New proof of Nagnibida's theorem

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Abstract. Using the Duhamel product for holomorphic functions we give a new proof of Nagnibida's theorem on unicellularity of integration operator $J_\alpha, (J_\alpha f)(z) = \int_\alpha^z f(t)dt$, acting in the space $\mathcal{Hol}(\Omega)$.

Let α be a fixed complex number, Ω be a simple-connected domain in the plane containing the number α , and assume that Ω is star-shaped with respect to the α in the sense that $z \in \Omega$ implies $\lambda z + (1 - \lambda)\alpha \in \Omega$ for each $\lambda \in (0, 1)$. Let $\mathcal{Hol}(\Omega)$ be the vector space of all functions holomorphic in Ω with the topology of compact convergence, and let J_α denotes the integration operator acting in $\mathcal{Hol}(\Omega)$ by formula

$$(J_\alpha f)(z) = \int_\alpha^z f(t)dt,$$

where integral is taken on the line segment with the ends α and z .

We denote by I^p the set of functions with a zero at the point α of order greater than or equal p , i.e.,

$$I^p \stackrel{\text{def}}{=} \left\{ f \in \mathcal{Hol}(\Omega) : f(\alpha) = f'(\alpha) = \dots = f^{(p-1)}(\alpha) = 0 \right\}$$

It is clear that all the subspaces I^p , $p \geq 1$, are closed J_α -invariant subspaces (i.e., $I^p \in \text{Lat}(J_\alpha)$, $p \geq 1$) and if $p > q$, then $I^p \subset I^q$. The operator J_α has been extensively studied by Nagnibida [?].

In this short communication we give a new proof of the following theorem of Nagnibida [3, Theorem 5] on unicellularity of the operator J_α in the space $\mathcal{Hol}(\Omega)$. We recall that an operator A acting in the topological vector space X is called unicellular if its lattice of invariant subspaces $\text{Lat}(A)$ is linearly ordered with respect to inclusion.

Theorem. $\text{Lat}(J_\alpha) = \{I^p : p \geq 1\}$, i.e., the operator J_α is unicellular in the space $\mathcal{Hol}(\Omega)$.

Proof. In the proof of theorem we shall essentially use the similar arguments of the paper [?]. For every $f, g \in \mathcal{Hol}(\Omega)$ let us define the following product:

$$(1) \quad \begin{aligned} \left(f \underset{\alpha}{\circledast} g \right)(z) &\stackrel{\text{def}}{=} \frac{d}{dz} \int_{\alpha}^z f(z + \alpha - t)g(t)dt \\ &= \int_{\alpha}^z f'(z + \alpha - t)g(t)dt + f(\alpha)g(z), \end{aligned}$$

where integral is taken on the line segment with the ends α and z . (For $\alpha = 0$ this is the so-called Duhamel product, see [?]). Clearly $f \underset{\alpha}{\circledast} g$ also belongs to $\mathcal{Hol}(\Omega)$, and with this multiplication $\mathcal{Hol}(\Omega)$ becomes an algebra. One can use results of operational calculus [?], [?] and analytic continuation to show that $\mathcal{Hol}(\Omega)$ is commutative and associative (it is actually clear from (1) that $f(z) \equiv 1$ is the unit for $\mathcal{Hol}(\Omega)$, and a simple change of variables shows commutativity). It is clear from (1) that $J_\alpha f(z) = (z - \alpha) \underset{\alpha}{\circledast} f(z)$. Then by considering this and the fact that the system $\{(z - \alpha)^n\}_{n=0}^{\infty}$ is complete in $\mathcal{Hol}(\Omega)$ (which is obtained from the well-known completeness criterion of Banach in the Frechet spaces and classical Runge's approximation theorem), it is not difficult to show that the problem of description of all closed J_α -invariant subspaces is equivalent to the description of all closed ideals of the algebra $\mathcal{Hol}(\Omega)$ with multiplication as $\underset{\alpha}{\circledast}$. We shall now describe the all closed ideals of the

algebra $\left(\mathcal{Hol}(\Omega), \otimes_{\alpha}\right)$ by showing that any nontrivial closed ideals has the form I^p for some $p \geq 1$. The proof will be done in several steps:

Step 1. Let $f \in \left(\mathcal{Hol}(\Omega), \otimes_{\alpha}\right)$. Then f is $\otimes_{\alpha}$ -invertible if and only if $f(\alpha) \neq 0$.

