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# Generalized weighted Besov spaces on the Bessel hypergroup

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**Abstract.** In this paper we study generalized weighted Besov type spaces on the Bessel-Kingman hypergroup. We give different characterizations of these spaces in terms of generalized convolution with a kind of smooth functions and by means of generalized translation operators. Also a discrete norm is given to obtain more general properties on these spaces.

## 1. Introduction

The Besov spaces has been defined by many ways on $\mathbb{R}^n$ ([8], [4], [14], [15], [12], [13]). This theory has been generalized using Hankel transform and the weight $t^\gamma$; $\gamma > 0$ ([5], [1],). Our objective is to find weights extending the case $t^\gamma$ to define generalized weighted Besov type spaces on the Bessel-Kingman hypergroup. In the present work we fix $\alpha > -\frac{1}{2}$ and we put $\mathbb{X} = [0, +\infty[$. For $1 \leq p, q \leq \infty$ and an adequate weight $\omega$ (see [2]) we

define the Besov type spaces $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ using the harmonic analysis associated with the Bessel operator:

$$\mathcal{L}_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}; \qquad x \in ]0,\infty[.$$

For $\alpha = \frac{n-1}{2}$; $n \in \mathbb{N}$ and $n \geq 2$, the operator $\mathcal{L}_\alpha$ is the radial part of the Laplacian operator defined on $\mathbb{R}^n$. We recall that $x \to j_\alpha(\lambda x)$ ($\lambda \in [0,\infty[$) is the unique solution of the following system (see [16])

$$\begin{cases} \mathcal{L}_\alpha u = -\lambda^2 u; \\ u(0) = 1, \ \dfrac{du}{dx}(0) = 0 \end{cases}$$

where $j_\alpha$ is the normalized Bessel function of first kind and order $\alpha$. The Fourier Bessel transform for a reasonable function $f$ is given by ([16]):

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{X}} j_\alpha(\lambda x) f(x) d\mu_\alpha(x), \qquad \text{for all } \lambda \in [0,\infty[ \tag{1.1}$$

where $d\mu_\alpha(x) = \dfrac{x^{2\alpha+1}dx}{2^\alpha \Gamma(\alpha+1)}$.

It is well known that the Fourier-Bessel transform is a topological isomorphism from $S_*(\mathbb{X})$ onto itself (see [16]) where $S_*(\mathbb{X})$ is the Schwartz subspace consisting of even functions. This paper deals with generalized weighted Besov-Bessel type spaces defined on $\mathbb{X}$ and it is organized as follows. In the first section, we collect some harmonic analysis properties on the Bessel hypergroup which are given in [16]. In the second section we introduce the generalized weighted Besov-Bessel type spaces $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$, we give some properties and inclusion results of these spaces with respect to the parameters $p$, $q$ and $\omega$. Completeness and density results are also given in this section. We introduce an equivalent discrete norm replacing the group $]0,+\infty[$ by the 2-powers group $\mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\}$ to give a new characterization of the space $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ and to develop other inclusion properties. In the third section we characterize the space $\Lambda^{\omega}_{p,q}(\mathbb{X}) = \dot{\Lambda}^{\omega}_{p,q}(\mathbb{X}) \cap L^p$ under some conditions on weights by equivalent norms using the difference $\Delta_x f = T_x^{(\alpha)} f - f$ and the generalized modulus of continuity $m_p(f,x) = \sup\limits_{0 \leq y \leq x} \|\Delta_y f\|_p$ associated with the translation operators $T_x^{(\alpha)}$. In proving these results, the main tool used here is the harmonic analysis on the Bessel hypergroup.

Finally, we mention that $C$ will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

## 2. Preliminaries

In this section we recall some basic results in harmonic analysis related to the Bessel operators (see [16]). We first begin by some notations.

