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Pseudodifferential operators on α-modulation spaces*

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Abstract. We study expansions of pseudodifferential operators from the Hörmander class in a special family of functions called brushlets. We prove that such operators have a sparse representation in a brushlet system. Using this sparsity, we show that a pseudodifferential operator extends to a bounded operator between α-modulation spaces. These spaces were introduced by Gröbner in [15]. They are, in some sense, intermediate spaces between the classical Besov and Modulation spaces.

1. Introduction

The problem of investigating boundedness of pseudodifferential operators (Ψdo's) in various function spaces has been considered by numerous authors. Bourdaud [4] and Gibbons [12] considered the boundedness on Besov spaces, and Päivärinta [21] and Bui [22] on Triebel-Lizorkin spaces. In [28, 29], Yamazaki generalized these results by considering paradifferential operators in anisotropic Besov and Triebel-Lizorkin spaces. More recently there

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has been a growing interest in analyzing pseudodifferential operators on modulation spaces, see, e.g., [25, 26, 17, 19, 5]. For a more detailed description on this topic we refer the reader to [16, Chapter 14].

The construction of smooth atoms or molecules such as wavelets, has made it possible to construct sparse representations of a whole class of operators, among those many Calderón-Zygmund and Ψdo's, and thus construct much simpler proofs of the above mentioned results. A major contribution to the study of singular integral operators was given by Frazier, Jawerth, Torres and Weiss, see e.g. [10, 11, 9, 27]. They studied the boundedness of Calderón-Zygmund operators on various function spaces, including Besov and Triebel-Lizorkin spaces. Their proofs were based on the fact that there exist retracts between these function spaces and corresponding sequence spaces. The classes of singular integral operators studied included Ψdo's with symbols belonging to the Hörmander class $S^s_{1,\delta}$ for $s \in \mathbb{R}$ and $0 \leq \delta < 1$ (see Section 3 for a definition of the Hörmander classes). In [26], Tachizawa used Wilson bases to obtain sparse representations of a large class of pseudodifferential operators. More recently, singular integral operators has been studied in a more general setting, see e.g. [13, 14].

This paper concerns Ψdo's of the Hörmander class $\mathrm{OPS}^s_{\rho,\delta}$ for $s \in \mathbb{R}$, $0 \leq \delta < 1$ and $0 < \rho \leq 1$. In all the above mentioned results on the boundedness of Ψdo's on Besov and Triebel-Lizorkin spaces the parameter ρ is fixed, $\rho = 1$. Very little is known when $\rho < 1$, except for a limited set of parameters (see Section 4). We will see in this paper that whenever $\rho \geq \delta$ the corresponding operator extends to a bounded operator between the so-called α-modulation spaces provided $\rho = \alpha$. This class of spaces includes the classical Besov spaces (for $\alpha = 1$) and the modulation spaces (for $\alpha = 0$).

It was pointed out in the papers [7, 6] that the Besov and modulation spaces are special cases of the decomposition type Banach spaces $D(Q, B, Y)$ introduced in [7]. In his Ph.D. thesis [15], Gröbner introduced the α-modulation spaces we will consider in this paper. These spaces correspond to a segmentation of the frequency axis according to a polynomial rule, see Section 2.1.

The arguments in the present paper follow the ideas of Frazier and Jawerth, and consist of the following steps. First, we notice that α-modulation spaces can be characterized by expansions in a special set of functions called a brushlet system, and that the spaces can be seen as a retract of weighted sequence spaces using the corresponding brushlet coefficients. Then, we observe that a Ψdo has a sparse representation in such a brushlet system. This, finally, converts the problem to show a boundedness result for an infinite matrix on some given sequence spaces.

The paper is organized as follows. In Section 2, we introduce the brushlet systems and the α-modulation spaces. In Section 3, we study brushlet expansions of certain Ψdo's. Using the sparsity of such expansions we show the main result in Section 4 (Theorem 4.3); that Ψdo's extend to bounded operators between univariate α-modulation spaces for given parameters.

Since the arguments in this paper are based on the univariate brushlet systems, we have not been able to generalize Theorem 4.3 to higher dimensions. It is still an open question how to construct multivariate brushlet systems giving rise to characterizations as in Theorem 2.8.

Let us introduce some notation. For a finite discrete set D we denote by $\#D$ the cardinality of D. $A \asymp B$ means that there exist two constants $c, C \in (0, \infty)$ such that $cA \leq B \leq CA$. For notational convenience, we use the shorthand notation L_p for the univariate Lebesgue space $L_p(\mathbb{R})$, $1 \leq p \leq \infty$.

2. Brushlet systems and α-modulation spaces

In this section we shall define what we will call a brushlet system. We will only focus on the properties needed in order to prove the results for Ψdo's in the subsequent sections. After the introduction of brushlet systems, we introduce the α-modulation spaces in Section 2.1. Finally, we give a brushlet characterization of α-modulation spaces at the end of the section.

Each brushlet basis is associated with a partition of the frequency axis. The partition can be chosen with almost no restrictions, but in order to have good properties of the associated basis we need to impose some growth conditions on the partition. We introduce the following definition.

