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Non-equivalent greedy and almost greedy bases in ℓ_p

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Abstract. For $1 < p < \infty$ and $p \neq 2$ we construct a family of mutually non-equivalent greedy bases in ℓ_p having the cardinality of the continuum. In fact, no basis from this family is equivalent to a rearranged subsequence of any other basis thereof. We are able to extend this statement to the spaces L_p and H_1. Moreover, the technique used in the proof adapts to the setting of almost greedy bases where similar results are obtained.

1. Introduction

A *greedy* basis (x_n) is one for which a certain version of the 'greedy algorithm' is optimal for n-term approximation. Greedy bases are characterized as being *unconditional* and *democratic*.

There are exactly three Banach spaces which have unique unconditional bases up to equivalence, namely c_0, ℓ_1, and ℓ_2 with their standard bases [8]. From the characterization of greedy bases mentioned above it follows

A part of this work was influenced by A. Kamont while she was visiting the Department of Mathematics, University of South Carolina, in 2001.

that these three spaces also have unique greedy bases. On the other hand, for $1 < p < \infty, p \neq 2$, we show that there exist uncountably many mutually non-equivalent greedy bases in both ℓ_p and $L_p[0,1]$. The space $L_1[0,1]$ does not have a greedy basis since it does not embed into a space with an unconditional basis [11]. Instead, we consider the dyadic Hardy space $H_1(\delta)$ and we show that it has uncountably many non-equivalent greedy bases. A first result in this direction was obtained by Anna Kamont and the authors are grateful for her communication over these initial ideas.

We now give precise definitions for the concepts mentioned above. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized basis of a Banach space X, and let $(x_n^*)_{n\in\mathbb{N}}$ be the biorthogonal sequence in X^*. For $x \in X$, we define the *greedy ordering for* x to be the unique map $\rho : \mathbb{N} \to \mathbb{N}$ such that $\{j : x_j^*(x) \neq 0\} \subseteq \rho(\mathbb{N})$ and such that if $j < k$ then

$$|x^*_{\rho(j)}(x)| > |x^*_{\rho(k)}(x)| \text{ or } |x^*_{\rho(j)}(x)| = |x^*_{\rho(k)}(x)| \text{ and } \rho(j) < \rho(k).$$

(To ensure uniqueness of ρ we suppose that ρ is bijective when $\{j : x_j^*(x) \neq 0\}$ is finite.) The m-th greedy approximation to x is then given by

$$G_m(x) = \sum_{j=1}^{m} x^*_{\rho(j)}(x) x_{\rho(j)}\,.$$

The error in the *best* m-term approximation is given by

$$\sigma_m(x) := \inf\left\{ \left\| x - \sum_{j\in A} \alpha_j x_j \right\| : \ |A| = m,\ \alpha_j \in \mathbb{R} \right\}.$$

A basis is called *greedy* if there exists a constant C such that for any $x \in X$ and $m \in \mathbb{N}$ we have

$$\|x - G_m(x)\| \leq C\sigma_m(x)\,.$$

The least such constant C is called the greedy constant of the basis and denoted $\mathrm{gc}(x_n)$.

A normalized basis is called *democratic* if there exists a constant Δ such that

$$\left\| \sum_{k\in B} e_k \right\| \leq \Delta \left\| \sum_{k\in A} e_k \right\| \quad \text{if } |B| \leq |A|\,.$$

The least such constant Δ is called the democratic constant and denoted $\Delta(x_n)$. If (x_n) is democratic then for any finite $A \subset \mathbb{N}$ we have

$$\left\| \sum_{n\in A} x_n \right\| \approx \phi(|A|),$$

where $\phi(n) = \sup_{|A|\le n} \left\| \sum_{j\in A} x_j \right\|$ is the *fundamental function* of (x_n).

A basis is *unconditional* if there exists a constant C such that for every sequence of signs $\varepsilon_n \in \{-1,+1\}$ and for every sequence of scalars (a_n) we have

$$\left\| \sum_{n=1}^{\infty} \varepsilon_n a_n x_n \right\| \le C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|.$$

The least such constant C is called the unconditional basis constant and denoted $\mathrm{ubc}(x_n)$.

Since the order of the elements of an unconditional basis is not particularly important in this context, we shall say that two normalized unconditional bases B and B' are equivalent if they are equivalent after rearrangement, i.e. if there exist enumerations (x_n) and (x'_n) of B and B' such that

$$\left\| \sum_{n\in\mathbb{N}} a_n x_n \right\| \approx \left\| \sum_{n\in\mathbb{N}} a_n x'_n \right\|$$

for all scalars (a_n). We shall also say that B and B' are *non-subequivalent* when B is not equivalent to a subsystem of B' and vice-versa.

We employ standard Banach space notation and terminology throughout (see [6]). For the convenience of the reader, however, we mention the notation that is used most heavily. We write $X \sim Y$ when X and Y are linearly isomorphic Banach spaces, and we say that X and Y are C-isomorphic if there exists a surjective isomorphism $T : X \to Y$ with $\|T\|\|T^{-1}\| \le C$. The Banach-Mazur distance $d(X,Y)$ is the infimum of such C. The closed unit ball of a Banach space X is denoted B_X, and the closed linear span of a sequence (x_n) in X is denoted $[x_n]$. For a sequence $(X_n, \|\cdot\|_n)$ of Banach spaces, the direct sum $(\sum_{n=1}^{\infty} \oplus X_n)_p$ is the space of all sequences (x_n) $(x_n \in X_n)$ equipped with the norm

$$\|(x_n)\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|_n^p \right)^{1/p}.$$

The following characterization of greedy bases can be found in [5].