*Notations.* Let $\alpha > -\frac{1}{2}$ and $\mathbb{X} = [0, \infty[$. We denote by

- $\mathbb{R}_+^* = ]0, +\infty[$.
- $\mathcal{C}_*(\mathbb{X})$ the space of even continuous functions on $\mathbb{R}$.
- $\mathcal{C}_*^\infty(\mathbb{X})$ the space of functions $f : \mathbb{R} \to \mathbb{C}$, even and $\mathcal{C}^\infty$ on $\mathbb{R}$.
- $\mathcal{D}_*(\mathbb{X})$ the subspace of $\mathcal{C}_*^\infty(\mathbb{X})$ consisting of functions with compact support.
- $\mathcal{C}_{*,0}(\mathbb{X})$ the space consisting of even continuous functions tending to zero at the infinity.
- $S_*(\mathbb{X})$ is the Schwartz subspace consisting of even functions.
- $\mathcal{D}'_*(\mathbb{X})$ (resp. $S'_*(\mathbb{X})$) the topological dual of $\mathcal{D}_*(\mathbb{X})$ (resp. $S_*(\mathbb{X})$).
- $S_{*,0}(\mathbb{X})$ the subspace of functions $\psi$ in $S_*(\mathbb{X})$ such that the support of $\mathcal{F}(\psi)$ is a compact subset of $]0, +\infty[$.
- $S^1_{*,0}(\mathbb{X})$ the subset of functions $\psi$ in $S_{*,0}(\mathbb{X})$ such that

$$\int_0^\infty |\mathcal{F}\psi(r\lambda)|^2 \frac{dr}{r} = 1, \quad \text{for all } \lambda \in ]0, \infty[.$$

  These functions are known as generalized wavelets on $\mathbb{X}$.
- $L^p(d\mu_\alpha) = L^p(\mathbb{X}, d\mu_\alpha)$, $1 \le p \le \infty$, the Lebesgue space associated to the measure $d\mu_\alpha$ and $\|.\|_p$ its usual norm given by:

$$\begin{aligned} \|f\|_p &= \left(\int_{\mathbb{X}} |f(x)|^p d\mu_\alpha(x)\right)^{\frac{1}{p}}, \quad \text{if } p \in [1, \infty[ \\ \|f\|_\infty &= \operatorname*{ess\,sup}_{x \in \mathbb{X}} |f(x)|. \end{aligned}$$

- $f_r(x) = r^{-(2\alpha+2)} f(\frac{x}{r})$; $r > 0$, the dilated of the function $f$ defined on $\mathbb{X}$ preserving the integral of $f$ with respect to the measure $d\mu_\alpha$, in the sense that

$$\int_{\mathbb{X}} f_r(x) d\mu_\alpha(x) = \int_{\mathbb{X}} f(x) d\mu_\alpha(x), \quad \text{for all } r > 0 \text{ and } f \in L^1(d\mu_\alpha). \tag{2.1}$$

**Definition 2.1.** • The generalized translation operators $T_x^{(\alpha)}$, $x \ge 0$, associated with the Bessel operators are defined for appropriate functions

$f$ by (see [16]):

$$(2.2)\ T_x^{(\alpha)} f(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^{\pi} f(\sqrt{x^2+y^2+2xy\cos\theta})(\sin\theta)^{2\alpha} d\theta, \quad \text{for all } \ y \in \mathbb{X}.$$

• The generalized convolution product associated with the Bessel operators is defined for a suitable pair of functions $f$ and $g$ by (see [16]):

$$f \#\, g(x) = \int_{\mathbb{X}} T_x^{(\alpha)} f(y) g(y) d\mu_\alpha(y), \quad \text{for all } \ x \in \mathbb{X}.$$

We recall that $(\mathbb{X}, \#, id_{\mathbb{X}})$ is an hypergroup in the sense of Jewett ([10], [7]) called Bessel-Kingman hypergroup and we mention that, to make difference between the space variables, the dual variables and the dilation variables, we shall denote them respectively by $x \in \mathbb{X}$, $\lambda \in [0, \infty[$ and $r > 0$.