Definition 2.1. A family $\mathcal{I}$ of intervals is called a *disjoint covering* of $\mathbb{R}$ if it consists of a countable set of pairwise disjoint half-open intervals $I = [\alpha_I, \alpha'_I)$, $\alpha_I < \alpha'_I$, such that $\cup_{I \in \mathcal{I}} I = \mathbb{R}$. If, furthermore, each interval in $\mathcal{I}$ has a unique adjacent interval in $\mathcal{I}$ to the left and to the right, and there exists a constant $A > 1$ such that

$$A^{-1} \leq \frac{|I|}{|I'|} \leq A \qquad \text{for all adjacent } I, I' \in \mathcal{I}, \tag{2.1}$$

we call $\mathcal{I}$ a *moderate disjoint covering* of $\mathbb{R}$.

Given a moderate disjoint covering $\mathcal{I}$ of $\mathbb{R}$, assign to each interval $I = [\alpha_I, \alpha'_I) \in \mathcal{I}$ a left and right cutoff radius $\varepsilon_I, \varepsilon'_I > 0$, satisfying

$$\begin{cases} \text{(i)} & \varepsilon'_I = \varepsilon_{I'} \text{ whenever } \alpha'_I = \alpha_{I'} \\ \text{(ii)} & \varepsilon_I + \varepsilon'_I \leq |I| \\ \text{(iii)} & \varepsilon_I \geq c|I|, \end{cases} \tag{2.2}$$

with $c > 0$ independent of I.

Example 2.2. If we let $\varepsilon_I = \frac{1}{2A}|I|$ and ε'_I be given by (i) in (2.2) then (ii) and (iii) are clearly satisfied.

We are now ready to define the brushlet system. For each $I \in \mathcal{I}$, we will construct a smooth bell function localized in a neighborhood of this interval. Take a non-negative ramp function $\rho \in C^r(\mathbb{R})$ for some $r \geq 1$, satisfying

$$\rho(\xi) = \begin{cases} 0 & \text{for } \xi \leq -1, \\ 1 & \text{for } \xi \geq 1, \end{cases} \tag{2.3}$$

with the property that

$$\rho(\xi)^2 + \rho(-\xi)^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

Define for each $I = [\alpha_I, \alpha'_I) \in \mathcal{I}$ the *bell function*

$$b_I(\xi) := \rho\left(\frac{\xi - \alpha_I}{\varepsilon_I}\right)\rho\left(\frac{\alpha'_I - \xi}{\varepsilon'_I}\right). \tag{2.5}$$

Notice that $\operatorname{supp}(b_I) \subset [\alpha_I - \varepsilon_I, \alpha'_I + \varepsilon'_I]$ and $b_I(\xi) = 1$ for $\xi \in [\alpha_I + \varepsilon_I, \alpha'_I - \varepsilon'_I]$. Now the set of local cosine functions

$$\hat{w}_{n,I}(\xi) = \sqrt{\frac{2}{|I|}}\, b_I(\xi) \cos\left(\frac{\pi(n + \frac{1}{2})}{|I|}(\xi - \alpha_I)\right), \quad n \in \mathbb{N}_0, \quad I \in \mathcal{I}, \tag{2.6}$$

constitute an orthonormal basis for L_2, see e.g. [1]. We call the collection $\{w_{n,I}\}_{I \in \mathcal{I}, n \in \mathbb{N}_0}$ a *brushlet system.* The brushlets also have an explicit representation in the time domain. Define the set of *central bell functions* $\{g_I\}_{I \in \mathcal{I}}$ by

$$\hat{g}_I(\xi) := \rho\left(\frac{|I|}{\varepsilon_I}\xi\right)\rho\left(\frac{|I|}{\varepsilon'_I}(1 - \xi)\right), \tag{2.7}$$

such that $b_I(\xi) = \hat{g}_I\big(|I|^{-1}(\xi - \alpha_I)\big)$, and let for notational convenience

$$e_{n,I} := \frac{\pi\big(n + \frac{1}{2}\big)}{|I|}, \qquad I \in \mathcal{I},\ n \in \mathbb{N}_0.$$

Then,

$$w_{n,I}(x) = \sqrt{\frac{|I|}{2}}\, e^{i\alpha_I x}\big\{g_I\big(|I|(x + e_{n,I})\big) + g_I\big(|I|(x - e_{n,I})\big)\big\}. \tag{2.8}$$

By a straight forward calculation it can be verified (see [2]) that there exists a constant $C < \infty$ independent of $I \in \mathcal{I}$, such that

$$|g_I(x)| \leq C(1+|x|)^{-r}, \tag{2.9}$$

with $r \geq 1$ given by the smoothness of the ramp function. Thus a brushlet $w_{n,I}$ essentially consists of two humps at $\pm e_{n,I}$.

2.1 Modulation spaces. We have now introduced the brushlet systems. In this section we will give the definition of α-modulation spaces and give the characterization of these space by brushlet coefficients. The α-modulation spaces, introduced by Gröbner in [1], is a family of spaces that contain the classical modulation and Besov spaces as special cases. The spaces are defined by a parameter $\alpha \in [0,1]$. This parameter determines a segmentation of the frequency axis from which the spaces are built. Let us be more specific. First, we define an α-covering of $\mathbb{R}$.

Definition 2.3. A family $\mathcal{I}$ of intervals $I \in \mathbb{R}$ is called an *admissible covering* of $\mathbb{R}$ if $\cup_{I\in\mathcal{I}} I = \mathbb{R}$ and $\#\{I \in \mathcal{I}\colon x \in I\} \leq 2$ for all $x \in \mathbb{R}$. Furthermore, if there exists a constant $0 \leq \alpha \leq 1$, such that $|I| \asymp (1+|\xi|)^\alpha$ for all $I \in \mathcal{I}$, and all $\xi \in I$, then $\mathcal{I}$ is called an *α-covering* of $\mathbb{R}$.

