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New frames of Besov and Triebel-Lizorkin spaces[1]

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Abstract. Let $s < 0$. The author obtains some new frames for Besov spaces $B^s_{pq}(\mathbb{R}^n)$ with $1 \le p,\ q \le \infty$ and Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$ with $1 < p < \infty$ and $1 < q \le \infty$ by a dual method via the subatomic characterizations of these spaces when $s > 0$.

1. Introduction

Subatomic (or quarkonial) characterizations for spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ are proved to be a very useful tool in many applications. For example, in [7, 8], Triebel obtained the estimates of entropy numbers for the compact embedding between Besov and Triebel-Lizorkin spaces on bounded domains in $\mathbb{R}^n$ by first establishing the subatomic characterizations of these spaces. These characterizations are essential different from the atomic characterizations in that the subatoms are independent of any given f, but the atoms do. Thus, the atomic decomposition characterizations are not enough in some applications. In fact, the set of subatoms can be regarded

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as a kind of frame of these function spaces, which behaves like a base in some sense. See also [3] for some new bases of these spaces and their applications in nonlinear approximation.

Let $s < 0$. The main purpose of this paper is to show how to obtain new frames of Besov spaces $B^s_{pq}(\mathbb{R}^n)$ with $1 \le p,\ q \le \infty$ and Triebel-Lizorkin spaces $F^s_{pq}(\mathbb{R}^n)$ with $1 < p < \infty$ and $1 < q \le \infty$ by a dual method via the quarkonial (subatomic) characterizations of these spaces when $s > 0$ in [8]. Such a frame characteristic for Besov spaces $B^s_{pp}(\mathbb{R}^n)$ with $1 < p \le \infty$ is already obtained by Triebel in [9] and is used to characterize the regularity of the distribution considered. Moreover, using this frame characterization, one can give a new proof for the interpolation theorem between these function spaces and for the estimates of entropy numbers of the compact embedding between corresponding function spaces restricted to a bounded domain in $\mathbb{R}^n$. We will not give any details on this since they can be proved by the standard procedures in [5, 7]; see also [2, 10] for some details.

Let us now recall the definitions of Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with

$$\varphi(x) = \begin{cases} 1, & |x| \le 1; \\ 0, & |x| \ge 3/2\,. \end{cases} \tag{1.1}$$

We put $\varphi_0(x) = \varphi(x)$, $\varphi_1(x) = \varphi(x/2) - \varphi(x)$ and

$$\varphi_k(x) = \varphi_1(2^{-k+1}x), \tag{1.2}$$

where $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. Then,

$$\sum_{k=0}^{\infty} \varphi_k(x) = 1$$

for all $x \in \mathbb{R}^n$. That means that $\{\varphi_k\}_{k=0}^{\infty}$ forms a dyadic resolution of unity in $\mathbb{R}^n$. Recall that $(\varphi_k \hat{f})^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in \mathcal{S}'(\mathbb{R}^n)$ by the Paley-Wiener-Schwartz theorem; see [6]. In particular, $(\varphi_k \hat{f})^\vee(x)$ makes sense pointwise.

Definition 1.1. Let $s \in \mathbb{R}$ and $0 < q \le \infty$. Let $\{\varphi_j\}_{j=0}^{\infty}$ be as above.

(i) Let $0 < p \le \infty$. Then the space $B^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{pq}(\mathbb{R}^n)} = \left\{ \sum_{k=0}^{\infty} 2^{ksq} \left\| \left(\varphi_k \hat{f}\right)^\vee \right\|^q_{L^p(\mathbb{R}^n)} \right\}^{1/q}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty$. Then the space $F^s_{pq}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^s_{pq}(\mathbb{R}^n)} = \left\| \left\{ \sum_{k=0}^{\infty} 2^{ksq} \left| \left(\varphi_k \hat{f}\right)^{\vee} \right|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

(with the usual modification if $q = \infty$) is finite.

We remark that both spaces are quasi-Banach spaces which are independent of the chosen function φ; see [6]. Moreover, if $p \geq 1$ and $q \geq 1$, then the space $B^s_{pq}(\mathbb{R}^n)$ and the space $F^s_{pq}(\mathbb{R}^n)$ are Banach spaces.

We need the dual spaces of spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. In the following, for $s \in \mathbb{R}$ and $1 \leq p,\ q \leq \infty$, we define the space $\mathring{B}^s_{pq}(\mathbb{R}^n)$ to be the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{pq}(\mathbb{R}^n)$ endowed with the same quasi-norm as $B^s_{pq}(\mathbb{R}^n)$, and the space $\mathring{F}^s_{pq}(\mathbb{R}^n)$ to be the completion of $\mathcal{S}(\mathbb{R}^n)$ in $F^s_{pq}(\mathbb{R}^n)$ endowed with the same quasi-norm as $F^s_{pq}(\mathbb{R}^n)$. Then we have the following dual theorem; see [6, pp. 178-180] and [4]. In the following, if $1 \leq r \leq \infty$, we define r' by $1/r + 1/r' = 1$, and if $0 < r < 1$, we define $r' = \infty$.

Lemma 1.1. (i) *Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $0 < q < \infty$. Then*

$$\left(B^s_{pq}(\mathbb{R}^n)\right)^* = B^{-s}_{p'q'}(\mathbb{R}^n)\,.$$

Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$ and $0 < q \leq \infty$. Then

$$\left(\mathring{B}^s_{pq}(\mathbb{R}^n)\right)^* = B^{-s}_{p'q'}(\mathbb{R}^n)\,.$$

(ii) *Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q < \infty$. Then*

$$\left(F^s_{pq}(\mathbb{R}^n)\right)^* = F^{-s}_{p'q'}(\mathbb{R}^n)\,.$$

Let $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then

$$\left(\mathring{F}^s_{pq}(\mathbb{R}^n)\right)^* = F^{-s}_{p'q'}(\mathbb{R}^n)\,.$$

Throughout this paper, C denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$.