**Properties.** The following results are standard in the sense that they appear in monographs on harmonic analysis like ([16]). For suitable function $f : \mathbb{X} \to \mathbb{C}$ we have

(1) $T_0^{(\alpha)} f(y) = f(y)$, $\forall\ y \in \mathbb{X}$.

(2) $T_x^{(\alpha)} f(y) = T_y^{(\alpha)} f(x)$, $\forall\ x, y \in \mathbb{X}$.

(3) $T_x^{(\alpha)} f(y) = \displaystyle\int_{\mathbb{X}} W_\alpha(x,y,z) f(z) d\mu_\alpha(z)$, $\forall\ x, y \in \mathbb{X}$,
where

$$W_\alpha(x,y,z) = \begin{cases} \dfrac{2^{1-\alpha}\,\Gamma^2(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \dfrac{[(x+y)^2 - z^2]^{\alpha-\frac{1}{2}}[z^2-(x-y)^2]^{\alpha-\frac{1}{2}}}{(xyz)^{2\alpha}}, \\ \qquad \text{if } \ |x-y| < z < x+y \\ 0, \quad \text{otherwise.} \end{cases}$$

(4) Let $f$ be in $L^p(d\mu_\alpha)$, $1 \le p \le \infty$. Then for all $x \in \mathbb{X}$, the function $T_x^{(\alpha)} f$ belongs to $L^p(d\mu_\alpha)$ and we have

$$\|T_x^{(\alpha)} f\|_p \le \|f\|_p.$$

(5) For $f$ in $L^{p_1}(d\mu_\alpha)$ and $g$ in $L^{p_2}(d\mu_\alpha)$; $1 \le p_1, p_2 \le \infty$ the function $f \#\, g$ belongs to $L^{p_3}(d\mu_\alpha)$, $\frac{1}{p_1} + \frac{1}{p_1} = 1 + \frac{1}{p_3}$, and we have

$$\|f \#\, g\|_{p_3} \le \|f\|_{p_1} \|g\|_{p_2}.$$

(6) Let $f$ be in $L^1(d\mu_\alpha)$. Then the function $\mathcal{F}(f)$ belongs to $\mathcal{C}_{*,0}(\mathbb{X})$ and we have

$$\|\mathcal{F}(f)\|_\infty \le \|f\|_1$$

where $\|\mathcal{F}(f)\|_\infty = \sup_{\lambda\in[0,\infty[} |\mathcal{F}(f)(\lambda)|$.

(7) Let $f$ and $g$ in $L^1(d\mu_\alpha)$, then we have

$$\mathcal{F}(f_\# g) = \mathcal{F}(f)\mathcal{F}(g).$$

(8) Let $f$ be in $L^1(d\mu_\alpha)$. Then, for all $x \in \mathbb{X}$ and $\lambda \in [0,\infty[$, we have

$$\mathcal{F}(T_x^{(\alpha)} f)(\lambda) = j_\alpha(\lambda x)\mathcal{F}(f)(\lambda).$$

**Remark 2.1.** Let $p \geq 1$. Then for all $f \in L^p(d\mu_\alpha)$ and $\psi \in S_*(\mathbb{X})$, $\langle f, \psi\rangle$ means the value of $f \in S'_*(\mathbb{X})$ on $\psi$ and it is given by:

$$\langle f, \psi\rangle = \int_0^\infty f(x)\psi(x)d\mu_\alpha(x). \tag{2.3}$$

Hence, we identify $L^p(\mathbb{X})$ with a subspace of $S'_*(\mathbb{X})$ via the formula (2.3).