Remark 2.4. Notice that for a disjoint covering $\mathcal{I}$, both this covering and the set $\{\mathrm{supp}(b_I)\}_{I\in\mathcal{I}}$ are admissible coverings. Moreover, if $\mathcal{I}$ is an α-covering too (called a disjoint α-covering), it is automatically a moderate covering.

Next, we define a partition of unity based on an admissible (α-)covering.

Definition 2.5. Given an admissible covering $\mathcal{I}$ of $\mathbb{R}$, a family $\Psi = \{\psi_I\}_{I\in\mathcal{I}}$ of nontrivial functions is called a *bounded admissible partition of unity* subordinate to $\mathcal{I}$, if the following conditions are satisfied: $\sup_{I\in\mathcal{I}} \|\psi_I\|_{\mathcal{F}L_1} < \infty$, $\mathrm{supp}(\psi_I) \subset I$ for all $I \in \mathcal{I}$, and $\sum_{I\in\mathcal{I}} \psi_I(\xi) = 1$ for all $\xi \in \mathbb{R}$.

We can now give the definition of an α-modulation space. The construction is based on a partition of unity subordinate to an α-covering of the frequency axis.

Definition 2.6. Given $1 \leq p, q \leq \infty$, $s \in \mathbb{R}$, and $0 \leq \alpha \leq 1$, let $\mathcal{I}$ be an α-covering of $\mathbb{R}$ and let Ψ be a corresponding partition of unity. Then we define the *α-modulation space* $M_q^{s,\alpha}(L_p)$ as the set of distributions $f \in S'(\mathbb{R})$ satisfying

$$\|f\|_{M_q^{s,\alpha}(L_p)} := \left(\sum_{I\in\mathcal{I}} (1+|\xi_I|)^{qs} \left\|\mathcal{F}^{-1}(\psi_I \mathcal{F} f)\right\|_{L_p}^q\right)^{1/q} < \infty,$$

with $\{\xi_I\}_{I\in\mathcal{I}}$ a sequence satisfying $\xi_I \in I$. For $q=\infty$ we have the usual change of the sum to sup over $I \in \mathcal{I}$.

It is easy to see that two different choices of the set $\{\xi_I\}_{I\in\mathcal{I}}$ give equivalent norms since we are using α-coverings.

Remark 2.7. Notice that for $\alpha > 0$, we have

$$\|f\|_{M_q^{s,\alpha}(L_p)} \asymp \Big(\sum_{I\in\mathcal{I}} |I|^{qs/\alpha} \big\|\mathcal{F}^{-1}(\psi_I \mathcal{F}f)\big\|_{L_p}^q\Big)^{1/q} < \infty,$$

and thus, in particular, $M_q^{s,1}(L_p) = B_q^s(L_p)$ for $1 \le p,q \le \infty$, and $s \in \mathbb{R}$. The other "extreme" $M_q^{s,0}(L_p)$ is the classical modulation space $M_q^s(L_p)$, so in this sense the α-modulation spaces are intermediate between the Besov spaces and the modulation spaces.

It was noticed in [3] that α-modulation spaces can be seen as a retract of the weighted sequence spaces $\ell_q\big(\mathcal{I}, |I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}}, \ell_p(\mathbb{N}_0)\big)$. More precisely, we have

Theorem 2.8. *Let $\mathcal{B} = \{w_{n,I}\}_{I\in\mathcal{I},n\in\mathbb{N}_0}$ be a brushlet system associated with an α-covering $\mathcal{I}$ for some $0 < \alpha \le 1$. Then $\mathcal{B}$ constitutes an unconditional basis for the α-modulation spaces $M_q^{s,\alpha}(L_p)$, $1 < p,q < \infty$, $s \in \mathbb{R}$, and we have the characterization*

$$\|f\|_{M_q^{s,\alpha}(L_p)} \asymp \Big(\sum_{I\in\mathcal{I}}\Big(\sum_{n\in\mathbb{N}_0} \big(|I|^{\frac{s}{\alpha}+\frac{1}{2}-\frac{1}{p}}|\langle f, w_{n,I}\rangle|\big)^p\Big)^{q/p}\Big)^{1/q}.$$

For more information on α-modulation spaces and the more general decomposition spaces we refer the reader to [15, 7, 6] and [3]. Finally, we should mention that Fornasier has studied Gabor frames in α-modulation spaces in his Ph.D. thesis [8].

3. Pseudodifferential operators and brushlet systems

We want to investigate brushlet expansions of certain pseudodifferential operators. Here, by a *pseudodifferential operator* we mean an operator T initially defined on $\mathcal{S}(\mathbb{R})$ by

$$Tf(x) := \int_{\mathbb{R}} \sigma(x,\xi)\hat{f}(\xi)e^{ix\xi}\,d\xi, \tag{3.1}$$

where σ is the corresponding symbol satisfying some regularity conditions. A symbol σ is said to belong to the *Hörmander class* $\mathrm{S}^s_{\rho,\delta}$ for some fixed

$s \in \mathbb{R}$, $\rho \in (0,1]$ and $\delta \in [0,1)$, if $\sigma \in C^\infty(\mathbb{R} \times \mathbb{R})$ and

$$|\partial_x^m \partial_\xi^n \sigma(x,\xi)| \le C_{n,m}(1+|\xi|)^{s-\rho n+\delta m} \qquad \text{for all } n,m \in \mathbb{N}_0. \tag{3.2}$$