2. Main results and their proofs

Let $Q_{\nu m}$ be the cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-\nu}m$, and with side length $2^{-\nu}$, where $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. We also denote by $\chi^{(p)}_{\nu m}$ the p-normalized characteristic function of the cube $Q_{\nu m}$, which means

$$\chi^{(p)}_{\nu m}(x) = \begin{cases} 2^{\nu n/p}, & x \in Q_{\nu m}\ ; \\ 0, & x \notin Q_{\nu m}\,, \end{cases}$$

where $\nu \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{Z}^n$ and $0 < p \leq \infty$. In what follows, we will denote $\mathbb{N} \cup \{0\}$ simply by $\mathbb{N}_0$.

The following definition of quarks (or subatoms) can be found in [8, p. 12]; see also [7].

Definition 2.1. Let ψ be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\text{supp } \psi \subset \{y \in \mathbb{R}^n : |y| < 2^r\} \tag{2.1}$$

for some $r \geq 0$, and

$$\sum_{m \in \mathbb{Z}^n} \psi(x - m) = 1$$

for all $x \in \mathbb{R}^n$. Let $s \in \mathbb{R}$, $0 < \beta \leq \infty$, $\beta \in \mathbb{N}_0^n$ and $\psi^\beta(x) = x^\beta \psi(x)$, where

$$x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$$

for $\beta = (\beta_1, \cdots, \beta_n)$. Then

$$(\beta qu)_{\nu m}(x) = 2^{-\nu(s-n/p)} \psi^\beta(2^\nu x - m), \ x \in \mathbb{R}^n,$$

is called to be an $(s,p) - \beta-$quark (or subatom) related to $Q_{\nu m}$. Here $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$.

Proposition 2.1. *Let $s < 0$, $1 < p < \infty$ and $1 < q \leq \infty$. There is a set of Schwartz functions, $\{\Psi_k^{\beta,l} : \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0, \ l \in \mathbb{Z}^n\}$, such that for any $f \in F_{pq}^s(\mathbb{R}^n)$,*

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \gamma_{k,l}^\beta \Psi_k^{\beta,l} \tag{2.2}$$

unconditionally in both the norm of $F_{pq}^s(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ when $1 < p, \ q < \infty$ and only in $\mathcal{S}'(\mathbb{R}^n)$ when $1 < p < \infty$ and $1 < q \leq \infty$, where, when $k \in \mathbb{N}$, for all $\alpha \in \mathbb{N}_0^n$,

$$\int_{\mathbb{R}^n} \Psi_k^{\beta,l}(x) x^\alpha \, dx = 0,$$

and $\{\gamma_{k,l}^\beta : \beta \in \mathbb{N}_0^n, \ k \in \mathbb{N}_0, \ l \in \mathbb{Z}^n\}$ is a sequence of numbers linearly depending on f and satisfying

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\| \left(\sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^\beta \chi_{kl}^{(p)}(\cdot) \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{F_{pq}^s(\mathbb{R}^n)} \tag{2.3}$$

with the same r as in (2.1), $\rho > r$ and a constant $C > 0$ independent of f.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (1.1) and (1.2). Let Q_k be the cube in $\mathbb{R}^n$ centered at the origin and with side-length $2\pi 2^k$. Let $\kappa \in \mathcal{S}(\mathbb{R}^n)$,

$$\kappa_k(x) = \kappa(2^{-k} x),$$

$\kappa_k(x) = 1$ if $x \in \operatorname{supp} \varphi_k$, and $\operatorname{supp} \kappa_k \subset Q_k$, where $k \in \mathbb{N}_0$. Let $f \in F^s_{pq}(\mathbb{R}^n)$ and $g \in F^{-s}_{p'q'}(\mathbb{R}^n)$. By the proofs of [8, Theorem 2.9 and Corollary 2.12], we know that

$$
\begin{aligned}
(2.4) \qquad g &= \sum_{k=0}^{\infty} (\varphi_k \hat{g})^{\vee}(x) \\
&= C \sum_{k=0}^{\infty} \sum_{\beta \in \mathbb{N}_0^n} \sum_{l \in \mathbb{Z}^n} \lambda^{\beta}_{k+\rho, 2^\rho l} (\beta q u)_{k+\rho, 2^\rho l}(x) 2^{-\rho(s+n/p')} + \cdots,
\end{aligned}
$$

where $\cdots$ stands for some similar terms which can be treated in the same way, $\rho > r$ is a fixed positive number, $(\beta q u)_{k+\rho, 2^\rho l}(x)$ is a $(-s, p') - \beta-$quarks according to Definition 2.1,

$$
(2.5) \qquad \lambda^{\beta}_{k+\rho, 2^\rho l} = 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} \left(g, \psi^{\beta,\rho}_{k+\rho, 2^\rho l} \right)
$$

and

$$
\psi^{\beta,\rho}_{k+\rho, 2^\rho l}(x) = \sum_{j \in \mathbb{Z}^n} \frac{D^\beta \kappa^{\vee}(l-j)}{\beta!} \varphi_k^{\vee}(2^{-k} j - x).
$$

