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An inequality for first-order differences*

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Abstract. A question about comparing norms of difference operators that was raised in [1] and presented at the Fourth ISAAC Congress is answered in the affirmative.

Let $1 \le p < \infty$ and suppose $f : [0, \infty) \to \mathbf{R}$. For $h > 0$, set

$$F(h) \equiv \|\Delta_h f\|_{L^p(0,h)} = \left(\int_0^h |f(x+h) - f(x)|^p \, dx \right)^{1/p}$$

and

$$G(h) \equiv \|\Delta_h f\|_{L^p(h,3h)} = \left(\int_h^{3h} |f(x+h) - f(x)|^p \, dx \right)^{1/p}.$$

We wish to compare F and G as functions of h. It is easy to see that pointwise comparisons of F and G are impossible for arbitrary f as F depends on values of f on the interval $[0, 2h]$ but G depends on values of

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f on the interval $[h, 4h]$. It is possible to compare the functions F and G in norm, however. A special case of [1, Corollary 2] shows that there is a constant A such that

$$\left(\int_0^\delta F(h)^\theta h^{-l\theta}\,\frac{dh}{h}\right)^{1/\theta} \le A\left(\int_0^\delta G(h)^\theta h^{-l\theta}\,\frac{dh}{h}\right)^{1/\theta}$$

holds for all measurable f. Here $l > 0$, $0 < \delta \le \infty$, and $1 \le \theta < \infty$. (The analogous result in the case $\theta = \infty$ is also given there.)

It does not follow from the results of [1] that this inequality still holds if the constant 3 in the definition of G is replaced by a smaller number. That is the substance of the question raised in [1, Remark 3]. Corollary 4, below, shows that the above inequality holds when 3 is replaced by any $\beta > 1$.

The key is the following lemma comparing terms of the form $\|\Delta_{th} f\|_{L^p(bh,ch)}$ where as usual

$$\|f\|_{L^p(b,c)} = \begin{cases} \left(\displaystyle\int_b^c |f(x)|^p\,dx\right)^{1/p}, & 0 < p < \infty; \\ \operatorname{ess\,sup}_{b\le x\le c} |f(x)|, & p = \infty. \end{cases}$$

Lemma 1. *Suppose $1 \le p \le \infty$. If b, c, t, B, C, and T are positive real numbers with $b < c$ and $B < C$ then there exists a finite sequence $s_1, s_2, \ldots, s_N$ of positive real numbers such that*

$$\|\Delta_t f\|_{L^p(b,c)} \le \sum_{n=1}^N \|\Delta_{s_n T} f\|_{L^p(s_n B, s_n C)}$$

for all measurable $f : [0,\infty) \to \mathbf{R}$. The s_n can be chosen so that

$$s_n < \frac{c+t}{B+T} \tag{1}$$

for $n = 1, 2, \ldots, N$.

Proof. We begin by supposing that we have a solution to the following problem: Find positive integers M and K and real numbers t_m, b_k, and $r_{m,k}$ for $m = 1, \ldots, M$ and $k = 1, \ldots, K$ such that

$$0 = t_0 < t_1 < \cdots < t_m = t, \tag{2}$$

$$b = b_0 < \cdots < b_{K-1} < c \le b_K, \tag{3}$$

$$\frac{b_k + r_{m,k}}{C} = \frac{t_{m-1} - r_{m,k}}{T} < \frac{t_m - r_{m,k}}{T} = \frac{b_{k-1} + r_{m,k}}{B}. \tag{4}$$

An application of Minkowski's inequality gives

$$\|\Delta_t f\|_{L^p(b,c)} \le \|\Delta_t f\|_{L^p(b_0,b_K)} \le \sum_{k=1}^{K} \|\Delta_t f\|_{L^p(b_{k-1},b_k)}$$

and, since

$$\Delta_t f(x) = \sum_{m=1}^{M} f(x+t_m) - f(x+t_{m-1}) = \sum_{m=1}^{M} \Delta_{t_m - t_{m-1}} f(x+t_{m-1}),$$

another application yields

$$\|\Delta_t f\|_{L^p(b_{k-1},b_k)} \le \sum_{m=1}^{M} \|\Delta_{t_m - t_{m-1}} f\|_{L^p(b_{k-1}+t_{m-1},b_k+t_{m-1})}$$

for each k. To reach the conclusion of the lemma it is enough to show that for each m and k,