We will denote the corresponding class of pseudodifferential operators by $\mathrm{OPS}^s_{\rho,\delta}$. It is well known that any $T \in \mathrm{OPS}^s_{\rho,\delta}$ satisfies $T\colon \mathcal{S} \to \mathcal{S}$. Moreover, if $T \in \mathrm{OPS}^s_{\rho,\delta}$ for some $\delta \le \rho$, then the operator T^* defined by $\langle Tu, v\rangle = \langle u, T^*v\rangle$ for all $u, v \in \mathcal{S}(\mathbb{R})$, belongs to $\mathrm{OPS}^s_{\rho,\delta}$ as well, see e.g. Theorem 18.1.7 with additional comment on page 94 in [18]. We refer the reader to [16], [18] or [24] for more information on pseudodifferential operators.

Remark 3.1. The map $\sigma \mapsto T$ given by (3.1) is usually referred to as the Kohn-Nirenberg correspondence and σ is called the Kohn-Nirenberg symbol. Given $\kappa \in S'$, consider the operator T_κ given by

$$T_\kappa f = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{\kappa}(\xi, u) e^{i\xi(x+u/2)} f(x+u)\, du d\xi.$$

Then T_κ is also a pseudodifferential operator. κ is called the Weyl symbol and the map $\kappa \mapsto T_\kappa$ is called the Weyl correspondence. In [16] it was noticed that if $\kappa \in M^0_1(L_\infty(\mathbb{R}^2))$ then T_κ is a bounded operator on $M^0_q(L_p)$ for $p, q \in [1,\infty]$. Similar results has been obtained in [17], [19] and [5]. In [23] sufficient conditions on the Weyl symbol was given to ensure boundedness of T_κ on certain weighted Sobolev spaces.

In this section we will see that a pseudodifferential operator $T \in \mathrm{OPS}^s_{\rho,\delta}$ of the Hörmander-type, with $0 \le \delta \le \rho = \alpha$ and $s \in \mathbb{R}$, has a sparse representation in a brushlet system corresponding to an α-covering $\mathcal{I}$. Given a brushlet system $\{w_{n,I}\}_{I\in\mathcal{I}, n\in\mathbb{N}_0}$, we let

$$\tau(n, I, n', I') := \langle T w_{n,I}, w_{n',I'}\rangle.$$

Since both T and its adjoint are pseudodifferential operators with symbols satisfying the same bounds we can, without loss of generality, assume that $|I'| \ge |I|$ in any bound on $\tau(n, I, n', I')$ based on the inequality (3.2).

Proposition 3.2. *Given $\alpha \in (0,1]$, let $\{w_{n,I}\}_{I\in\mathcal{I}, n\in\mathbb{N}_0}$ be a brushlet system associated with an α-covering $\mathcal{I}$, such that g_I satisfies (2.9) for some $r > 2$. Given $T \in \mathrm{OPS}^{s_0}_{\alpha,\delta}$, $s_0 \in \mathbb{R}$, $0 \le \delta \le \alpha$, there exists a constant C*

independent of $n, n' \in \mathbb{N}_0$ *and* $I, I' \in \mathcal{I}$ *such that for* $|I'| \geq |I|$,

(3.3)
$$|\tau(n, I, n', I')| \leq C|I|^{s_0/\alpha}\left(\frac{|I'|}{|I|}\right)^{-(r-2)\theta-1/2}\left(1+|I|^2|k_{n,I}-k_{n',I'}|^2\right)^{-N(1-\theta)}$$

for any $\theta \in [0,1]$ *and integer* $N < (r-1)/2$.

Proof. Clearly, it suffices to show (3.3) for $\theta = 0$ and $\theta = 1$. Fix $n, n' \in \mathbb{N}_0$ and $I, I' \in \mathcal{I}$ with $|I'| \geq |I|$. Notice that $\langle T w_{n,I}, w_{n',I'}\rangle$ is given by the sum of four terms of the form

(3.4)
$$\sqrt{\tfrac{|I'|}{|I|}}\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(x,\xi)\overline{g_{I'}(|I'|(x \pm k_{n',I'}))}\hat{g}_I(|I|^{-1}(\xi-\alpha_I)) \times e^{-i\alpha_{I'}x}e^{\pm ik_{n,I}(\xi-\alpha_I)}e^{ix\xi}d\xi\, dx.$$

We will consider only one of the terms in (3.4), since the following estimates holds almost verbatim for the other three. By a change of coordinates (3.4) (with '+'-signs) equals

$$\sqrt{\tfrac{|I|}{|I'|}}\int_{\mathbb{R}}\int_{\mathbb{R}}\sigma(|I'|^{-1}x-k_{n',I'},|I|\xi+\alpha_I)\hat{g}_I(\xi)f(x,\xi)e^{i|I||I'|^{-1}x\xi}d\xi\, dx, \tag{3.5}$$

where

$$f(x,\xi) := \overline{g_{I'}(x)}e^{i(k_{n',I'}-|I'|^{-1}x)(\alpha_{I'}-\alpha_I)}e^{i|I|(k_{n,I}-k_{n',I'})\xi}.$$