Moreover,

(2.6)

$$
\left\| \left(\sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left| \lambda^{\beta}_{k+\rho, 2^\rho l} \chi^{(p')}_{k+\rho, 2^\rho l}(\cdot) \right|^{q'} \right)^{1/q'} \right\|_{L^{p'}(\mathbb{R}^n)} \le C_\rho 2^{-\rho|\beta|} \|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)},
$$

where $C_\rho > 0$ is a constant independent of g and β, and $\|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)}$ is the usual norm as in Definition 2.1. We also note that the series in (2.4) converge unconditionally in $L^{p'}(\mathbb{R}^n)$; see [8, p. 14]. In the following, we suppose $g \in \mathcal{S}(\mathbb{R}^n)$, then it is easy to show that the series in (2.4) in this case also converge in $\mathcal{S}(\mathbb{R}^n)$; see [8, pp. 23-24] again. Thus, we have

$$
\begin{aligned}
(f, g) &= C \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho(s+n/p')} \lambda^{\beta}_{k+\rho, 2^\rho l} \left(f, (\beta q u)_{k+\rho, 2^\rho l} \right) + \cdots \\
&= C \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho(s+n/p')} \left(f, (\beta q u)_{k+\rho, 2^\rho l} \right) \\
&\qquad \times 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} \left(g, \psi^{\beta,\rho}_{k+\rho, 2^\rho l} \right) + \cdots \\
&= \Bigg(C \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho(s+n/p')} 2^{-\rho|\beta|} 2^{-(k+\rho)(s+n/p')} \\
&\qquad \times \left(f, (\beta q u)_{k+\rho, 2^\rho l} \right) \psi^{\beta,\rho}_{k+\rho, 2^\rho l}, g \Bigg) + \cdots
\end{aligned}
$$

for all $g \in \mathcal{S}(\mathbb{R}^n)$. Therefore,

$$\begin{aligned} f \quad =& C \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} 2^{-\rho(s+n/p')} 2^{-\rho|\beta|} \\ & \times 2^{-(k+\rho)(s+n/p')} (f, (\beta q u)_{k+\rho, 2^\rho l})\, \psi^{\beta,\rho}_{k+\rho,2^\rho l} + \cdots \end{aligned}$$

in $\mathcal{S}'(\mathbb{R}^n)$.

Now let

$$\gamma^{\beta,\rho}_{k+\rho,2^\rho l} = C 2^{-\rho(s+n/p')} 2^{-\rho|\beta|} (f, (\beta q u)_{k+\rho,2^\rho l})$$

and

$$\Psi^{\beta,\rho}_{k+\rho,2^\rho l} = 2^{-(k+\rho)(s+n/p')} \psi^{\beta,\rho}_{k+\rho,2^\rho l}.$$

We then have

$$f = \sum_{\beta \in \mathbb{N}_0^n} \sum_{k=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \gamma^{\beta,\rho}_{k+\rho,2^\rho l} \Psi^{\beta,\rho}_{k+\rho,2^\rho l} + \cdots \tag{2.7}$$

in $\mathcal{S}'(\mathbb{R}^n)$. Let us now verify (2.3). We first establish some estimates. Let $\{\varphi_k\}_{k=0}^{\infty}$ be the same as in (1.1) and (1.2). Obviously, we have

$$(f, (\beta q u)_{k+\rho,2^\rho l}) = \sum_{j=0}^{\infty} \left(\left(\varphi_j \hat{f} \right)^{\vee}, (\beta q u)_{k+\rho,2^\rho l} \right).$$

We also have the following trivial estimate, that is, for all $j \in \mathbb{N}_0$,

$$\begin{aligned} & \left| \left(\left(\varphi_j \hat{f} \right)^{\vee}, (\beta q u)_{k+\rho,2^\rho l} \right) \right| \chi^{(p)}_{k+\rho,2^\rho l}(x) \\ & = \left| \int_{\mathbb{R}^n} \left(\varphi_j \hat{f} \right)^{\vee} (y) (\beta q u)_{k+\rho,2^\rho l}(y)\, dy \right| \chi^{(p)}_{k+\rho,2^\rho l}(x) \\ & \le C 2^{k(s+n/p')} 2^{r|\beta|} \int_{|y-2^{-k}l| \le 2^{r-k-\rho}} \left| \left(\varphi_j \hat{f} \right)^{\vee} (y) \right| dy \chi^{(p)}_{k+\rho,2^\rho l}(x) \\ & \le C 2^{k(s+n/p')} 2^{r|\beta|} 2^{-kn} M \left(\left(\varphi_j \hat{f} \right)^{\vee} \right) (x) \chi^{(p)}_{k+\rho,2^\rho l}(x), \end{aligned} \tag{2.8}$$

where M is the Hardy-Littlewood maximal function and $C > 0$ is a constant independent of k, j, l and x.