It might be observed that a long list of properties of the classical distributions on $\mathbb{R}^n$ remains valid also in our context. On the other hand it is not difficult to verify that the Fourier-Bessel transform and the Bessel operators are symmetric in the following sense

$$\int_0^\infty f\mathcal{F}(g)d\mu_\alpha = \int_0^\infty \mathcal{F}(f)gd\mu_\alpha$$

and

$$\int_0^\infty \mathcal{L}_\alpha fgd\mu_\alpha = \int_0^\infty \mathcal{L}_\alpha gfd\mu_\alpha, \text{ for all } f, g \in S_*(\mathbb{X}).$$

Therefore, we extend naturally the Fourier-Bessel transform and the operator $\mathcal{L}_\alpha$ on $S'_*(\mathbb{X})$ as follows:

For all $T \in S'_*(\mathbb{X})$ and $\psi \in S_*(\mathbb{X})$ we put

$$\langle \mathcal{F}(T), \psi\rangle = \langle T, \mathcal{F}(\psi)\rangle \quad \text{and} \quad \langle \mathcal{L}_\alpha T, \psi\rangle = \langle T, \mathcal{L}_\alpha\psi\rangle.$$

It has been mentioned in [16], p. 108 that for all $f \in \mathcal{D}'_*(\mathbb{X})$ (resp. $S'_*(\mathbb{X})$) and $\psi \in \mathcal{D}_*(\mathbb{X})$ (resp. $S_*(\mathbb{X})$), the generalized convolution product associated with Bessel operators of $f$ and $\psi$ is the function in $\mathcal{C}^\infty_*(\mathbb{X})$ given by $\psi_\# f(x) =< f, T_x^\alpha\psi >$.

**Definition 2.2.** A weight will be a measurable function $\omega : \mathbb{R}^*_+ \to \mathbb{R}^*_+$ satisfying

$$\exists\ C > 0; \quad \forall\ r, \rho > 0 \quad \Big(\frac{1}{2} \leq \frac{r}{\rho} \leq 2 \Rightarrow w(r) \leq Cw(\rho)\Big). \tag{2.4}$$

An elementary example of weight is the function defined on $\mathbb{R}_+^*$ by $w_{\gamma,\beta}(r) = r^\gamma(1+|Logr|)^\beta;\ \alpha,\beta\in\mathbb{R}$.

## 3. Generalized Weighted Besov-Bessel Type Spaces

In what follows we endow the group $\mathbb{R}_+^*$ with the invariant measure $\frac{dr}{r}$. We introduce the generalized weighted Besov-Bessel type space $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ and we give some properties.

**Definition 3.1.** Let $1\le p,\ q\le\infty,\ \psi\in S^1_{*,0}(\mathbb{X})$ and $\omega$ a weight. We define the generalized weighted Besov-Bessel type space $\dot{\Lambda}^{\omega,\psi}_{p,q}(\mathbb{X})$ as the set of all $f\in S'_*(\mathbb{X})$ such that

$$f=\int_0^\infty f_{\#}\psi_{r\#}\psi_r\frac{dr}{r} \tag{3.1}$$

and $\mathbf{A}^{\omega,\psi}_{p,q}(f)<\infty$, where

$$\mathbf{A}^{\omega,\psi}_{p,q}(f)=\begin{cases}\left(\displaystyle\int_0^\infty\Big(\frac{\|f_{\#}\psi_r\|_p}{\omega(r)}\Big)^q\frac{dr}{r}\right)^{\frac{1}{q}}, & \text{if}\quad 1\le q<\infty,\\ \operatorname*{ess\,sup}_{r>0}\Big(\dfrac{\|f_{\#}\psi_r\|_p}{\omega(r)}\Big), & \text{if}\quad q=\infty.\end{cases}$$

**Remark 3.1.** (1) We begin by mentioning that the definition of the generalized weighted Besov-Bessel type space $\dot{\Lambda}^{\omega,\psi}_{p,q}(\mathbb{X})$ given here is the same than that introduced by Chemin in the classical case for $w(r)=r^\gamma$ (see [9]) which is generalized by Bahouri, Gérard and Xu on the Heisemberg group (see [6]) and by M. Assal and H. Ben Abdallah on the Laguerre hypergroup (see [3]). We do not choose the classical definition introduced by Peetre (see [12]) in which $\dot{\Lambda}^{\omega,\psi}_{p,q}(\mathbb{X})$ is defined as a set of distributions modulo polynomials.

(2) We note here that the expression (3.1) is independent, in $S'_*(\mathbb{X})$, of the choice of $\psi$ in $S^1_{*,0}$.