Let $L_\xi := 1-\partial_\xi^2$ and notice that

$$f(x,\xi) = \left[1+|I|^2(k_{n,I}-k_{n',I'})^2\right]^{-N}L_\xi^N f(x,\xi).$$

Thus, by partial integration (3.5) equals

$$\sqrt{\tfrac{|I|}{|I'|}}\left[1+|I|^2(k_{n,I}-k_{n',I'})^2\right]^{-N} \times \int_{\mathbb{R}}\int_{\mathbb{R}}\left\{L_\xi^N\left[\sigma(|I'|^{-1}x-k_{n',I'},|I|\xi+\alpha_I)\hat{g}_I(\xi)e^{i|I||I'|^{-1}x\xi}\right]\right\}f(x,\xi)d\xi\, dx.$$

Since $|I|\xi+\alpha_I \in \operatorname{supp}(b_I)$ whenever $\xi \in \operatorname{supp}(\hat{g}_I)$, $\left(1+\left||I|\xi+\alpha_I\right|\right) \asymp |I|^{1/\alpha}$ for any $\xi \in \operatorname{supp}(\hat{g}_I)$. This fact and the bound (3.2) (with $\rho = \alpha$) gives

$$\left|\partial_\xi^l\left[\sigma(x',|I|\xi+\alpha_I)\right]\right| \leq C_l|I|^l\left(1+\left||I|\xi+\alpha_I\right|\right)^{s_0-l\alpha} \leq C_l'|I|^{s_0/\alpha}$$

for all $x' \in \mathbb{R}$ and $\xi \in \operatorname{supp}(\hat{g}_I)$. Observe that,

$$\begin{aligned}
&\big|\partial_\xi^m\big[\sigma(|I'|^{-1}x - k_{n',I'}, |I|\xi + \alpha_I)\hat{g}_I(\xi)e^{i|I||I'|^{-1}x\xi}\big]\big| \\
&\quad \le \sum_{l=0}^{m}\binom{m}{l}\big|\partial_\xi^l\big[\sigma(|I'|^{-1}x - k_{n',I'}, |I|\xi + \alpha_I)e^{i|I||I'|^{-1}x\xi}\big]\big||\hat{g}_I^{(m-l)}(\xi)| \\
&\quad \le C\chi_{\operatorname{supp}(\hat{g}_I)}(\xi)\sum_{l=0}^{m}\binom{m}{l}\sum_{l'=0}^{l}\binom{l}{l'}C_{l-l'}|I|^{s_0/\alpha}\left(\frac{|I|}{|I'|}\right)^{l'}|x|^{l'} \\
&\quad \le C'|I|^{s_0/\alpha}(1+|x|)^m\chi_{\operatorname{supp}(\hat{g}_I)}(\xi)\left(2+\frac{|I|}{|I'|}\right)^m \\
&\quad \le C''|I|^{s_0/\alpha}(1+|x|)^m\chi_{\operatorname{supp}(\hat{g}_I)}(\xi),
\end{aligned}$$

since $|I'| \ge |I|$. With these estimates on the derivatives we obtain

$$\begin{aligned}
&\big|L_\xi^N\sigma(|I'|^{-1}x - k_{n',I'}, |I|\xi + \alpha_I)\hat{g}_I(\xi)e^{i|I||I'|^{-1}x\xi}\big| \\
&\qquad\qquad \le C''|I|^{s_0/\alpha}(1+|x|)^{2N}\chi_{\operatorname{supp}(\hat{g}_I)}(\xi).
\end{aligned}$$

Now, since the same type of arguments hold for the other three terms in (3.4) we have the following bound for $N < (r-1)/2$,

$$\begin{aligned}
|\langle Tw_{n,I}, w_{n',I'}\rangle| &\le 4C''\sqrt{\frac{|I|}{|I'|}}|I|^{s_0/\alpha}\big[1+|I|^2|k_{n,I}-k_{n',I'}|^2\big]^{-N} \\
&\qquad \times \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|x|)^{2N}\chi_{\operatorname{supp}(\hat{g}_I)}(\xi)|g_{I'}(x)|d\xi\,dx. \\
&\le C\sqrt{\frac{|I|}{|I'|}}|I|^{s_0/\alpha}\big(1+|I|^2|k_{n,I}-k_{n',I'}|^2\big)^{-N}. \qquad (3.6)
\end{aligned}$$

This is the bound (3.3) for $\theta = 0$.

It only remains to prove (3.3) for $\theta = 1$. Define the two functions $w_{n,I}^+$ and $w_{n,I}^-$ by

$$w_{n,I}^{\pm}(x) = \sqrt{\tfrac{|I|}{2}}e^{i\alpha_I x}g_I(|I|(x \mp k_{n,I})),$$

such that $w_{n,I} = w_{n,I}^+ + w_{n,I}^-$ for any $n \in \mathbb{N}_0$ and $I \in \mathcal{I}$. Notice that for a fixed $x_0 \in \mathbb{R}$ the functions

$$\partial_\xi^k\big[\partial_x^l\sigma(x_0,\xi)\hat{w}_{n,I}(\xi)\big], \qquad 0 \le k, l \le r,$$

are supported inside $\operatorname{supp}(\hat{w}_{n,I})$, yielding

$$\int_{\mathbb{R}} x^k[\partial_x^l\sigma(x_0,\cdot)\hat{w}_{n,I}]^\vee(x)\overline{w_{n',I'}^{\pm}(x)}\,dx = 0, \qquad 0 \le k, l \le r$$

for all $I, I' \in \mathcal{I}$ such that b_I and $b_{I'}$ have disjoint support. Thus, by a Taylor expansion of $\sigma(\cdot, \xi)$ and by (3.2) we obtain

$$\begin{aligned}&|\langle T w_{n,I}, w^{\pm}_{n',I'}\rangle| \\ &\quad \le C_r \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|\xi|)^{s_0+\delta(r-2)} |x \mp k_{n',I'}|^{r-2} |\hat{w}_{n,I}(\xi)| |w^{\pm}_{n',I'}(x)| \, d\xi \, dx,\end{aligned}$$