$$\|\Delta_{t_m - t_{m-1}} f\|_{L^p(b_{k-1}+t_{m-1},b_k+t_{m-1})} \tag{5}$$

can be dominated by a sum of terms of the form

$$\|\Delta_{sT} f\|_{L^p(sB,sC)}, \quad \text{for } s > 0. \tag{6}$$

Fix m and k and write $r = r_{m,k}$. Rearranging terms in (4) gives

$$b_{k-1} + r = (t_m + b_{k-1})/(1 + T/B) > 0$$

and we see that $x + r > 0$ for all $x \in (b_{k-1}, b_k)$. Thus we may write

$$\begin{aligned}
\Delta_{t_m - t_{m-1}} f(x+t_{m-1}) &= f(x+t_m) - f(x+t_{m-1}) \\
&= f(x+t_m) - f(x+r) + f(x+r) - f(x+t_{m-1}) \\
&= \Delta_{t_m - r} f(x+r) - \Delta_{t_{m-1} - r} f(x+r)
\end{aligned}$$

for such x. A third application of Minkowski's inequality shows that (5) is no greater than

$$\|\Delta_{t_m - r} f\|_{L^p(b_{k-1}+r,b_k+r)} + \|\Delta_{t_{m-1} - r} f\|_{L^p(b_{k-1}+r,b_k+r)}.$$

Using (4), with $s' = (t_m - r)/T$ and $s = (t_{m-1} - r)/T$, we can dominate the last expression by

$$\|\Delta_{s'T} f\|_{L^p(s'B,s'C)} + \|\Delta_{sT} f\|_{L^p(sB,sC)},$$

which is a sum of terms of the form (6) as required.

To complete the proof we provide a solution to the problem (2)-(4) and verify that s and s' satisfy the upper bound (1). Since $0 < B < C$, both

$$\mu \equiv \frac{C(B+T)}{B(C+T)} \quad \text{and} \quad \nu \equiv \frac{C+T}{B+T}$$

are greater than 1. Choose a positive integer M so that $t < b(\mu^M - 1)$ and set $\varepsilon = t/(\mu^M - 1)$. For each m and k define

$$t_m = \varepsilon(\mu^m - 1) \quad \text{and} \quad b_k = \nu^k(b - \varepsilon) + \varepsilon.$$

Evidently, (2) is satisfied. The choice of M ensures that $0 < \varepsilon < b$ so the b_k's are increasing and unbounded. Therefore we can choose K to satisfy (3).

The inequality in (4) is automatically satisfied and the two equations in (4) reduce to

$$r_{m,k} = \frac{Ct_{m-1} - Tb_k}{C+T} \quad \text{and} \quad r_{m,k} = \frac{Bt_m - Tb_{k-1}}{B+T}.$$

A routine calculation shows that these two expressions coincide so that either may be used to define $r_{m,k}$ so that (4) is satisfied. Finally, we estimate s and s' (for fixed m and k) by

$$s < s' = \frac{t_m - r}{T} = \frac{b_{k-1} + t_m}{B+T} < \frac{c+t}{B+T}.$$

This completes the proof. □

Lemma 1 remains valid with the L^p-norm replaced by any Banach function space norm since only the triangle inequality is needed. Moreover, it is easily extended to quasinormed function spaces at the expense of an additional constant. In particular, if $0 < p < 1$ one obtains

$$\|\Delta_t f\|^p_{L^p(b,c)} \le \sum_{n=1}^{N} \|\Delta_{s_n T} f\|^p_{L^p(s_n B, s_n C)}.$$

and hence

$$\|\Delta_t f\|_{L^p(b,c)} \le N^{1/p-1} \sum_{n=1}^{N} \|\Delta_{s_n T} f\|_{L^p(s_n B, s_n C)}.$$

Our main result shows that the norms of F and G and other similar expressions are all comparable in great generality. For convenience we

introduce notation for the following weighted Lebesgue norms of functions of the variable h.

$$\|g(h)\|_{L_l^\theta(0,\delta)} = \begin{cases} \left(\int_0^\delta |g(h)|^\theta h^{-l\theta}\, dh/h\right)^{1/\theta}, & 1 \le \theta < \infty; \\ \operatorname{ess\,sup}_{0<h<\delta} h^{-l}|g(h)|, & \theta = \infty. \end{cases}$$

Here $g(h)$ is understood to be formula involving the variable h rather than a function name. In addition to homogeneity, the triangle inequality, and the Fatou property of these norms we will need the following dilation property, easily proved by a change of variable. For any $s > 0$,

$$\|g(h)\|_{L_l^\theta(0,\delta)} = s^l \|g(h/s)\|_{L_l^\theta(0,s\delta)}. \tag{7}$$

Note also that $\|g(h)\|_{L_l^\theta(0,\delta_1)} \le \|g(h)\|_{L_l^\theta(0,\delta_2)}$ whenever $\delta_1 \le \delta_2$.