(3) If $f$ belongs to $L^2(\mathbb{X})$, then (3.1) holds in $L^2(\mathbb{X})$, which is a consequence of Plancherel's formula (see [16]).

(4) The expression (3.1) is not true in $S'_*(\mathbb{X})$ if $f$ is a polynomial function on $\mathbb{X}$. Indeed in this case, for all $r>0$, we have $f_{\#}\psi_r=0$. Thus, let $\mathbb{P}_*[\mathbb{X}]$ the subspace of even polynomials, then $\mathbb{P}_*[\mathbb{X}]$ is invariant under the generalized translation operators $T_x^{(\alpha)};\ x\in\mathbb{X}$ and we have $f_{\#}\psi_r=0$ for all $\psi\in S^1_{*,0}$. To prove the above result, it suffice to take $P(y)=y^{2k};\ k\in\mathbb{N}$.

Then it holds

$$\begin{aligned} T_x^{(\alpha)}P(y) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \left(x^2+y^2+2xy\cos\theta\right)^k(\sin\theta)^{2\alpha}d\theta \\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\sum_{j=0}^{k} 2^j\mathfrak{C}_k^j\left(x^2+y^2\right)^{k-j}(xy)^j \\ &\quad\times\int_0^\pi(\cos\theta)^j(\sin\theta)^{2\alpha}d\theta. \end{aligned}$$

The result follows using the fact that $\int_0^\pi(\cos\theta)^j(sin\theta)^{2\alpha}d\theta = 0$ for all odd integer $j$. Now, using the fact that $\mathcal{F}(\psi)$ belongs to $S_{*,0}(\mathbb{X})$ and the invariance of $\mathbb{P}_*[\mathbb{X}]$ under $T_x^{(\alpha)}$ we obtain $f_\#\psi_r = 0$.

The following proposition gives the independence of the space $\dot{\Lambda}_{p,q}^{\omega,\psi}(\mathbb{X})$ on the choice of $\psi$ in $S_{*,0}^1$.

**Proposition 3.1.** *Let $1 \le p, q \le \infty$ and $\omega$ a weight. Then the space $\dot{\Lambda}_{p,q}^{\omega,\psi}(\mathbb{X})$ is independent of the choice of the function $\psi$ in $S_{*,0}^1(\mathbb{X})$.*

Not that the function $w(r) = r^\gamma$ satisfies the condition (2.4) for all $\gamma \in \mathbb{R}$.

The main idea in the proof of the above proposition is based on the following lemma.

**Lemma 3.1.** *Let $w$ a weight. Then the following assertions are equivalent*

(i) $\exists\ C > 0;\ \forall\ r,\ \rho > 0\ \left(\frac{1}{2} \le \frac{r}{\rho} \le 2 \Rightarrow w(r) \le Cw(\rho)\right)$.

(ii) *For all $\alpha, \beta$ such that $0 < \alpha < \beta$, $\exists C > 0;\ \forall r,\ \rho > 0,\ \left(\alpha \le \frac{r}{\rho} \le \beta \Rightarrow w(r) \le Cw(\rho)\right)$.*

*Proof of Proposition* 3.1. Assume $\psi$ and $\phi$ be a pair of functions belonging to $S_{*,0}^1(\mathbb{X})$. To get the desired result it suffices to prove that $\dot{\Lambda}_{p,q}^{\omega,\psi}(\mathbb{X}) \subseteq \dot{\Lambda}_{p,q}^{\omega,\phi}(\mathbb{X})$. Since $\mathcal{F}\phi$ and $\mathcal{F}\psi$ are supported by some compact subsets of $]0,\infty[$, then there exist $\alpha$, $\beta$ $(0 < \alpha < \beta)$ such that $\phi_{\rho\#}\psi_r = 0$ if $\frac{r}{\rho} \notin [\alpha, \beta]$. Moreover we have $\|\phi_{\rho\#}\psi_r\|_1 \le \|\phi\|_1\|\psi\|_1$ and we obtain for all $f \in \dot{\Lambda}_{p,q}^{\omega,\psi}(\mathbb{X})$

$$f_\#\phi_\rho = \int_{\alpha\rho}^{\beta\rho}(f_\#\psi_r)_\#\psi_{r\#}\phi_\rho\frac{dr}{r}.$$