Now let $\varphi_{-1} \equiv 0$. Then, for all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$,

$$\sum_{\nu=-1}^{1} \varphi_{j+\nu} \varphi_j(x) = \varphi_j(x).$$

We then have

(2.9)
$$\begin{aligned}
&\left(\left(\varphi_j\hat{f}\right)^\vee, (\beta qu)_{k+\rho,2^\rho l}\right)\\
&\quad=\sum_{\nu=-1}^{1}\left(\varphi_{j+\nu}^\vee * \left(\varphi_j\hat{f}\right)^\vee, (\beta qu)_{k+\rho,2^\rho l}\right)\\
&\quad=\sum_{\nu=-1}^{1}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\varphi_{j+\nu}^\vee(u-y)\left(\varphi_j\hat{f}\right)^\vee(y)\,dy\right](\beta qu)_{k+\rho,2^\rho l}(u)\,du\\
&\quad=\sum_{\nu=-1}^{1}\int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}\varphi_{j+\nu}^\vee(u-y)(\beta qu)_{k+\rho,2^\rho l}(u)\,du\right]\left(\varphi_j\hat{f}\right)^\vee(y)\,dy.
\end{aligned}$$

We now claim that for $j\ge 2$, $\nu=-1,\ 0,\ 1$, and any $L\in\mathbb{N}_0$,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\varphi_{j+\nu}^\vee(u-y)(\beta qu)_{k+\rho,2^\rho l}(u)\,du\right|\\
&\qquad\le C_L\frac{2^{k(s+n/p')}2^{(k-j)(L+1)}2^{r|\beta|}2^{[(k-j)\vee 0](n+1)}}{(1+2^k|y-2^{-k}l|)^{n+1}},
\end{aligned}\tag{2.10}$$

where $a\vee b=\max(a,b)$ and $C_L>0$ is a constant independent of j, k, β, l and y.

We first note that for $\alpha\in\mathbb{N}_0^n$,

$$\int_{\mathbb{R}^n}\varphi_{j+\nu}^\vee(u-y)u^\alpha\,du=0.\tag{2.11}$$

To prove our claim, without loss of generality, we only show the case $\nu=0$. By (2.11), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\varphi_j^\vee\quad(u-y)(\beta qu)_{k+\rho,2^\rho l}(u)\,du\right|\\
&\qquad=2^{(j-1)n}\left|\int_{\mathbb{R}^n}\varphi_1^\vee(2^{j-1}u-2^{j-1}y)\left[(\beta qu)_{k+\rho,2^\rho l}(u)\right.\right.\\
&\qquad\quad\left.\left.-\sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|\le L}\frac{1}{\alpha!}D^\alpha(\beta qu)_{k+\rho,2^\rho l}(y)(u-y)^\alpha\right]\,du\right|\\
&\qquad\le C2^{jn}\sum_{\alpha\in\mathbb{N}_0^n,\,|\alpha|=L+1}\int_{\mathbb{R}^n}|\varphi_1^\vee(2^{j-1}u-2^{j-1}y)|\\
&\qquad\quad\times|D^\alpha(\beta qu)_{k+\rho,2^\rho l}((1-\theta)u+\theta y)||u-y|^{L+1}\,du,
\end{aligned}\tag{2.12}$$

where $\theta\in(0,1)$.

By Definition 2.1, we have that for $|\alpha|=L+1$,

$$|D^\alpha(\beta qu)_{k+\rho,2^\rho l}((1-\theta)u+\theta y)|\le C2^{k(s+n/p')}2^{k(L+1)}2^{r|\beta|},\tag{2.13}$$

and if $D^\alpha(\beta qu)_{k+\rho,2^\rho l}((1-\theta)u+\theta y)\neq 0$, then

$$|2^{k+\rho}[(1-\theta)u+\theta y]-2^\rho l|<2^r. \tag{2.14}$$

From (2.14), it follows that

$$\begin{aligned} 2^k|y-2^{-k}l| &\le 2^k|y-u|+2^k|u-2^{-k}l| \\ &\le 2^k|y-u|+2^k\theta|u-y|+2^k2^r2^{-k-\rho} \\ &\le 2^{k+1}|u-y|+2^r2^{-\rho}. \end{aligned} \tag{2.15}$$

By (2.13), (2.15) and (2.12), we then have

$$\begin{aligned} &\left|\int_{\mathbb{R}^n}\varphi^\vee_{j+\nu}(u-y)(\beta qu)_{k+\rho,2^\rho l}(u)\,du\right| \\ &\quad\le C\frac{2^{jn}2^{r|\beta|}2^{k(L+1)}2^{k(s+n/p')}}{(1+2^k|y-2^{-k}l|)^{n+1}} \\ &\qquad\times\int_{\mathbb{R}^n}\left|\varphi^\vee_1(2^{j-1}u-2^{j-1}y)\right||u-y|^{L+1}\left(1+2^k|u-y|\right)^{n+1}\,du \\ &\quad= C\frac{2^{r|\beta|}2^{k(L+1)}2^{k(s+n/p')}2^{-j(L+1)}}{(1+2^k|y-2^{-k}l|)^{n+1}} \\ &\qquad\times\int_{\mathbb{R}^n}|\varphi^\vee_1(u)||u|^{L+1}\left(1+2^{k-j}|u|\right)^{n+1}\,du \\ &\quad\le C\frac{2^{r|\beta|}2^{(k-j)(L+1)}2^{k(s+n/p')}2^{[(k-j)\vee 0](n+1)}}{(1+2^k|y-2^{-k}l|)^{n+1}}. \end{aligned}$$

Thus, our claim holds.

By (2.10) and (2.9), for $j\ge k$, we have

$$\begin{aligned} &\left|\left(\left(\varphi_j\hat{f}\right)^\vee,(\beta qu)_{k+\rho,2^\rho l}\right)\right|\chi^{(p)}_{k+\rho,2^\rho l}(x) \\ &\quad\le C_L2^{k(s+n/p')}2^{(k-j)(L+1)}2^{r|\beta|}\chi^{(p)}_{k+\rho,2^\rho l}(x)\int_{\mathbb{R}^n}\frac{|(\varphi_j\hat{f})^\vee(y)|}{(1+2^k|y-x|)^{n+1}}\,dy \\ &\quad\le C_L2^{k(s+n/p')}2^{(k-j)(L+1)}2^{r|\beta|}2^{-kn}M\left(\left(\varphi_j\hat{f}\right)^\vee\right)(x)\chi^{(p)}_{k+\rho,2^\rho l}(x), \end{aligned} \tag{2.16}$$

where $C_L>0$ is a constant independent of k, j, β, l and x.