Proof. Since $\left(f \otimes_{\alpha} g\right)(\alpha) = f(\alpha)g(\alpha)$, it is clear that if f is $\otimes_{\alpha}$ -invertible, then $f(\alpha) \neq 0$. We now prove that if $f(\alpha) \neq 0$ then f is $\otimes_{\alpha}$ -invertible. Without loss of generality, we assume that $f(\alpha) = 1$. Then, obviously, $f(z) = 1 - h(z)$, where $h(z) = 1 - f(z)$ and $h(0) = 0$. Let Ω' be an arbitrary bounded domain, star-shaped with respect to the point α , and such that its closure is contained in Ω . Choose the number $M > 0$ such that $|h'(z)| \leq M$ for $z \in \Omega'$. Then

$$|h(z)| = \left| \int_{\alpha}^z h'(t) dt \right| \leq Mr_{\alpha}$$

for $z \in \Omega'$, $|z - \alpha| = r_{\alpha}$. Let $h_{\alpha}^{[n]}$ denote the $\otimes_{\alpha}$ -product of h with itself n times for $n \geq 0$, where $h_{\alpha}^{[0]}(z) \equiv 1$. We shall prove by induction that

$$|h_{\alpha}^{[n]}(z)| \leq \frac{M^n r_{\alpha}^n}{n!}$$

and

$$\left| \frac{d}{dz} h_{\alpha}^{[n]}(z) \right| \leq \frac{M^n r_{\alpha}^{n-1}}{(n-1)!}$$

for $z \in \Omega'$. Indeed, for this purpose, assume that

$$|h_{\alpha}^{[k]}(z)| \leq \frac{M^k r_{\alpha}^k}{k!}$$

and

$$\left| \frac{d}{dz} h_{\alpha}^{[k]}(z) \right| \leq \frac{M^k r_{\alpha}^{k-1}}{(k-1)!}$$

for $z \in \Omega'$. Then, by considering that $h(\alpha) = 0$, we obtain that

$$|h_{\alpha}^{[k+1]}(z)| = \left| \frac{d}{dz} \int_{\alpha}^z h(z + \alpha - t) h_{\alpha}^{[k]}(t) dt \right|$$

$$\begin{aligned}
&= \left| \int_{\alpha}^z h'(z + \alpha - t) h_{\alpha}^{[k]}(t) dt \right| \\
&\leq \int_{\alpha}^z |h'(z + \alpha - t)| |h_{\alpha}^{[k]}(t)| |dt| \\
&\leq \frac{M^{k+1}}{k!} \int_{\alpha}^z |t - \alpha|^k |d(t - \alpha)| \\
&= \frac{M^{k+1}}{k!} \int_0^{r_{\alpha}} \rho_{\alpha}^k d\rho_{\alpha} \\
&= \frac{M^{k+1} r_{\alpha}^{k+1}}{(k+1)!}
\end{aligned}$$

and

$$\begin{aligned}
\left| \frac{d}{dz} h_{\alpha}^{[k+1]}(z) \right| &= \left| \int_{\alpha}^z h''(z + \alpha - t) h_{\alpha}^{[k]}(t) dt + h'(\alpha) h_{\alpha}^{[k]}(z) \right| \\
&= \left| \int_{\alpha}^z h'(z + \alpha - t) \frac{d}{dt} h_{\alpha}^{[k]}(t) dt \right| \\
&\leq \frac{M^{k+1}}{(k-1)!} \int_0^{r_{\alpha}} \rho_{\alpha}^{k-1} d\rho_{\alpha} \\
&= \frac{M^{k+1} r_{\alpha}^k}{k!}
\end{aligned}$$

