rank distance space spanned by $B = \{B_1, B_2\}$, where $B_1 = [(1, -1)^T, (2, 1)^T]$ and $B_2 = [(1, 1)^T, (-2, 1)^T]$. Clearly, B is linearly independent. Consider an arbitrary nonzero element X from C. Let it be $X = [(x + y, y - x), (2x - 2y, x + y)]$, where $x, y \in \mathbb{R}$. The matrix representation of X is

$$T(X) = \begin{pmatrix} x + y & 2x - 2y \\ y - x & x + y \end{pmatrix}. \quad (26)$$

The determinant of $T(X)$ is $(x - y)^2 + 2x^2 + 2y^2$, which is zero only if $x = y = 0$. Hence, the rank of $T(X)$ is 2. C has minimum rank distance $r = 2$. Also, the dimension of C is 2. Here, $n = N = 2$, $r = 2$ and $k = n(N - r + 1) = 2$, i.e. dimension of C achieves the upper bound.

Example 3. Let $V^n = (\mathbb{R}^2)^n$, $n = 2t, t \in \mathbb{N}$ be the rank distance space. We construct an RARD code with minimum distance $r = 2$. Let C be the rank distance space spanned by $B = \{B_1, B_2, B_3, \ldots, B_{2t}\}$, where the elements B_i for $1 \leq i \leq 2t$ are defined as follows:

$$B_i = \begin{cases} \left[\mathbf{O}_1^T, \mathbf{O}_2^T, \mathbf{O}_{j-1}^T, \mathbf{E}_1^T, \mathbf{E}_2^T, \mathbf{O}_j^T, \ldots, \mathbf{O}_{2t-2}^T\right], & \text{if } i \text{ is odd}, \\ \left[\mathbf{O}_1^T, \mathbf{O}_2^T, \mathbf{O}_{j-1}^T, -\mathbf{E}_2^T, \mathbf{E}_1^T, \mathbf{O}_j^T, \ldots, \mathbf{O}_{2t-2}^T\right], & \text{if } i \text{ is even}. \end{cases} \quad (27)$$

Clearly, B is linearly independent over $\mathbb{R}$. The span of B gives a linear RARD code. To find its minimum distance, consider an arbitrary nonzero element X in C. X can be expressed in terms of the elements of B.

$$\text{i.e. } X = \sum_{i=1}^{2t} a_{1i} B_i, \quad (28)$$

where $B_i \in B$ and $a_{1i} \in \mathbb{R}$. Then, the matrix representation of X is

$$T(X) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(2t-1)} & a_{12t} \\ -a_{12} & a_{11} & -a_{13} & \cdots & -a_{1(2t)} & a_{1(2t-1)} \end{pmatrix}. \quad (29)$$

We can easily see that every such matrix has rank 2, except when all of the a_{ij}'s are zero. Hence, the rank of $T(X)$ is 2. Thus, C has minimum rank distance $r = 2$. Since B is linearly independent, the dimension of the space is $2t$. Here, $N = 2$, $n = 2t$, and $r = 2$. So $k = n(N - r + 1) = 2t$. Thus, the dimension of C achieves the upper bound.

4. Conclusion

While the role of rank distance codes defined over finite fields is significant in dealing with various applications, codes defined over $\mathbb{R}$ are advantageous in some applications where the errors in the received data are not discrete or quantized. Real rank distance codes are well suited for many applications involving analog signals or continuous data.

In this work, we have considered the problem of the construction of a rank distance code over $\mathbb{R}$. We have constructed a class of linear real array rank distance (RARD) codes. RARD codes can be used in many applications such as network coding, distributed data storage, compressed sensing, and cryptography whenever the code alphabet is the real field $\mathbb{R}$. In this process, we obtained an interesting subclass of equidistant constant rank codes which can have many applications including network coding. We obtained an upper bound on the dimension of RARD codes which led the way to define an optimal RARD code. We constructed two examples of optimal RARD codes.

Future research on RARD codes may focus on the construction of a class maximum distance separable RARD codes to ensure the maximum error correction capability.

Data Availability

The data that support the findings of this study are available from the corresponding author upon request.

Disclosure

The last author is a postdoctoral fellow at the University of Kerala, India.

Conflicts of Interest

The authors hereby declare that there are no potential conflicts of interest.

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