We now have

$$\begin{aligned} &\left\{\sum_{k=0}^\infty\sum_{l\in\mathbb{Z}^n}\left[\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\right|\chi^{(p)}_{k+\rho,2^\rho l}(x)\right]^q\right\}^{1/q} \\ &\quad= C\left\{\sum_{k=0}^\infty\sum_{l\in\mathbb{Z}^n}2^{-\rho|\beta|q}\left[|(f,(\beta qu)_{k+\rho,2^\rho l})|\,\chi^{(p)}_{k+\rho,2^\rho l}(x)\right]^q\right\}^{1/q} \end{aligned}$$

$$
\begin{aligned}
&= C\left\{\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} 2^{-\rho|\beta|q}\left[\left|\sum_{j=0}^{\infty}\left(\left(\varphi_j\hat{f}\right)^{\vee},(\beta qu)_{k+\rho,2^\rho l}\right)\right|\chi^{(p)}_{k+\rho,2^\rho l}(x)\right]^q\right\}^{1/q}\\
&\le C\left\{\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} 2^{-\rho|\beta|q}\left[\left|\sum_{j=0}^{k+1}\left(\left(\varphi_j\hat{f}\right)^{\vee},(\beta qu)_{k+\rho,2^\rho l}\right)\right|\chi^{(p)}_{k+\rho,2^\rho l}(x)\right]^q\right\}^{1/q}\\
&\quad +C\left\{\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} 2^{-\rho|\beta|q}\left[\left|\sum_{j=k+2}^{\infty}\left(\left(\varphi_j\hat{f}\right)^{\vee},(\beta qu)_{k+\rho,2^\rho l}\right)\right|\chi^{(p)}_{k+\rho,2^\rho l}(x)\right]^q\right\}^{1/q}\\
&= I_1+I_2.
\end{aligned}
$$

For I_1, by (2.8), Hölder's inequality and $s<0$, we have

(2.17)

$$
\begin{aligned}
I_1 &\le C\left\{\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} 2^{-\rho|\beta|q}\right.\\
&\qquad\left.\times\left[\sum_{j=0}^{k+1}2^{k(s+n/p')}2^{r|\beta|}2^{kn/p-kn}M\left(\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)\chi_{Q_{k+\rho,2^\rho l}}(x)\right]^q\right\}^{1/q}\\
&= C2^{(r-\rho)|\beta|}\left\{\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k+1}M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)2^{(k-j)s}\right]^q\right\}^{1/q}\\
&\le C2^{(r-\rho)|\beta|}\left\{\sum_{k=0}^{\infty}\sum_{j=0}^{k+1}\left[M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)\right]^q 2^{(k-j)sq/2}\right\}^{1/q}\\
&\le C2^{(r-\rho)|\beta|}\left\{\sum_{j=0}^{\infty}\left[M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)\right]^q\right\}^{1/q},
\end{aligned}
$$

where $C>0$ is a constant independent of β and x.

For I_2, we choose $L\in\mathbb{N}_0$ such that $L+1>-s$. We then, by (2.16) and Hölder's inequality, have that

(2.18)

$$
\begin{aligned}
I_2 &\le C_L\left\{\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} 2^{-\rho|\beta|q}\left[\sum_{j=k+2}^{\infty}2^{k(s+n/p')}2^{(k-j)(L+1)}2^{r|\beta|}2^{kn/p-kn}\right.\right.\\
&\qquad\left.\left.\times\ M\left(\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)\chi_{Q_{k+\rho,2^\rho l}}(x)\right]^q\right\}^{1/q}\\
&\le C_L2^{(r-\rho)|\beta|}\left\{\sum_{k=0}^{\infty}\left[\sum_{j=k+2}^{\infty}M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)2^{(k-j)(L+1+s)}\right]^q\right\}^{1/q}
\end{aligned}
$$

$$\leq C_L 2^{(r-\rho)|\beta|}\left\{\sum_{k=0}^{\infty}\sum_{j=k+2}^{\infty}\left[M\Big(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\Big)(x)\right]^q 2^{(k-j)(L+1+s)q/2}\right\}^{1/q}$$

$$\leq C_L 2^{(r-\rho)|\beta|}\left\{\sum_{j=0}^{\infty}\left[M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(x)\right]^q\right\}^{1/q},$$

where $C_L > 0$ is a constant independent of β and x.

By (2.17), (2.18) and the Fefferman-Stein vector-valued maximal inequality in [1], we have that the right hand side of (2.3) is controlled by

$$C\left\|\left\{\sum_{j=0}^{\infty}\left[M\left(2^{js}\left(\varphi_j\hat{f}\right)^{\vee}\right)(\cdot)\right]^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)}$$

$$\leq C\left\|\left\{\sum_{j=0}^{\infty}2^{jsq}\left|\left(\varphi_j\hat{f}\right)^{\vee}\right|^q\right\}^{1/q}\right\|_{L^p(\mathbb{R}^n)} = C\|f\|_{F^s_{pq}(\mathbb{R}^n)},$$

where $C > 0$ is a constant independent of f. Thus, (2.3) holds.