Theorem A. *A basis (x_n) is greedy if and only if it is unconditional and democratic. Moreover, the greedy constant $\mathrm{gc}(x_n)$ can be estimated using the democratic constant $\Delta(x_n)$ and the unconditional constant $\mathrm{ubc}(x_n)$:*

$$\mathrm{gc}(x_n) \le 4\Delta(x_n)(\mathrm{ubc}(x_n))^3.$$

We shall use the fact that greedy bases remain greedy under isomorphisms. More precisely, we have the following result.

Lemma 1.1. *If $(e_n)_{n\in\mathbb{N}}$ is a normalized greedy basis of X and $T : X \longrightarrow Y$ is an isomorphism. Then $f_n = Te_n/\|Te_n\|$ defines a normalized greedy basis of Y and*

(i) $\mathrm{ubc}(f_n) \leq \|T^{-1}\| \cdot \|T\| \cdot \mathrm{ubc}(e_n)$,

(ii) $\Delta(f_n) \leq (\|T^{-1}\| \cdot \|T\| \cdot \mathrm{ubc}(e_n))^2 \Delta(e_n)$.

Proof. For any signs (ε_n) and bounded sequence of scalars (a_n) we obtain

$$\begin{aligned}
\left\|\sum_{n\in\mathbb{N}} \varepsilon_n a_n f_n\right\|_Y &= \left\|\sum_{n\in\mathbb{N}} \varepsilon_n a_n T e_n\right\| Te_n\|^{-1}\|_Y \\
&= \left\|T\left(\sum_{n\in\mathbb{N}} \varepsilon_n a_n e_n\right\| Te_n\|^{-1}\right)\|_Y \\
&\leq \mathrm{ubc}(e_n)\cdot\|T\|\cdot\left\|\sum_{n\in\mathbb{N}} a_n e_n\right\| Te_n\|^{-1}\|_X \\
&\leq \mathrm{ubc}(e_n)\max_{n\in\mathbb{N}}|a_n|\cdot\|T\|\cdot\|T^{-1}\|\cdot\left\|\sum_{n\in\mathbb{N}} a_n f_n\right\|_Y
\end{aligned}$$

This concludes (i). To see (ii), take finite sets $A, B \subset \mathbb{N}$ with $|A| \leq |B|$. Then we have

$$\begin{aligned}
\left\|\sum_{n\in B} f_n\right\|_Y &= \left\|T\left(\sum_{n\in B} e_n\|Te_n\|^{-1}\right)\right\|_Y \\
&\leq \|T\|\cdot\mathrm{ubc}(e_n)\cdot\max_{n\in B}\|Te_n\|^{-1}\cdot\left\|\sum_{n\in B} e_n\right\|_X \\
&\leq \|T\|\cdot\|T^{-1}\|\cdot\mathrm{ubc}(e_n)\cdot\Delta(e_n)\cdot\left\|\sum_{n\in A} e_n\right\|_X \\
&\leq \|T\|^2\cdot\|T^{-1}\|\cdot\mathrm{ubc}(e_n)^2\cdot\Delta(e_n)\cdot\left\|\sum_{n\in A} e_n\|Te_n\|^{-1}\right\|_X \\
&= \|T\|^2\cdot\|T^{-1}\|\cdot\mathrm{ubc}(e_n)^2\cdot\Delta(e_n)\cdot\left\|\sum_{n\in A} T^{-1}f_n\right\|_X \\
&\leq \|T\|^2\cdot\|T^{-1}\|^2\cdot\mathrm{ubc}(e_n)^2\Delta(e_n)\cdot\left\|\sum_{n\in A} f_n\right\|_Y . \qquad \square
\end{aligned}$$

The previous lemma yields at once the following result which will be used repeatedly.

Lemma 1.2. *Suppose that for a family of Banach spaces* $(X_\alpha)_{\alpha\in A}$ *and a further space* Y *we have* $\sup_{\alpha\in A} d(X_\alpha, Y) \le C$, *i.e. the* X_α *spaces are uniformly isomorphic to* Y. *Moreover, assume that each* X_α *has a normalized greedy basis* (x_n^α) *such that the greedy constants are uniformly bounded. Then* Y *has a family of normalized greedy bases* (y_n^α) *equivalent to* (x_n^α) *with uniformly bounded greedy constants.*

2. Greedy bases for ℓ_p, L_p and H_1

Let $\mathcal{S} = \{[(k-1)2^{-n}, k2^{-n}] : n \in \mathbb{N}_0, 1 \le k \le 2^n\}$ denote the collection of dyadic intervals in $[0,1]$. Then for every $I \in \mathcal{S}$, let h_I denote the L_p-normalized Haar function supported on I, i.e. $h_I(t) \equiv |I|^{-1/p}$ on the left half of I and $h_I(t) \equiv -|I|^{-1/p}$ on the right half of I. Write H_n for the dyadic intervals from the n-th Haar level, i.e. $H_n = \{I \in \mathcal{S} : |I| = 2^{-n}\}$. Then the Haar functions on each level span an isometric copy of $\ell_p^{2^n}$:

$$\left\| \sum_{I\in H_n} a_I h_I \right\|_{L_p} = \left(\sum_{I\in H_n} |a_I|^p (|I|^{-1/p})^p |I| \right)^{1/p} = \left(\sum_{I\in H_n} |a_I|^p \right)^{1/p}.$$

We will start with a useful lemma concerning equi-integrable sets in L_p. Then we will apply this lemma to the unit balls of certain finite-dimensional subspaces of $L_p[0,1]$.