Now, on one hand

$$\begin{aligned}&\int_{\mathbb{R}} (1+|\xi|)^{s_0+\delta(r-2)} |\hat{w}_{n,I}(\xi)| \, d\xi \\ &\qquad \le C|I|^{-1/2} \int_{\mathbb{R}} (1+|\xi|)^{s_0+\delta(r-2)} |\hat{g}_I(|I|^{-1}(\xi-\alpha_I))| \, d\xi \\ &\qquad = C|I|^{1/2} \int_{\mathbb{R}} (1+||I|\xi+\alpha_I|)^{s_0+\delta(r-2)} |\hat{g}_I(\xi)| \, d\xi \le C'|I|^{s_0/\alpha+r-3/2},\end{aligned}$$

since $\delta \le \alpha$, and on the other hand

$$\begin{aligned}&\int_{\mathbb{R}} |x \mp k_{n',I'}|^{r-2} |w^{\pm}_{n',I'}(x)| \, dx \\ &\qquad \le C|I'|^{1/2} \int_{\mathbb{R}} |x \mp k_{n',I'}|^{r-2} |g_{I'}(|I'|(x \mp k_{n',I'}))| \, dx \\ &\qquad \le C'|I'|^{-1/2} \int_{\mathbb{R}} |I'|^{-r+2} |x|^{r-2} |g_{I'}(x)| \, dx \le C''|I'|^{3/2-r}.\end{aligned}$$

Hence, for any $I, I' \in \mathcal{I}$ such that b_I and $b_{I'}$ have disjoint support, we have

$$|\langle T w_{n,I}, w_{n',I'}\rangle| \le C|I|^{s_0/\alpha} \left(\tfrac{|I'|}{|I|}\right)^{-r+3/2}. \tag{3.7}$$

Recall that whenever $\operatorname{supp}(b_I) \cap \operatorname{supp}(b_{I'}) \neq \emptyset$, then $|I| \asymp |I'|$. Thus, combining (3.6) and (3.7) the result follows. $\square$

Remark 3.3. Proposition 3.2 generalizes the corresponding sparsity results for wavelet expansions of pseudodifferential operators $T \in \mathrm{OPS}^{s_0}_{1,\delta}$, $s_0 > \mathbb{R}$, $0 \le \delta < 1$, see e.g. [24].

4. Pseudodifferential operators and α-modulation spaces

It is well known that a pseudodifferential operator $T \in \mathrm{OPS}^{s_0}_{1,\delta}$, $s_0 \in \mathbb{R}$, $0 \le \delta < 1$ extends to a bounded operator from $F^{s+s_0}_q(L_p(\mathbb{R}^d))$ to $F^s_q(L_p(\mathbb{R}^d))$ and from $B^{s+s_0}_q(L_p(\mathbb{R}^d))$ to $B^s_q(L_p(\mathbb{R}^d))$ for $1 < p, q < \infty$ and $s > 0$, see e.g. [27, Chapter 5] and [28]. Furthermore, for $\delta \le \rho$, $0 < \rho \le 1$, and $s = (\rho - 1)/2$, $T \in \mathrm{OPS}^s_{\rho,\delta}$ is of weak type $(1,1)$, $T\colon H_1 \to L_1$ and $T\colon L_\infty \to \mathrm{BMO}$, see [24]. Also, if $|1/p - 1/2| \le s/(\rho - 1)$, $1 < p < \infty$, then $T\colon L_p \to L_p$.

However, these last examples are very restricted concerning the parameter s and the proofs are quite involved. Using brushlet expansions we can construct a rather simple proof for the boundedness of pseudodifferential operators, with a general parameter s, applied on α-modulation spaces. But first we state two technical lemmas that will be of use in obtaining the boundedness result.

Lemma 4.1. *Given $0 < \alpha < 1$, let $\mathcal{I}$ and $\mathcal{I}'$ be two α-coverings. For each $I \in \mathcal{I}$ let*

$$A_I = \{I' \in \mathcal{I}' \colon I' \cap I \neq \emptyset\}.$$

Then there exists a constant d_A such that $\#A_I \le d_A$ independent of I.

See [3] for a proof.

Lemma 4.2. *Given $0 < \alpha \le 1$, let $\mathcal{I}$ be a disjoint α-covering. Fix an $I \in \mathcal{I}$ and let $\gamma = (1-\alpha)/\alpha$. Then,*

$$\sum_{I' \in \mathcal{I}\colon |I'| \le |I|} |I'|^q \le C|I|^{q+\gamma}$$

for any $q > -\gamma$, and

$$\sum_{I' \in \mathcal{I}\colon |I'| \ge |I|} |I'|^{-q} \le C|I|^{-q+\gamma}$$

for any $q > \gamma$.