We still need to verify that the series in (2.2) converge in the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1 < p,\ q < \infty$. By (2.3) and the following Proposition 2.2, we know that the series in (2.2) converge in the norm of $F^s_{pq}(\mathbb{R}^n)$ to some, say, $h \in F^s_{pq}(\mathbb{R}^n)$. Moreover, by Proposition 2.2 below, we know that the series in (2.2) also converge in $\mathcal{S}'(\mathbb{R}^n)$ to h. Thus, $h = f$ in $\mathcal{S}'(\mathbb{R}^n)$ and $h \in F^s_{pq}(\mathbb{R}^n)$. Then, by Definition 1.1, we have also $h = f$ in the norm of $F^s_{pq}(\mathbb{R}^n)$. This just means that the series in (2.2) converge to f in the norm of $F^s_{pq}(\mathbb{R}^n)$.

This finishes the proof of Proposition 2.1. □

Proposition 2.2. *Let $s < 0$, $1 < p < \infty$, $1 < q \leq \infty$ and $\{\Psi^{\beta,l}_k :\ \beta \in \mathbb{N}^n_0,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ be the same as in Proposition 2.1. If $\{\gamma^{\beta}_{k,l} :\ \beta \in \mathbb{N}^n_0,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ is a sequence of complex numbers and*

$$\sup_{\beta\in\mathbb{N}^n_0} 2^{(\rho-r)|\beta|}\left\|\left(\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta}_{k,l}\chi^{(p)}_{k,l}\right|^q\right)^{1/q}\right\|_{L^p(\mathbb{R}^n)} < \infty, \tag{2.19}$$

where $\rho > r$ and r is the same as in (2.1), then

$$\sum_{\beta\in\mathbb{N}^n_0}\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\gamma^{\beta}_{k,l}\Psi^{\beta,l}_k \tag{2.20}$$

unconditionally converge in both $\mathcal{S}'(\mathbb{R}^n)$ and the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1 < p,\ q < \infty$ and only in $\mathcal{S}'(\mathbb{R}^n)$ when $1 < p < \infty$ and $1 < q \leq \infty$ to some $f \in F^s_{pq}(\mathbb{R}^n)$.

Moreover,

$$\|f\|_{F^s_{pq}(\mathbb{R}^n)} \le C \sup_{\beta\in\mathbb{N}_0^n} 2^{(\rho-r)|\beta|} \left\| \left(\sum_{k=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \left|\gamma^{\beta}_{k,l}\chi^{(p)}_{k,l}\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}, \tag{2.21}$$

where $C>0$ is a constant independent of f.

Proof. Obviously, when $1<p,\ q<\infty$, we only need to show that the series in (2.20) converge in the norm of $F^s_{pq}(\mathbb{R}^n)$ since $F^s_{pq}(\mathbb{R}^n)\subset\mathcal{S}'(\mathbb{R}^n)$; see [6, Theorem 2.3.3]. By noting (2.7), where $\{\Psi^{\beta,l}_k:\ \beta\in\mathbb{N}_0^n,\ k\in\mathbb{N}_0,\ l\in\mathbb{Z}^n\}$ comes from, and by using the same notation as in (2.7), we can re-write (2.20) into

$$\sum_{\beta\in\mathbb{N}_0^n}\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n} \gamma^{\beta,\rho}_{k+\rho,2^\rho l}\Psi^{\beta,\rho}_{k+\rho,2^\rho l} \tag{2.22}$$

and (2.19) into

$$\sup_{\beta\in\mathbb{N}_0^n} 2^{(\rho-r)|\beta|} \left\| \left(\sum_{k=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}\right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty. \tag{2.23}$$

Let us now show the conclusions for the series in (2.22) under the assumption (2.23). To prove that the first series in (2.22) converges in the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1<p,\ q<\infty$, for $L\in\mathbb{N}$, we set

$$S_L = \sum_{l\in\mathbb{Z}^n,\ |l|\le L} \gamma^{\beta,\rho}_{k+\rho,2^\rho l}\Psi^{\beta,\rho}_{k+\rho,2^\rho l}.$$

We now use a dual argument. Let $g\in\mathcal{S}(\mathbb{R}^n)$. By (2.5) and (2.6), we then have that for $L_1,\ L_2\in\mathbb{N}$ and $L_1<L_2$,

(2.24)

$$\begin{aligned}
&|(S_{L_2}-S_{L_1},g)| \\
&= \left| \sum_{l\in\mathbb{Z}^n,\ L_1<|l|\le L_2} \gamma^{\beta,\rho}_{k+\rho,2^\rho l}\left(\Psi^{\beta,\rho}_{k+\rho,2^\rho l},g\right)\right| \\
&= C\left| \sum_{l\in\mathbb{Z}^n,\ L_1<|l|\le L_2} 2^{\rho|\beta|}2^{-(k+\rho)(s+n/p')}\gamma^{\beta,\rho}_{k+\rho,2^\rho l}2^{-\rho|\beta|}\left(\psi^{\beta,\rho}_{k+\rho,2^\rho l},g\right)\right| \\
&= C2^{\rho|\beta|}\left| \sum_{l\in\mathbb{Z}^n,\ L_1<|l|\le L_2} \gamma^{\beta,\rho}_{k+\rho,2^\rho l}\lambda^{\beta}_{k+\rho,2^\rho l}\int_{\mathbb{R}^n}\chi^{(p)}_{k+\rho,2^\rho l}(x)\chi^{(p')}_{k+\rho,2^\rho l}(x)\,dx\right| \\
&\le C2^{\rho|\beta|}\int_{\mathbb{R}^n}\left[\sum_{l\in\mathbb{Z}^n,\ L_1<|l|\le L_2}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(x)\right|^q\right]^{1/q}
\end{aligned}$$

$$\times \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\lambda^{\beta}_{k+\rho,2^\rho l}\chi^{(p')}_{k+\rho,2^\rho l}(x)\right|^{q'} \right]^{1/q'} dx$$

$$\le C2^{\rho|\beta|} \left\| \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^{q} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}$$

$$\times \left\| \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\lambda^{\beta}_{k+\rho,2^\rho l}\chi^{(p')}_{k+\rho,2^\rho l}(\cdot)\right|^{q'} \right]^{1/q'} \right\|_{L^{p'}(\mathbb{R}^n)}$$

$$\le C\|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)} \left\| \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^{q} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)}.$$