Lemma 2.1. *Let* $1 \le p < \infty$. *Suppose that* F *is a linear subspace of* $L_p(\mu)$ *such that* $\{|f|^p : f \in B_F\} \subset L_1(\mu)$ *is equi-integrable. Then for any* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that for any* $f \in F$ *and* $g \in L_p(\mu)$ *with* $|\operatorname{supp}(g)| < \eta$ *we have*

$$\frac{1}{1+\varepsilon} \left(\|f\|_p^p + \|g\|_p^p \right)^{1/p} \le \|f+g\|_p \le (1+\varepsilon) \left(\|f\|_p^p + \|g\|_p^p \right)^{1/p}.$$

Proof. Suppose that B_F is equi-integrable. Pick $\varepsilon > 0$. We can assume that $\varepsilon < 1$. Take any $0 < \delta < \varepsilon$. By equi-integrability we find $\eta > 0$ such that $\mu(A) \le \eta$ implies $\int_A |f|^p d\mu \le \delta^p$ for all $f \in B_F$. Pick $f \in F$ and $g \in L_p$ such that. By homogeneity we may assume that $\|f\| = 1$. Take any $g \in L_p(\mu)$ and let $A = \operatorname{supp}(g)$. Suppose that $\mu(A) \le \eta$. We consider two cases. First, assume $\|g\| \ge \sqrt{\delta}$. Then in particular $\sqrt{\delta} \ge \delta$. Therefore,

$$\begin{aligned} \|f+g\|_p^p &= \|(f+g)|_A\|_p^p + \|f|_{A^c}\|_p^p \\ &\ge |\|g\|_p - \|f|_A\|_p|^p + (1-\delta^p) \end{aligned}$$

$$
\begin{aligned}
&= (\|g\|_p - \|f|_A\|_p)^p + 1 - \delta^p \\
&\geq (\|g\| - \delta)^p + 1 - \delta^p \\
&= \left(\|g\|_p - \sqrt{\delta}\sqrt{\delta}\right)^p + 1 - \delta^p \\
&\geq \left(\|g\|_p - \sqrt{\delta}\|g\|_p\right)^p + 1 - \delta^p \\
&= \|g\|_p^p(1-\sqrt{\delta})^p + 1 - \delta^p \\
&\geq \left(\|g\|_p^p + 1\right)\min\{(1-\sqrt{\delta})^p, 1-\delta^p\} \\
&= \left(\|g\|_p^p + 1\right)(1-\sqrt{\delta})^p .
\end{aligned}
$$

Also,

$$
\begin{aligned}
\|f+g\|_p^p &= \|(f+g)|_A\|_p^p + \|f|_{A^c}\|_p^p \\
&\leq (\|g\|_p + \|f|_A\|_p)^p + 1 \\
&\leq (\delta + \|g\|_p)^p + 1 \\
&\leq \left(\sqrt{\delta}\sqrt{\delta} + \|g\|_p\right)^p + 1 \\
&\leq \|g\|_p^p(1+\sqrt{\delta})^p + 1 \\
&\leq (\|g\|_p^p + 1)(1+\sqrt{\delta})^p .
\end{aligned}
$$

Now suppose that $\|g\|_p \leq \sqrt{\delta}$. Then

$$
\begin{aligned}
\|f+g\|_p^p &\geq \|f|_{A^c}\|_p^p \\
&\geq 1 - \delta^p \\
&= (1-\delta^{p/2})(1+\delta^{p/2}) \\
&\geq (1-\delta^{p/2})(1+\|g\|_p^p) .
\end{aligned}
$$

Moreover, in this case

$$
\begin{aligned}
\|f+g\|_p^p &= \|(f+g)|_A\|_p^p + \|f|_{A^c}\|_p^p \\
&\leq (\delta + \sqrt{\delta})^p + 1 \\
&\leq \left((\delta+\sqrt{\delta})^p) + 1\right)(1 + \|g\|_p^p) .
\end{aligned}
$$

Letting $\delta \to 0$ gives the result. □

Now we consider subsequences of the Haar system in $L_p[0,1]$ for $1 < p < \infty$. It is a well-known theorem of Gamlen and Gaudet [3] that the closed linear span of any subsequence of the Haar system is isomorphic to $L_p[0,1]$ or ℓ_p. We shall construct a subsequence of the levels of the Haar system such that the norm of the linear span of a collection of Haar functions from these levels behaves like the unit vector basis of ℓ_p when the number of

functions from each Haar level is proportional to the total number of Haar functions from the previous level in the subsequence.

Lemma 2.2. *For every $\varepsilon > 0$ there is an increasing sequence of integers (n_k) such that for all $S \subset \cup_{k=1}^{\infty} H_{n_k}$ with $|S \cap H_{n_k}| \leq 2|H_{n_{k-1}}|$ and $(a_k) \in \ell_p$ we have*

$$\frac{1}{1+\varepsilon}\left(\sum_{I\in S}|a_I|^p\right)^{1/p} \leq \left\|\sum_{I\in S} a_I h_I\right\|_{L_p} \leq (1+\varepsilon)\left(\sum_{I\in S}|a_I|^p\right)^{1/p}.$$

Proof. We construct n_k inductively. Pick $\varepsilon > 0$. Choose $\varepsilon_1 = 0$ and any $n_1 \in \mathbb{N}$. Then we have

$$\left\|\sum_{I\in H_{n_1}} a_I h_I\right\|_p = \left(\sum_{I\in H_{n_1}} |a_I|^p\right)^{1/p}.$$

Pick $\varepsilon_k > 0$ for $k \geq 2$ such that $\prod_{k=1}^{\infty}(1+\varepsilon_k) = 1+\varepsilon$. Let us assume that $n_1, \ldots, n_{k-1}$ have been chosen such that for all $S \subset \cup_{j=1}^{k-1} H_{n_j}$, with $|S \cap H_{n_j}| \leq 2|H_{n_{j-1}}|$ for $1 \leq j \leq k-1$, and for all scalars $(a_I)_{I\in S}$ we have

$$\prod_{j=1}^{k-1}(1+\varepsilon_j)^{-1}\left(\sum_{I\in S}|a_I|^p\right) \leq \left\|\sum_{i\in S} a_I h_I\right\|_p^p \leq \prod_{j=1}^{k-1}(1+\varepsilon_j)\left(\sum_{I\in S}|a_I|^p\right).$$