Thus the series $g(z) = \sum_{n=0}^{\infty} h_{\alpha}^{[n]}(z)$ and $\sum_{n=0}^{\infty} \frac{d}{dz} h_{\alpha}^{[n]}(z)$ are majorized respectively by the series

$$\sum_{n=0}^{\infty} \frac{M^n r_{\alpha}^n}{n!} = e^{Mr_{\alpha}} \leq e^{Md}$$

and

$$M \sum_{n=1}^{\infty} \frac{M^{n-1} r_{\alpha}^{n-1}}{(n-1)!} = Me^{Mr_{\alpha}} \leq Me^{Md},$$

where d is the diameter of Ω' . Consequently, g is holomorphic in Ω' and because of the arbitrariness of Ω' , g is holomorphic in Ω . Finally

$$f \underset{\alpha}{\circledast} g = (1 - h) \underset{\alpha}{\circledast} g = (1 - h) \underset{\alpha}{\circledast} \sum_{n=0}^{\infty} h_{\alpha}^{[n]} = \mathbf{1}.$$

Step 2. Let $f, g \in \mathcal{Hol}(\Omega)$. Suppose f has a zero at the point α of order p , g has a zero of order q , and $p \leq q$. Then there exists $h \in \mathcal{Hol}(\Omega)$ with a zero of order $q - p$ such that $f \underset{\alpha}{\circledast} h = g$.

Proof. Since $f^{(p)} \in \mathcal{Hol}(\Omega)$ and $f^{(p)}(\alpha) \neq 0$, by Step 1 there exists $\tilde{h} \in \mathcal{Hol}(\Omega)$ such that $f^{(p)} \underset{\alpha}{\circledast} \tilde{h} = \mathbf{1}$. Then $g = g \underset{\alpha}{\circledast} f^{(p)} \underset{\alpha}{\circledast} \tilde{h}$. From (1) and integration by parts p times we obtain $g \underset{\alpha}{\circledast} f^{(p)} = g^{(p)} \underset{\alpha}{\circledast} f$. Thus $g = g^{(p)} \underset{\alpha}{\circledast} \tilde{h} \underset{\alpha}{\circledast} f = f \underset{\alpha}{\circledast} h$, where $h \stackrel{\text{def}}{=} g^{(p)} \underset{\alpha}{\circledast} \tilde{h}$.

Step 3. $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ is a local ring and any nontrivial closed ideal in $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ has the form I^p for some $p \geq 1$.

Proof. If $f \in \left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ and $f(\alpha) = 0$, then it follows from (1) that $\left(f \underset{\alpha}{\circledast} g\right)(\alpha) = 0$ for $g \in \left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$. Thus I^1 form an ideal. From the Step 1 this ideal must be maximal and is only maximal ideal of $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$. Thus $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ is local ring. Each I^p is an ideal (closed) of $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ as can be seen from (1). We now prove that other nontrivial ideals not exist. Indeed, if I is any nontrivial ideal in $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$, then I has an element f of lowest order p . By Step 2, if $g \in I^p$, then there exists $h \in \mathcal{Hol}(\Omega)$ such that $g = h \underset{\alpha}{\circledast} f$, and hence $I^p \subseteq I$. If $I^p \neq I$, then I has an element of order less than or equal to $p - 1$, a contradiction. Thus $I = I^p$. Consequently, the only ideals of $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ are the $I^p, p \geq 1$.

In addition we observe that the sequence

$$\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right) = I^0 \supset I^1 \supset I^2 \supset \dots \supset \{0\}$$

is a filtration of $\left(\mathcal{Hol}(\Omega), \underset{\alpha}{\circledast}\right)$ and that I^p/I^{p+1} is isomorphic to the complex numbers.

Thus, by Step 3 the proof of theorem is established. $\square$

Remark. Note that the invertibility criterion with respect to the Duhamel product $\underset{\alpha}{\circledast}$ can also be easily deduced from the corresponding result of Wigley [4, Theorem] for $\alpha = 0$. Indeed, there is a canonical algebra isomorphism Γ between the spaces $\mathcal{Hol}(\Omega)$ and $\mathcal{Hol}(\Omega_0)$ where $\Omega_0 = \Omega - \alpha$, when endowed with their corresponding Duhamel products.

The map is obviously defined by $\Gamma f(z) = f(z + \alpha)$. Besides, Γ implements a similarity between the corresponding integration operators, so from the invariant subspace point of view, it is enough to assume that $\alpha = 0$. $\square$

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