By Lemma 1.1, when $1<p<\infty$ and $1<q\le\infty$, we further have

(2.25)

$$\|S_{L_2}-S_{L_1}\|_{F^s_{pq}(\mathbb{R}^n)} = \sup_{\|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)}\le 1} |(S_{L_2}-S_{L_1},g)|$$

$$\le C \left\| \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^{q} \right]^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

where $C>0$ is a constant independent of L_1 and L_2. By (2.23), we know that if $1<p,\ q<\infty$, as $L_1,\ L_2\to\infty$,

$$\|S_{L_2}-S_{L_1}\|_{F^s_{pq}(\mathbb{R}^n)}\to 0. \tag{2.26}$$

Thus, $\{S_L\}_{L\in\mathbb{N}}$ is a Cauchy sequence in $F^s_{pq}(\mathbb{R}^n)$ when $1<p,\ q<\infty$. Since $F^s_{pq}(\mathbb{R}^n)$ is a Banach space, we then know that $\{S_L\}_{L\in\mathbb{N}}$ converges as $L\to\infty$ in the norm of $F^s_{pq}(\mathbb{R}^n)$. This means that the interior series in (2.22) converges in the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1<p,\ q<\infty$. However, when $1<p<\infty$ and $q=\infty$, from (2.23) and (2.25), we can not deduce (2.26) still holds. But, by (2.6), we know

$$\left\| \left[\sum_{l\in\mathbb{Z}^n,\, L_1<|l|\le L_2} \left|\lambda^{\beta}_{k+\rho,2^\rho l}\chi^{(p')}_{k+\rho,2^\rho l}(\cdot)\right|^{q'} \right]^{1/q'} \right\|_{L^{p'}(\mathbb{R}^n)} \to 0, \tag{2.27}$$

as $L_1,\ L_2\to\infty$. From (2.27) and (2.24), it still follows that for any given $g\in\mathcal{S}(\mathbb{R}^n)$,

$$|(S_{L_2}-S_{L_1},g)|\to 0,$$

as $L_1,\ L_2\to\infty$. This just means that the first series in (2.22) converges in $\mathcal{S}'(\mathbb{R}^n)$ when $1<p<\infty$ and $q=\infty$. Similarly, we can show the other series in (2.22) also converge in the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1<p,\ q<\infty$

and in $\mathcal{S}'(\mathbb{R}^n)$ when $1 < p < \infty$, $1 < q \le \infty$. Thus, the series in (2.22) converge in the norm of $F^s_{pq}(\mathbb{R}^n)$ when $1 < p,\ q < \infty$ and in $\mathcal{S}'(\mathbb{R}^n)$ when $1 < p < \infty$ and $1 < q \le \infty$ to some, say, $f \in F^s_{pq}(\mathbb{R}^n)$. We now verify (2.21). Let $g \in \mathcal{S}(\mathbb{R}^n)$. By (2.5) and (2.6), we then have

$$\begin{aligned}
|(f,g)| &= \left|\sum_{\beta\in\mathbb{N}_0^n}\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\left(\Psi^{\beta,\rho}_{k+\rho,2^\rho l},g\right)\right| \\
&= C\sum_{\beta\in\mathbb{N}_0^n}2^{\rho|\beta|}\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\lambda^{\beta}_{k+\rho,2^\rho l}\right|\int_{\mathbb{R}^n}\chi^{(p)}_{k+\rho,2^\rho l}(x)\chi^{(p')}_{k+\rho,2^\rho l}(x)\,dx \\
&\le C\sum_{\beta\in\mathbb{N}_0^n}2^{\rho|\beta|}\int_{\mathbb{R}^n}\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(x)\right|^q\right]^{1/q} \\
&\quad\times\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\lambda^{\beta}_{k+\rho,2^\rho l}\chi^{(p')}_{k+\rho,2^\rho l}(x)\right|^{q'}\right]^{1/q'}dx \\
&\le C\sum_{\beta\in\mathbb{N}_0^n}2^{\rho|\beta|}\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\
&\quad\times\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\lambda^{\beta}_{k+\rho,2^\rho l}\chi^{(p')}_{k+\rho,2^\rho l}(\cdot)\right|^{q'}\right]^{1/q'}\right\|_{L^{p'}(\mathbb{R}^n)} \\
&\le C\|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)}\sum_{\beta\in\mathbb{N}_0^n}\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where $C > 0$ is a constant independent of f and g. By Lemma 1.1 again, we have

$$\begin{aligned}
\|f|F^s_{pq}(\mathbb{R}^n)\| &= \sup_{\|g\|_{F^{-s}_{p'q'}(\mathbb{R}^n)}\|\le 1}|(f,g)| \\
&\le C\sum_{\beta\in\mathbb{N}_0^n}\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\
&\le C\sum_{\beta\in\mathbb{N}_0^n}\frac{1}{2^{(\rho-r)|\beta|}} \\
&\quad\times\sup_{\beta\in\mathbb{N}_0^n}2^{(\rho-r)|\beta|}\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)} \\
&\le C\sup_{\beta\in\mathbb{N}_0^n}2^{(\rho-r)|\beta|}\left\|\left[\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\left|\gamma^{\beta,\rho}_{k+\rho,2^\rho l}\chi^{(p)}_{k+\rho,2^\rho l}(\cdot)\right|^q\right]^{1/q}\right\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

where $C > 0$ is a constant independent of f.