Given $n_1, \ldots, n_{k-1}$, apply Lemma 2.1 to ε_k and the finite-dimensional space spanned by the Haar levels $H_{n_1}, \ldots, H_{n_{k-1}}$,

$$F = \operatorname{span}\{h_I : I \in H_{n_j}, j = 1, \ldots, k-1\},$$

to obtain η_k. Then choose $n_k > n_{k-1}$ large enough to ensure that $2^{n_{k-1}-n_k+1} \leq \eta_k$. We will now verify that the sequence (n_k) has the desired property. To see this, let $S \subset \cup_{j=1}^{\infty} H_{n_j}$ be a set of dyadic intervals such that $|S \cap H_{n_k}| \leq 2|H_{n_{k-1}}|$ for all $k \in \mathbb{N}$. Then

$$|\operatorname{supp}(\sum_{I\in S\cap H_{n_k}} a_I h_I)| \leq |S \cap H_{n_k}|2^{-n_k} \leq 22^{n_{k-1}}2^{-n_k} \leq \eta_k.$$

Therefore, by the choice of η_k and Lemma 2.1

$$\left\|\sum_{j=1}^{k}\sum_{I\in S\cap H_{n_j}} a_I h_I\right\|_p^p = \left\|\sum_{j=1}^{k-1}\sum_{I\in S\cap H_{n_j}} a_I h_I + \sum_{S\cap H_{n_k}} a_I h_I\right\|_p^p$$

$$\leq (1+\varepsilon_k)\left(\left\|\sum_{j=1}^{k-1}\sum_{I\in S\cap H_{n_j}} a_I h_I\right\|_p^p + \left\|\sum_{I\in S\cap H_{n_k}} a_I h_I\right\|_p^p\right)$$
$$\leq (1+\varepsilon_k)\prod_{j=1}^{k-1}(1+\varepsilon_j)\sum_{j=1}^{k}\sum_{I\in S\cap H_{n_j}} |a_I|^p$$
$$= \prod_{j=1}^{k}(1+\varepsilon_j)\sum_{j=1}^{k}\sum_{I\in S\cap H_{n_j}} |a_I|^p.$$

A similar calculation shows that

$$\left\|\sum_{j=1}^{k}\sum_{I\in S\cap H_{n_j}} a_I h_I\right\|_{L_p}^p \geq \prod_{j=1}^{k}(1+\varepsilon_j)^{-1}\sum_{j=1}^{k}\sum_{I\in S\cap H_{n_j}} |a_I|^p. \qquad \square$$

Now we are ready to prove the main theorem of this section.

Theorem 2.1. *For $1 < p < \infty$, $p \neq 2$, there exist $\mathfrak{c}$ mutually non-subequivalent greedy bases of ℓ_p with uniformly bounded greedy constants.*

Remark 2.1. Clearly, Theorem 2.1 implies that ℓ_p has $\mathfrak{c}$ mutually non-equivalent greedy bases. It was pointed out by a referee of an earlier version of the paper that this weaker fact could be obtained by applying methods from [13, 14, 15]. In fact, the proof of Theorem 2.1 shows that the only equivalent subsequences of the bases constructed are equivalent to the unit vector basis of ℓ_p.

The proof we actually shows that the only equivalent Without this ingredient, the above conclusion could be obtained to a large extend by applying methods from [13, 14, 15]. This had been pointed out by a referee of an earlier version of this article.

Proof. First, choose (n_k) as in Lemma 2.2. Denote by $\mathcal{M}$ a continuum family of mutually almost disjoint infinite subsets of $\{n_k : k \in \mathbb{N}\}$ having the cardinality of the continuum. Such a family exists, e.g. take for each real number s a sequence of rational numbers (q_n^s) such that $q_n^s \longrightarrow s$. Then identify $\mathbb{Q}$ with the set $\{n_k\}$ and take the family $\{q_n^s : n \in \mathbb{N}\}_{s\in\mathbb{R}}$. Each element of $\mathcal{M}$ is represented as an increasing sequence $(m_k)_{k=1}^{\infty}$. Moreover, $\mathcal{M}$ has cardinality $\mathfrak{c}$ and any two distinct sequences $(m_k), (m_k') \in \mathcal{M}$ have finite intersection, i.e.

$$|\{m_k : k \in \mathbb{N}\} \cap \{m_k' : k \in \mathbb{N}\}| < \infty.$$

Fix $(m_k) \in \mathcal{M}$. For $k \in \mathbb{N}$, define $F_1 = H_{m_1}, F_2 = H_{m_2} \cup H_{m_3}, F_3 = H_{m_4} \cup H_{m_5} \cup H_{m_6}, \ldots$, etc. Let

$$\mathcal{F}_j = \operatorname*{span}_{L_p}\{h_I : I \in F_j\}.$$

By a theorem of Paul Müller [10] we have that the Banach-Mazur distances from ℓ_p are uniformly bounded, i.e. $d(\mathcal{F}_j, \ell_p^{\dim \mathcal{F}_j}) \leq K_p$ where K_p is an absolute constant. Hence,

$$\left(\sum_{j=1}^{\infty} \oplus \mathcal{F}_j\right)_p \approx \left(\sum_{j=1}^{\infty} \oplus \ell_p^{\dim \mathcal{F}_j}\right)_p \approx \ell_p$$