Theorem 2. *Suppose $1 \le p \le \infty$, $1 \le \theta \le \infty$, $l > 0$, $a \in \mathbf{R}$, and $0 < \delta \le \infty$. If $t > 0$, $0 \le b < c$, $T > 0$, and $0 \le B < C$ then, for all measurable $f : [a,\infty) \to \mathbf{R}$,*

$$\Big\| \|\Delta_{th} f\|_{L^p(a+bh,a+ch)} \Big\|_{L_l^\theta(0,\delta)} \le A \Big\| \|\Delta_{Th} f\|_{L^p(a+Bh,a+Ch)} \Big\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)}. \tag{8}$$

The constant A above depends only on t, b, c, T, B, C and l.

Proof. By translating the function f the theorem is easily reduced to the case $a = 0$ so we assume $a = 0$ henceforth.

We first prove the theorem in the case that b and B are positive. Lemma 1 yields $s_1, s_2, \ldots, s_N$ such that $s_n < (c+t)/(B+T)$ for all n and

$$\|\Delta_t f\|_{L^p(b,c)} \le \sum_{n=1}^N \|\Delta_{s_n T} f\|_{L^p(s_n B, s_n C)} \tag{9}$$

for all f. For each h, define the dilation f_h by $f_h(x) = f(hx)$ and observe that $\Delta_t f_h(x) = \Delta_{th} f(hx)$. Changes of variable show that

$$\|\Delta_t f_h\|_{L^p(b,c)} = h^{-1/p} \|\Delta_{th} f\|_{L^p(bh,ch)}$$

and

$$\|\Delta_{s_n T} f_h\|_{L^p(s_n B, s_n C)} = h^{-1/p} \|\Delta_{s_n T h} f\|_{L^p(s_n Bh, s_n Ch)}$$

so inequality (9), applied to f_h, implies

$$\|\Delta_{th} f\|_{L^p(bh,ch)} \le \sum_{n=1}^N \|\Delta_{s_n T h} f\|_{L^p(s_n Bh, s_n Ch)}.$$

It is important to point out that N and $s_1, s_2, \ldots, s_N$ do not depend on h. We use the triangle inequality and property (7) to get

$$\begin{aligned}\left\|\|\Delta_{th}f\|_{L^p(bh,ch)}\right\|_{L_l^\theta(0,\delta)} &\le \sum_{n=1}^{N}\left\|\|\Delta_{s_nTh}f\|_{L^p(s_nBh,s_nCh)}\right\|_{L_l^\theta(0,\delta)}\\ &= \sum_{n=1}^{N} s_n^l \left\|\|\Delta_{Th}f\|_{L^p(Bh,Ch)}\right\|_{L_l^\theta(0,s_n\delta)}\\ &\le \left(\sum_{n=1}^{N} s_n^l\right)\left\|\|\Delta_{Th}f\|_{L^p(Bh,Ch)}\right\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)} .\end{aligned}$$

This proves the theorem in the case that b and B are positive and it is a simple matter to show that it remains valid when $B = 0$ since replacing B by zero only makes the right hand side of (8) larger.

The case $b = 0$ requires some additional argument. For each integer $j \ge 0$, apply (7) with $s = 1/2$ to get

$$\begin{aligned}N_j &\equiv \left\|\|\Delta_{th}f\|_{L^p(2^{-j}ch,ch)}\right\|_{L_l^\theta(0,\delta)}\\ &= 2^{-l}\left\|\|\Delta_{2th}f\|_{L^p(2^{1-j}ch,2ch)}\right\|_{L_l^\theta(0,\delta/2)} .\end{aligned} \tag{10}$$