This finishes the proof of Proposition 2.2. □

Combining Proposition 2.1 with Proposition 2.2 leads us to the following theorem.

Theorem 2.1. *Let $s < 0$, $1 < p < \infty$, $1 < q \le \infty$ and $\{\Psi_k^{\beta,l} : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ be the same as in Proposition 2.1. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is an element of $F_{pq}^s(\mathbb{R}^n)$ if, and only if, it can be represented as in (2.2) with a sequence of complex numbers $\{\gamma_{k,l}^\beta : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ satisfying (2.19), where $\rho > r$ and r is the same as in (2.1), and (2.2) holds unconditionally in $\mathcal{S}'(\mathbb{R}^n)$. Furthermore,*

$$\|f\|_{F_{pq}^s(\mathbb{R}^n)} \sim \inf \left\{ \sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\| \left(\sum_{k=0}^\infty \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^\beta \chi_{k,l}^{(p)} \right|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \right\},$$

where the infimum is taken over all admissible representations (2.2).

By a proof similar to that for Proposition 2.1 and Proposition 2.2, respectively, we can establish the following theorems on the spaces $B_{pq}^s(\mathbb{R}^n)$. We omit the details.

Proposition 2.3. *Let $s < 0$ and $1 \le p,\ q \le \infty$. There is a set of Schwartz functions, $\{\Psi_k^{\beta,l} : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$, such that for any $f \in B_{pq}^s(\mathbb{R}^n)$, (2.2) holds unconditionally in both the norm of $B_{pq}^s(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ when $1 \le p,\ q < \infty$ and only in $\mathcal{S}'(\mathbb{R}^n)$ when $1 \le p,\ q \le \max(p,q) = \infty$, where, when $k \in \mathbb{N}$, for all $\alpha \in \mathbb{N}_0^n$,*

$$\int_{\mathbb{R}^n} \Psi_k^{\beta,l}(x) x^\alpha \, dx = 0,$$

and $\{\gamma_{k,l}^\beta : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ is a sequence of numbers linearly depending on f and satisfying

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\{ \left(\sum_{k=0}^\infty \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^\beta \right|^p \right)^{q/p} \right\}^{1/q} \le C \|f\|_{B_{pq}^s(\mathbb{R}^n)}$$

with the same r as in (2.1), $\rho > r$ and a constant $C > 0$ independent of f.

Proposition 2.4. *Let $s < 0$, $1 \le p,\ q \le \infty$ and $\{\Psi_k^{\beta,l} : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ be the same as in Proposition 2.3. If $\{\gamma_{k,l}^\beta : \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ is a sequence of complex numbers and*

$$\sup_{\beta \in \mathbb{N}_0^n} 2^{(\rho - r)|\beta|} \left\{ \left(\sum_{k=0}^\infty \sum_{l \in \mathbb{Z}^n} \left| \gamma_{k,l}^\beta \right|^p \right)^{q/p} \right\}^{1/q} < \infty, \tag{2.28}$$

where $\rho > r$ and r is the same as in (2.1), then (2.20) unconditionally converge in both $\mathcal{S}'(\mathbb{R}^n)$ and the norm of $B_{pq}^s(\mathbb{R}^n)$ when $1 \le p,\ q < \infty$ and

only in $\mathcal{S}'(\mathbb{R}^n)$ *when* $1 \le p,\ q \le \max(p,q) = \infty$ *to some* $f \in B^s_{pq}(\mathbb{R}^n)$. *Moreover,*

$$\|f\|_{B^s_{pq}(\mathbb{R}^n)} \le C \sup_{\beta\in\mathbb{N}_0^n} 2^{(\rho-r)|\beta|} \left\{ \left(\sum_{k=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \left|\gamma^\beta_{k,l}\right|^p \right)^{q/p} \right\}^{1/q},$$

where $C > 0$ *is a constant independent of* f.

Combining Proposition 2.1 with Proposition 2.2 yields the following result.

Theorem 2.2. *Let* $s < 0$, $1 \le p,\ q \le \infty$ *and* $\{\Psi^{\beta,l}_k :\ \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ *be the same as in Proposition* 2.3. *Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *is an element of* $B^s_{pq}(\mathbb{R}^n)$ *if, and only if, it can be represented as in* (2.2) *with a sequence of complex numbers* $\{\gamma^\beta_{k,l} :\ \beta \in \mathbb{N}_0^n,\ k \in \mathbb{N}_0,\ l \in \mathbb{Z}^n\}$ *satisfying* (2.28), *where* $\rho > r$ *and* r *is the same as in* (2.1), *and* (2.2) *holds unconditionally in* $\mathcal{S}'(\mathbb{R}^n)$. *Furthermore,*

$$\|f\|_{B^s_{pq}(\mathbb{R}^n)} \sim \inf \left\{ \sup_{\beta\in\mathbb{N}_0^n} 2^{(\rho-r)|\beta|} \left\{ \left(\sum_{k=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \left|\gamma^\beta_{k,l}\right|^p \right)^{q/p} \right\}^{1/q} \right\},$$

where the infimum is taken over all admissible representations (2.2).

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