with uniform isomorphism constants. Write $\mathcal{I} = \cup_{k=1}^{\infty} H_{m_k}$. The sequence $(h_I)_{I \in \mathcal{I}}$ is a basis of $X = \left(\sum_{j=1}^{\infty} \oplus \mathcal{F}_j\right)_p \approx \ell_p$. First, let us show that $\operatorname{ubc}(h_I)_{I \in \mathcal{I}} \leq C_p$, where C_p is the unconditional basis constant of the Haar system in $L_p[0,1]$. For any scalars (a_I) and signs (ε_I) we have

$$\begin{aligned}
\left\|\sum \varepsilon_I a_I h_I\right\|_X &= \left(\sum_{j=1}^{\infty} \left\|\sum_{I \in F_j} \varepsilon_I a_I h_I\right\|_{L_p}^p\right)^{1/p} \\
&\leq C_p \left(\sum_{j=1}^{\infty} \left\|\sum_{I \in F_j} a_I h_I\right\|_{L_p}^p\right)^{1/p} \\
&= C_p \left\|\sum a_I h_I\right\|_X .
\end{aligned}$$

Since the Haar system is democratic in L_p with fundamental function $\varphi(n) \approx n^{1/p}$, we obtain for any finite $A \subset \cup_{k=1}^{\infty} H_{m_k}$ that

$$\begin{aligned}
\left\|\sum_{I \in A} h_I\right\|_X &= \left(\sum_{j=1}^{\infty} \left\|\sum_{I \in A \cap F_j} h_I\right\|_{L_p}^p\right)^{1/p} \\
&\approx \left(\sum_{j=1}^{\infty} |A \cap F_j|\right)^{1/p} \\
&= |A|^{1/p} .
\end{aligned}$$

Therefore, (h_I) is a democratic basis of X with democratic constant bounded by some absolute constant Δ_p.

Since $X = \left(\sum_{j=1}^{\infty} \oplus \mathcal{F}_j\right)_p$ is uniformly isomorphic to ℓ_p and (h_I) is a basis for X with uniformly bounded unconditionality and democratic constants, it follows from Theorem A and Lemma 1.2 that (h_I) corresponds to a basis of ℓ_p with uniformly bounded greedy constant.

Next we show that $(h_I)_{I\in\mathcal{I}}$ is not equivalent to the unit vector basis of ℓ_p. To see this, notice first that a subsequence of the Rademacher functions can be obtained as a block basis of $(h_I)_{I\in\mathcal{I}}$,

$$\sum_{I\in H_{m_k}} |I|^{1/p} h_I = r_{m_k}\,.$$

Write $(h_I)_{I\in\mathcal{I}} = (h_{m_k j})_{kj}$. To derive a contradiction, assume that $(h_{m_k j})_{kj}$ is equivalent to the unit vector basis of ℓ_p. For each $N \in \mathbb{N}$, let $p_N = N(N-1)/2 + 1$ and $q_N = N(N+1)/2$. Then $F_N = \cup_{k=p_N}^{q_N} H_{n_k}$ and for scalars $(a_k)_{k=p_N}^{q_N}$ we have by Khintchine's inequality

$$\begin{aligned}\left(\sum_{k=p_N}^{q_N} |a_k|^2\right)^{1/2} &\approx \left\|\sum_{k=p_N}^{q_N} a_k r_{m_k}\right\|_{L_p} \\ &= \left\|\sum_{k=p_N}^{q_N}\sum_{j=1}^{2^{m_k}} 2^{-m_k/p} a_k h_{m_k j}\right\|_{L_p} \\ &= \left\|\sum_{k=p_N}^{q_N}\sum_{j=1}^{2^{m_k}} 2^{-m_k/p} a_k h_{m_k j}\right\|_{X}\,.\end{aligned}$$

Since we are assuming that the basis $(h_{m_k j})_{kj}$ of X is equivalent to the unit vector basis of ℓ_p, we obtain that this last expression is equivalent to

$$\left(\sum_{k=p_N}^{q_N}\sum_{j=1}^{2^{m_k}} |2^{-m_k/p} a_k|^p\right)^{1/p} = \left(\sum_{k=p_N}^{q_N} 2^{m_k} |2^{-m_k/p} a_k|^p\right)^{1/p}\,.$$

Thus,

$$\left(\sum_{k=p_N}^{q_N} |a_k|^2\right)^{1/2} \approx \left(\sum_{k=p_N}^{q_N} |a_k|^p\right)^{1/p}$$

with uniform equivalence constants. Clearly, this is not possible for all $N \in \mathbb{N}$ and scalars (a_k), so we obtain the desired contradiction to our assumption that $(h_I)_{I\in\mathcal{I}}$ was equivalent to the unit vector basis of ℓ_p.

Next we show that distinct elements of $\mathcal{M}$ correspond to non-equivalent bases of ℓ_p. To see this, take distinct elements $(m_k), (m'_k) \in \mathcal{M}$ and define the collections (F_k), (F'_k), $\mathcal{I}$, and $\mathcal{I}'$ as above.

Let $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{I}'$ and let us assume that there exists a bijection $I \mapsto I'$ from $\mathcal{J}$ to $\mathcal{J}'$ such that $(h_I)_{I\in\mathcal{J}}$ and $(h_{I'})_{I'\in\mathcal{J}'}$ are equivalent subsystems, i.e.

$$\left\|\sum_{I\in\mathcal{J}} a_I h_I\right\| \approx \left\|\sum_{I'\in\mathcal{J}'} a_I h_{I'}\right\|$$

for any sequence of scalars $(a_I)_{I\in\mathcal{I}}$. For convenience let $(h_{I_k})_{k\in\mathbb{N}}$ be any enumeration of $(h_I)_{I\in\mathcal{J}}$, and let $(h_{I'_k})_{k\in\mathbb{N}}$ be the corresponding enumeration of $(h_I)_{I\in\mathcal{J}'}$ given by the bijection $I \mapsto I'$. Since (m_k) and (m'_k) are almost disjoint it follows that I_k and I'_k belong to different levels of the Haar system except for finitely many values of k. After removing this finite exceptional set, we may assume for each $k \in \mathbb{N}$ that in the standard ordering of the dyadic intervals either $I_k < I'_k$ (in which case the Haar level of I_k precedes the Haar level of I'_k) or $I'_k < I_k$.