Proof. For $\alpha = 1$ the result is well known, see e.g. [2]. Suppose $0 < \alpha < 1$. For $k \in \mathbb{N}$ let $a_k = k^{1/(1-\alpha)}$ and $\Delta_k = |a_{k+1} - a_k|$. Then it is straight forward to show that $\Delta_k \asymp a_k^\alpha$, i.e., $\{[-a_{k+1}, -a_k), [-1,1), [a_k, a_{k+1})\}_{k\in\mathbb{N}}$ is an α-covering. Moreover, for any $N \in \mathbb{N}$

$$\sum_{k=1}^N \Delta_k^q \le C \sum_{k=1}^N k^{q/\gamma} \le C' N^{q/\gamma+1},$$

with C' is independent of N. Hence,

$$\sum_{k=1}^{N} \Delta_k^q \leq C'' \Delta_N^{(q/\gamma+1)\gamma} = \Delta_N^{q+\gamma}.$$

For a given $k \in \mathbb{N}$ let $A_k = \{I \in \mathcal{I} : I \cap [a_k, a_{k+1}) \neq \emptyset\}$. By Lemma 4.1 $\#A_k \leq d_A < \infty$ independent of $k \in \mathbb{N}$. Let $N \in \mathbb{N}$ be the smallest positive integer such that $I \subset [-a_{N+1}, a_{N+1})$. Then,

$$\begin{aligned}\sum_{I' : |I'| \leq |I|} |I'|^q &\leq \sum_{k=0}^{N} \sum_{I' \in A_k} |I'|^q \leq C \sum_{k=0}^{N} \sum_{I' \in A_k} \Delta_k^q \\ &\leq C' \sum_{k=0}^{N} \Delta_k^q \leq C'' \Delta_N^{q+\gamma} \leq C''' |I|^{q+\gamma},\end{aligned}$$

using that $|I'| \asymp \Delta_k$ for $I' \in A_k$. This proves the first inequality in the lemma. The second estimate can be proven by similar arguments, which we leave to the reader. □

We can now state and prove our main result, which concludes the paper.

Theorem 4.3. *Let $0 < \alpha \leq 1$, $s_0 \in \mathbb{R}$, $0 \leq \delta \leq \alpha$, and $\delta < 1$. A pseudodifferential operator $T \in \mathrm{OPS}_{\alpha,\delta}^{s_0}$ extends to a bounded operator from $M_q^{s,\alpha}(L_p)$ to $M_q^{s-s_0-\beta,\alpha}(L_p)$ for any $s \in \mathbb{R}$, $1 < p, q < \infty$ and $\beta := 1 - \alpha \in [0, 1)$.*

Remark 4.4. Notice that Theorem 4.3 includes the boundedness result for Ψdo's on Besov spaces mentioned above.

Proof. Let $\{w_{n,I}\}_{I \in \mathcal{I}, n \in \mathbb{N}_0}$ be a brushlet system corresponding to an α-covering $\mathcal{I}$. Given $f \in M_q^s(L_p)$ with brushlet expansion $f = \sum_{n,I} c_{n,I} w_{n,I}$. Then according to Theorem 2.8 there exists a constant $0 < C < \infty$ such that

$$\Big[\sum_{I \in \mathcal{I}} \Big(\sum_{n \in \mathbb{N}_0} |x_{I,n}|^p \Big)^{q/p} \Big]^{1/q} \leq C \|f\|_{B_q^s(L_p)}, \tag{4.1}$$

where $x_{I,n} = |I|^{s/\alpha+1/2-1/p} c_{n,I}$. Let formally

$$Tf = \sum_{n \in \mathbb{N}_0, I \in \mathcal{I}} d_{n,I} w_{n,I}.$$

Assume the parameter r corresponding to the brushlet system has been chosen so large that we can fix a constant $\theta > 0$ such that

$$\frac{\max(s', -s)}{\alpha(r-2)} < \theta < 1 - \frac{1}{N}, \qquad s' = s - s_0,$$

where N is an integer satisfying $0 < 2N < r - 1$. Then Proposition 3.2 implies that

$$\begin{aligned}
|d_{n,I}| \le C' \Big(&\sum_{|I'| \ge |I|} \sum_{n'} \Big(\tfrac{|I|}{|I'|}\Big)^{(r-2)\theta + 1/2} |I|^{s_0/\alpha} \\
&\qquad \times \big(1 + |I|^2 |k_{n,I} - k_{n',I'}|^2\big)^{-(1-\theta)N} c_{n',I'} \\
&+ \sum_{|I'| < |I|} \sum_{n'} \Big(\tfrac{|I'|}{|I|}\Big)^{(r-2)\theta + 1/2} |I'|^{s_0/\alpha} \\
&\qquad \times \big(1 + |I'|^2 |k_{n,I} - k_{n',I'}|^2\big)^{-(1-\theta)N} c_{n',I'} \Big) \\
= |I|^{-s'/\alpha - 1/2 + 1/p} \Big(&\sum_{|I'| \ge |I|} A_{I,I'} x_{I'} + \sum_{|I'| < |I|} \tilde{A}_{I,I'} x_{I'} \Big),
\end{aligned} \tag{4.2}$$

where $A_{I,I'} = (a_{n,n'})_{n,n' \in \mathbb{N}_0}$,

$$a_{n,n'} = \left(\frac{|I|}{|I'|}\right)^{(r-2)\theta + 1 - 1/p + s/\alpha} \left(1 + \pi^2 \left|(n + 1/2) - \tfrac{|I|}{|I'|}(n' + 1/2)\right|^2\right)^{-(1-\theta)N},$$

and $\tilde{A}_{I,I'} = (\tilde{a}_{n,n'})_{n,n' \in \mathbb{N}_0}$,

$$\tilde{a}_{n,n'} = \left(\frac{|I'|}{|I|}\right)^{(r-2)\theta + 1/p - s'/\alpha} \left(1 + \pi^2 \left|\tfrac{|I'|}{|I|}(n + 1/2) - (n' + 1/2)\right|^2\right)^{-(1-\theta)N}.$$