Note that $N_0 = 0$. To estimate N_j for $j \ge 1$ we break up $\Delta_{2th}f$ as

$$\Delta_{2th}f(x) = \Delta_{th}f(x+th) + \Delta_{th}f(x)\chi_{(ch,\infty)}(x) + \Delta_{th}f(x)\chi_{(0,ch)}(x)$$

and use Minkowski's inequality to obtain

$$\begin{aligned}\|\Delta_{2th}f\|_{L^p(2^{1-j}ch,2ch)} &\le \|\Delta_{th}f\|_{L^p(2^{1-j}ch+th,2ch+th)}\\ &\quad+\|\Delta_{th}f\|_{L^p(ch,2ch)} + \|\Delta_{th}f\|_{L^p(2^{1-j}ch,ch)}.\end{aligned} \tag{11}$$

Using the first part of the proof we get constants A_1 and A_2 such that

$$\begin{aligned}&\left\|\|\Delta_{th}f\|_{L^p(2^{1-j}ch+th,2ch+th)}\right\|_{L_l^\theta(0,\delta/2)}\\ &\quad\le \left\|\|\Delta_{th}f\|_{L^p(th,2ch+th)}\right\|_{L_l^\theta(0,\delta/2)} \le A_1\left\|\|\Delta_{Th}f\|_{L^p(Bh,Ch)}\right\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)}\end{aligned}$$

and

$$\begin{aligned}\left\|\|\Delta_{th}f\|_{L^p(ch,2ch)}\right\|_{L_l^\theta(0,\delta/2)} &\le A_2\left\|\|\Delta_{Th}f\|_{L^p(Bh,Ch)}\right\|_{L_l^\theta(0,\frac{2c+t}{B+T}\frac{\delta}{2})}\\ &\le A_2\left\|\|\Delta_{Th}f\|_{L^p(Bh,Ch)}\right\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)} .\end{aligned}$$

With $K = (A_1 + A_2) \left\| \|\Delta_{Th} f\|_{L^p(Bh,Ch)} \right\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)}$ we use the estimate (11) in (10) to get

$$N_j \le 2^{-l} \left(K + \left\| \|\Delta_{th} f\|_{L^p(2^{1-j}ch,ch)} \right\|_{L_l^\theta(0,\delta/2)} \right) \le 2^{-l}(K + N_{j-1}).$$

A simple induction, starting with $N_0 = 0$, shows that $N_j \le K/(2^l - 1)$ for all $j \ge 0$ and by the Fatou property we have

$$\left\| \|\Delta_{th} f\|_{L^p(0,ch)} \right\|_{L_l^\theta(0,\delta)} = \lim_{j\to\infty} N_j \le \frac{A_1 + A_2}{2^l - 1} \left\| \|\Delta_{Th} f\|_{L^p(Bh,Ch)} \right\|_{L_l^\theta(0,\frac{c+t}{B+T}\delta)}.$$

This completes the proof. □

If $\delta = \infty$ Theorem 2 becomes a general equivalence.

Corollary 3. *Suppose* $1 \le p \le \infty$, $1 \le \theta \le \infty$, $l > 0$, *and* $a \in \mathbf{R}$. *If* $t > 0$, $0 \le b < c$, $T > 0$, *and* $0 \le B < C$ *then*

$$\left\| \|\Delta_{th} f\|_{L^p(a+bh,a+ch)} \right\|_{L_l^\theta(0,\infty)} \approx \left\| \|\Delta_{Th} f\|_{L^p(a+Bh,a+Ch)} \right\|_{L_l^\theta(0,\infty)}$$

for all measurable $f : [a,\infty) \to \mathbf{R}$. *The constants in this equivalence depend only on* t, b, c, T, B, C *and* l.

The answer to the question raised in [1, Remark 3] is the special case $b = 0$, $c = t = B = T = 1$, $C = \beta$ of Theorem 2.

Corollary 4. *Suppose* $1 \le p \le \infty$, $1 \le \theta \le \infty$, $l > 0$, $a \in \mathbf{R}$, *and* $0 < \delta \le \infty$. *If* $\beta > 1$ *then there exists a constant* A *depending only on* l *and* β *such that*

$$\left\| \|\Delta_h f\|_{L^p(a,a+h)} \right\|_{L_l^\theta(0,\delta)} \le A \left\| \|\Delta_h f\|_{L^p(a+h,a+\beta h)} \right\|_{L_l^\theta(0,\delta)}$$

for all measurable $f : [a,\infty) \to \mathbf{R}$.

The techniques used in this paper do not seem to extend to higher order differences. It is natural to ask whether or not Theorem 2 holds with Δ replaced by Δ^k for $k > 1$.

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