For each k We can estimate by disjointness of (m_k) and (m'_k) :

$$|\{I'_j : I_j < I'_j\} \cap H_{n_k}| \le \sum_{j=1}^{k-1} |H_{n_j}| \le 2^{n_{k-1}+1} = 2|H_{n_{k-1}}|\,.$$

Therefore, we have

$$|\{I'_j : I_j < I'_j\} \cap H_{n_k}| \le 2|H_{n_{k-1}}|\,.$$

and Lemma 2.2 applies. Thus,

$$\begin{aligned}\left\|\sum_{\{j:I_j<I'_j\}} a_j h_{I_j}\right\|_X &\approx \left\|\sum_{\{j:I_j<I'_j\}} a_j h_{I'_j}\right\|_{X'} \\ &= \left(\sum_{k=1}^{\infty}\left\|\sum_{F'_k\cap\{j:I_j<I'_j\}} a_j h_{I'_j}\right\|_{L_p}^p\right)^{1/p} \\ &\approx \left(\sum_{k=1}^{\infty}\sum_{F'_k\cap\{j:I_j<I'_j\}} |a_j|^p\right)^{1/p} = \left(\sum_{\{j:I_j<I'_j\}} |a_j|^p\right)^{1/p}.\end{aligned}$$

Similarly,

$$\left\| \sum_{\{j:I_j>I'_j\}} a_j h_{I_j} \right\|_X \approx \left(\sum_{\{j:I_j>I'_j\}} |a_j|^p \right)^{1/p}.$$

Using the unconditionality of (h_{I_j}), we obtain

$$\begin{aligned} \left\| \sum_{j\in\mathbb{N}} a_j h_{I_j} \right\|_X &\approx \left\| \sum_{\{j:I_j<I'_j\}} a_j h_{I_j} \right\|_X + \left\| \sum_{\{j:I_j>I'_j\}} a_j h_{I_j} \right\|_X \\ &\approx \left(\sum_{\{j:I_j<I'_j\}} |a_j|^p \right)^{1/p} + \left(\sum_{\{j:I_j>I'_j\}} |a_j|^p \right)^{1/p} \\ &\approx \left(\sum_{j\in\mathbb{N}} |a_j|^p \right)^{1/p}. \end{aligned}$$

Thus we have proved that a subsystem $(h_I)_{I\in\mathcal{J}}$ is equivalent to a subsystem $(h_{I'})_{I'\in\mathcal{J}'}$ if and only if they are both equivalent to the unit vector basis of ℓ_p. But we have already seen that the system $(h_I)_{I\in\mathcal{I}}$ is not equivalent to the unit vector basis of ℓ_p. Consequently, $(h_I)_{I\in\mathcal{I}}$ and $(h_{I'})_{I'\in\mathcal{I}'}$ are not equivalent to each other. □

Remark 2.2. The proof shows that distinct systems $(h_I)_{I\in\mathcal{I}}$ and $(h_{I'})_{I'\in\mathcal{I}'}$ are non-equivalent in the strong sense that their only common equivalent subsystems are equivalent to the unit vector basis of ℓ_p. On the other hand, it is well known that every basis of ℓ_p or L_p contains a subsequence that is equivalent to the unit vector basis of ℓ_p.

Now we look at the case of $L_p[0,1]$. We need the following 'uniform' version of a theorem of Gamlen and Gaudet [3]. We have not been able to find an explicit statement of this result in the literature, but the proof is essentially contained in [7]. In particular, the proof requires the uniform version of the 'Pełczyński decomposition' argument that is explained in [6, p. 56].

Theorem B ([3]). *Let (m_j) be any increasing sequence of positive integers and let X_p be the closed linear span of $\{h_I : I \in \cup_{k=1}^{\infty} H_{m_k}\}$ in $L_p[0,1]$. Then, for $1 < p < \infty$, $d(X_p, L_p[0,1]) \le K_p$, where K_p is an absolute constant.*

Theorem 2.2. *For $1 < p < \infty, p \neq 2$, there exist $\mathfrak{c}$ mutually non-subequivalent greedy bases of L_p with uniformly bounded greedy constants.*

Proof. Let $(h_{n_k j})_{kj}$ be a sequence of Haar functions as given by Lemma 2.2. As before, let $\mathcal{M}$ be a family of mutually almost disjoint infinite subsets of $\{n_k : k \in \mathbb{N}\}$ having the cardinality of the continuum. For each $(m_k) \in \mathcal{M}$, we consider the set of Haar functions $\mathcal{I} = \{h_I : I \in \cup_{k=1}^{\infty} H_{m_k}\}$. The greedy constant of each $\mathcal{I}$ is clearly bounded above by the greedy constant of the Haar basis for $L_p[0,1]$. By Theorem B the closed linear span of each $\mathcal{I}$ is K_p-isomorphic to $L_p[0,1]$. So by Lemma 1.2 each $\mathcal{I}$ gives rise to a greedy basis of $L_p[0,1]$ with uniformly bounded greedy constant.