Notice that $\{x_{I',n'}\}_{n' \in \mathbb{N}_0}$ is an $\ell_p(\mathbb{N}_0)$ sequence for each $I' \in \mathcal{I}$ according to (4.1). To estimate the terms in (4.2) we will use the following observation based on Schur's lemma. Let $A = (a_{n,m})_{n,m \in \mathbb{N}_0}$ be a double infinite matrix given by

$$a_{n,m} = (1 + |b_1 n - b_2 m|)^{-N}$$

for some $b_1, b_2 > 0$ and $N > 1$. Then A defines a bounded operator on $\ell_p(\mathbb{N}_0)$, $1 \le p \le \infty$, with norm bounded by $C b_1^{-1/p} b_2^{-1+1/p}$ for some constant C depending only on N.

$u_{n,I}$ Let $Y_{I'} = \|\{x_{I',n'}\}_{n'\in\mathbb{N}_0}\|_{\ell_p}$. Since $(1-\theta)N > 1$, Minkowski's inequality yields

$$\Big(\sum_{n\in\mathbb{N}_0}\Big|\sum_{|I'|\geq|I|} A_{I,I'}x_{I'}\Big|^p\Big)^{1/p} \leq \sum_{|I'|\geq|I|}\Big(\sum_{n\in\mathbb{N}_0}|A_{I,I'}x_{I'}|^p\Big)^{1/p}$$
$$\leq C\sum_{|I'|\geq|I|}\Big(\frac{|I|}{|I'|}\Big)^{(r-2)\theta+s/\alpha}Y_{I'},$$

and

$$\Big(\sum_{n\in\mathbb{N}_0}\Big|\sum_{|I'|<|I|} \tilde{A}_{I,I'}x_{I'}\Big|^p\Big)^{1/p} \leq \sum_{|I'|<|I|}\Big(\sum_{n\in\mathbb{N}_0}|\tilde{A}_{I,I'}x_{I'}|^p\Big)^{1/p}$$
$$\leq C\sum_{|I'|<|I|}\Big(\frac{|I'|}{|I|}\Big)^{(r-2)\theta-s'/\alpha}Y_{I'}.$$

Let $\gamma = (1-\alpha)/\alpha$, $a = (r-2)\theta + s/\alpha$ and $b = (r-2)\theta - s'/\alpha$. Using (4.2) and the two estimates above, we get

(4.3)
$$\Big(\sum_{I\in\mathcal{I}}\Big(\sum_{n\in\mathbb{N}_0}|I|^{\frac{s'}{\alpha}-\gamma+\frac{1}{2}-\frac{1}{p}}|d_{n,I}|^p\Big)^{q/p}\Big)^{1/q}$$
$$\leq \Big(\sum_{I\in\mathcal{I}}|I|^{-q\gamma}\Big(\sum_{|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a Y_{I'} + \sum_{|I'|<|I|}\Big(\tfrac{|I'|}{|I|}\Big)^b Y_{I'}\Big)^q\Big)^{1/q}.$$

Since $a > 0$, an application of Hölder's inequality and Lemma 4.2 yields

$$\Big(\sum_{|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a Y_{I'}\Big)^q \leq \Big(\sum_{I':\,|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a\Big)^{q-1}\Big(\sum_{|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a Y_{I'}^q\Big)$$
$$\leq C|I|^{(q-1)\gamma}\Big(\sum_{|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a Y_{I'}^q\Big),$$

and a second application of Lemma 4.2 gives

$$\sum_{I\in\mathcal{I}}|I|^{-\gamma}\sum_{|I'|\geq|I|}\Big(\tfrac{|I|}{|I'|}\Big)^a Y_{I'}^q = \sum_{I'\in\mathcal{I}}\Big(\sum_{I:\,|I'|\geq|I|}|I|^{-\gamma}\Big(\tfrac{|I|}{|I'|}\Big)^a\Big)Y_{I'}^q$$
$$\leq C\sum_{I'\in\mathcal{I}}Y_{I'}^q.$$

Using similar estimates we get

$$\sum_{I\in\mathcal{I}} |I|^{-q\gamma}\Big(\sum_{|I'|<|I|}\Big(\tfrac{|I'|}{|I|}\Big)^b Y_{I'}\Big)^q \le C'\sum_{I'\in\mathcal{I}} Y_{I'}^q.$$

Thus, we have the following bound according to (4.3)

$$\begin{aligned}\Big(\sum_{I\in\mathcal{I}}\Big(\sum_{n\in\mathbb{N}_0}|I|^{\frac{s'}{\alpha}-\gamma+\frac{1}{2}-\frac{1}{p}}|d_{n,I}|^p\Big)^{q/p}\Big)^{1/q} &\le C\Big(\sum_{I'\in\mathcal{I}} Y_{I'}^q\Big)^{1/q}\\ &= C\Big(\sum_{I'\in\mathcal{I}}\Big(\sum_{n'\in\mathbb{N}_0}|x_{n',I'}|^p\Big)^{q/p}\Big)^{1/q},\end{aligned}$$

and the result now follows from Theorem 2.8. □

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