Next we show that distinct elements $(m_k), (m'_k) \in \mathcal{M}$ correspond to non-equivalent unconditional bases $\mathcal{I}$ and $\mathcal{I}'$. The argument is very similar to the ℓ_p case. Let $\mathcal{J} \subseteq \mathcal{I}$ and $\mathcal{J}' \subseteq \mathcal{I}'$, and as before let us assume that there exists a bijection $I \mapsto I'$ from $\mathcal{J}$ to $\mathcal{J}'$ such that $(h_I)_{I\in\mathcal{J}}$ and $(h_{I'})_{I'\in\mathcal{J}'}$ are equivalent subsystems. Let $(h_{I_k})_{k\in\mathbb{N}}$ be any enumeration of $(h_I)_{I\in\mathcal{J}}$ and let $(h_{I'_k})_{k\in\mathbb{N}}$ be the corresponding enumeration of $(h_{I'})_{I'\in\mathcal{J}'}$ given by the bijection $I \mapsto I'$. Since (m_k) and (m'_k) are almost disjoint it follows that I_k and I'_k belong to different levels of the Haar system except for finitely many values of k. By Lemma 2.2 we have as before that

$$\left\| \sum_{\{k:I_k<I'_k\}} a_k h_{I_k} \right\|_{L_p} \approx \left(\sum_{\{k:I_k<I'_k\}} |a_k|^p \right)^{1/p}$$

and

$$\left\| \sum_{\{k:I_k>I'_k\}} a_k h_{I_k} \right\|_{L_p} \approx \left(\sum_{\{k:I_k>I'_k\}} |a_k|^p \right)^{1/p}.$$

Thus, by unconditionality, we have

$$\left\| \sum_{k\in\mathbb{N}} a_k h_{I_k} \right\|_{L_p} \approx \left(\sum_{k\in\mathbb{N}} |a_k|^p \right)^{1/p}.$$

This proves that the subsystems $(h_I)_{I\in\mathcal{J}}$ and $(h_{I'})_{I'\in\mathcal{J}'}$ are equivalent if and only if they are both equivalent to the unit vector basis of ℓ_p. Consequently, $(h_I)_{I\in\mathcal{I}}$ and $(h_{I'})_{I'\in\mathcal{I}'}$ are not equivalent to each other.

Note that an unconditional basis in L_p always has a subsequence equivalent to the unit vector basis of ℓ_p. Also, there are greedy bases in L_p which are not equivalent to a subsequence of the Haar system (see [1] for $p < 2$ and [13] for $p > 2$), and hence are not equivalent to any general Haar system [4]. □

The space $L_1[0,1]$ does not have an unconditional basis. However, an analogous result holds for the dyadic Hardy space $H_1(\delta)$. The proof requires

the following analogue of Theorem B which is implicit in a theorem of Müller [9].

Theorem C ([9]). *Let (m_j) be any increasing sequence of positive integers and let X be the closed linear span of $\{h_I : I \in \cup_{k=1}^{\infty} H_{m_k}\}$ in $H_1(\delta)$. Then $d(X, H_1(\delta)) \le K$, where K is an absolute constant.*

Theorem 2.3. *There exist $\mathfrak{c}$ mutually non-subequivalent greedy bases of $H_1(\delta)$ with uniformly bounded greedy constants.*

Proof. Observe that for $x = \sum_{k=1}^{\infty} a_k h_k \in H_1(\delta)$, Khintchine's inequality gives

$$\|x\|_{H_1} = \left\| \left(\sum_{k=1}^{\infty} |a_k|^2 h_k^2 \right)^{1/2} \right\|_{L_1} \approx \left\| \sum_{k=1}^{\infty} a_k h_k(t) r_k(s) \right\|_{L_1([0,1]^2)},$$

where (r_k) are the Rademacher functions. Notice that $|\operatorname{supp}(h_k(t)r_k(s))| = |\operatorname{supp}(h_k)|$, so Lemma 2.1 applies with $p = 1$. So we may choose a sequence (n_k) satisfying the analogue of Lemma 2.2 for $H_1(\delta)$. As before, let $\mathcal{M}$ be a maximal family of mutually almost disjoint infinite subsets of $\{n_k : k \in \mathbb{N}\}$ having the cardinality of the continuum. For each $(m_k) \in \mathcal{M}$, we consider the set of Haar functions $\mathcal{I} = \{h_I : I \in \cup_{k=1}^{\infty} H_{m_k}\}$. The greedy constant of each $\mathcal{I}$ is clearly bounded above by the greedy constant of the Haar basis for $H_1(\delta)$. By Theorem C, the closed linear span of each $\mathcal{I}$ is K-isomorphic to $H_1(\delta)$. So by Lemma 1.2 each $\mathcal{I}$ gives rise to a greedy basis of $H_1(\delta)$ with a uniformly bounded greedy constant. Analogously to the cases ℓ_p and L_p, distinct elements of $\mathcal{M}$ correspond to non-equivalent unconditional bases of $H_1(\delta)$ since equivalent subsequences of these bases are equivalent to the unit vector basis of ℓ_1. $\square$

3. Almost greedy bases

A basis (x_n) is called *almost greedy* as introduced in [1] if there exists a constant C such that for all $x \in X$ and $n \in \mathbb{N}$ we have

$$\|x - G_m(x)\| \le C \inf \left\{ \left\| x - \sum_{j \in A} x_j^*(x) x_j \right\| : |A| = m \right\}.$$

Clearly, a greedy basis is almost greedy. It was proved in [2] that X has an almost greedy basis if X has a basis and contains a complemented subspace S with a symmetric basis and $S \not\approx c_0$. It follows that almost

greedy bases are much more plentiful than greedy bases. However, from the standpoint of $m-$term approximation, almost greedy bases are almost as good as greedy bases as indicated by the following theorem that was proved in [2].

Theorem 3.1. *Let (x_n) be an almost greedy bases for X. For $x \in X$ and $m \geq 1$ let $\tilde{G}_m(x)$ be the best approximation to x from the linear span of $\{x_{\rho(1)}, \ldots, x_{\rho(m)}\}$. Then*

$$\|x - \tilde{G}_m(x)\| \leq C\sigma_m(x)$$

where C is a constant that depends only on the basis (x_n).

This theorem shows that for an almost greedy basis the first m basis elements chosen by the 'thresholding greedy algorithm' span an almost optimal subspace for finding $m-$term approximation.

Our main results displays an abundance of almost greedy bases (in the absence of greedy bases).

Theorem 3.2. *Suppose that X is an infinite-dimensional Banach space with a basis, and let $1 \leq p < \infty$. Then there exist $\mathfrak{c}$ mutually non-subequivalent almost greedy bases of $X \oplus \ell_p$.*

Proof. The existence of a conditional basis $(f_j)_{j=1}^{\infty}$ for X was established in [12]. Thus, there are integers $0 = M_0 < M_1 < \ldots$ such that the equivalence constant between the unit vector basis of $\ell_p^{m_j}$ and $(f_k)_{k=M_{j-1}+1}^{N_j}$ where $n_j = N_j - N_{j-1}$ approaches infinity as j increases. The construction in [2] gives the following. For every increasing sequence of integers (m_k) there is a basis of $X \oplus \ell_p$ of the form $(e_{k,j})$, where $(e_{k,j})_{j=1}^{m_k}$ is a basis for $F_k = \ell_p^{m_k} \oplus [f_j]_{j=N_{k-1}+1}^{N_k}$ and $\sum \oplus F_k$ is a F.D.D. for $X \oplus \ell_p$. In addition, $(f_j)_{j=N_{k-1}+1}^{N_k}$ will be represented as a block basis of $(e_{k,j})_{j=1}^{m_k}$. Moreover, provided that (m_k) is sufficiently rapidly increasing, the construction from [2] gives that if $\Lambda \subseteq \cup_{l \geq k}(e_{l,j})_{j=1}^{m_l}$ and $|\Lambda| \leq m_1 + m_2 + \cdots + m_{k-1}$ then $(e_{l,k})_{(l,k)\in\Lambda}$ is equivalent to the unit vector basis of $\ell_p^{|\Lambda|}$. Now, consider a family $\mathcal{M}$ of mutually almost disjoint subsequences of such a sufficiently increasing sequence (m_k). For each $(m_k) \in \mathcal{M}$ we do the above construction to get a basis $\cup_{k=1}^{\infty}(e_{k,j})_{j=1}^{m_k}$ of $X \oplus \ell_p$. Suppose that (m_k) and (m'_k) are two members of $\mathcal{M}$. Then, for some $N \in \mathbb{N}$ we have that $(m_k)_{k\geq N}$ and $(m'_k)_{k\geq N}$ are disjoint. Suppose $k > N$ and $(e_{k,j})_{j=1}^{m_k}$ is $C-$equivalent to

$$\{e'_{(k,j)} : (k,j) \subseteq \Lambda\} \subseteq \cup_{k=1}^{\infty}(e_{k,j})_{j=1}^{m'_k}$$

Then $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1 = \{e'_{k,j} : m'_k > m_k\}$ and $\Lambda_2 = \{e'_{k,j} : m'_k < m_k\}$

Hence, $(e'_{k,j})_{(k,j)\in\Lambda_1}$ is uniformly equivalent to the unit vector basis of $\ell_p^{|\Lambda_1|}$ and $(e'_{k,j})_{(k,j)\in\Lambda_2}$ is equivalent to the unit vector basis of $\ell_p^{|\Lambda_2|}$. Since $\sum\limits_{k=1}^{\infty} \oplus F'_k$ is a F.D.D. we obtain

$$\left\| \sum_{(k,j)\in\Lambda} a_{k,j} e'_{k,j} \right\| \approx \max \left(\left\| \sum_{(k,j)\in\Lambda_1} a_{k,j} e'_{k,j} \right\|, \left\| \sum_{(k,j)\in\Lambda_2} a_{k,j} e'_{k,j} \right\| \right)$$
$$\approx \left(\sum_{(k,j)\in\Lambda} |a_{k,j}|^p \right)^{1/p}.$$

Therefore, $(e_{k,j})_{j=1}^{m_k}$ is equivalent to the unit vector basis of $\ell_p^{m_k}$. This is a contradiction for large k since $(f_j)_{j=M_{k-1}+1}^{M_k}$ is equivalent to a block basis of $(e_{k,j})_{j=1}^{m_k}$. □

Corollary 3.1. *Suppose that X has a basis and contains a complemented copy of ℓ_p for $1 \le p < \infty$. Then X has a continuum of mutually non-subequivalent almost greedy bases.*

Remark 3.1. These bases are conditional. Recall that two conditional bases (e_k) and (f_k) are *affinely equivalent* if there exists a sequence (a_k) such that for all (b_k) we have $\left\| \sum\limits_{k=1}^{\infty} b_k e_k \right\| \approx \left\| \sum\limits_{k=1}^{\infty} a_k b_k f_k \right\|$ (see [12]). The bases constructed above are not affinely subequivalent in the sense that no subsequences from two bases are affinely equivalent, so we may replace "non-subequivalent" by "non-affinely-subequivalent" in the statement of Corollary 3.1 .

Since L_1 contains a complemented copy of ℓ_1 we can conclude the following.

Corollary 3.2. *L_1 has a continuum of mutually non-subequivalent almost greedy bases.*

For $X = \ell_p$ we can prove the analogue of the result stated in Remark 2.1. The proof is similar to that of Theorem 2.1, so we omit the details.

Corollary 3.3. *Let $1 \le p < \infty$. Then ℓ_p has a continuum of conditional almost greedy bases so that their only common subsequences are equivalent to the unit vector basis of ℓ_p.*

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