Using the fact that $w$ satisfies (2.4) and Lemma 3.1 we get

$$\frac{\|f_\#\phi_\rho\|_p}{w(\rho)} \le C\int_\alpha^\beta \frac{\|f_\#\psi_{r\rho}\|_p}{w(r\rho)}\frac{dr}{r}.$$

And so, Minkowski's inequality leads to

$$\begin{aligned}\Big(\int_0^\infty \Big(\frac{\|f_\# \phi_\rho\|_p}{w(\rho)}\Big)^q \frac{d\rho}{\rho}\Big)^{1/q} &\leq C\int_\alpha^\beta \Big(\int_0^\infty \Big(\frac{\|f_\# \psi_{r\rho}\|_p}{w(r\rho)}\Big)^q \frac{d\rho}{\rho}\Big)^{1/q}\frac{dr}{r}\\ &= C\Big(Log\frac{\beta}{\alpha}\Big)\Big(\int_0^\infty \Big(\frac{\|f_\# \psi_\rho\|_p}{w(\rho)}\Big)^q \frac{d\rho}{\rho}\Big)^{1/q}.\end{aligned}$$

This completes the proof of the proposition. □

**Remark 3.2.** In view of their independence of the function $\psi$ in $S^1_{*,0}$ the spaces $\dot{\Lambda}^\omega_{p,q}(\mathbb{X})$; $1 \leq p,q \leq \infty$ will be denoted indifferently with or without $\psi$ which will be chosen adequately.

The following proposition deals with an elementary example of weights transmitting its homogeneity to the space $\dot{\Lambda}^\omega_{p,q}(\mathbb{X})$.

**Proposition 3.2.** *Let $1 \leq p,q \leq \infty$, and $\omega(\rho) = \rho^\gamma$; ($\gamma \in \mathbb{R}$). Then the Besov-Bessel type space $\dot{\Lambda}^\omega_{p,q}(\mathbb{X})$ is homogeneous of degree $d(p,\gamma) = \frac{2\alpha+2}{p} - \gamma$ in the sense that, for all $f \in \dot{\Lambda}^\omega_{p,q}(\mathbb{X})$*

$$\mathbf{A}^\omega_{p,q}(d_r f) = r^{\frac{2\alpha+2}{p}-\gamma}\mathbf{A}^\omega_{p,q}(f), \quad \textit{for all } \ r > 0$$

*where $d_r f$ is the distribution given by $\langle d_r f, \psi\rangle = \langle f, \psi_{1/r}\rangle$ for all $\psi \in S_*(\mathbb{X})$.*

*Proof.* Let $1 \leq p,q \leq \infty$ and $\gamma \in \mathbb{R}$. Assume $f$ in $\dot{\Lambda}^\omega_{p,q}(\mathbb{X})$, then

$$\begin{aligned}\mathbf{A}^\omega_{p,q}(d_r f) &= \Big\|\frac{\|(d_r f)_\# \psi_\rho\|_p}{\rho^\gamma}\Big\|_{L^q(\frac{d\rho}{\rho})}\\ &= r^{\frac{2\alpha+2}{p}}\Big\|\frac{\|f_\# \psi_{\frac{\rho}{r}}\|_p}{\rho^\gamma}\Big\|_{L^q(\frac{d\rho}{\rho})}\\ &= r^{\frac{2\alpha+2}{p}-\gamma}\Big\|\frac{\|f_\# \psi_\rho\|_p}{\rho^\gamma}\Big\|_{L^q(\frac{d\rho}{\rho})}\\ &= r^{\frac{2\alpha+2}{p}-\gamma}\mathbf{A}^\omega_{p,q}(f).\end{aligned}$$

□

**Definition 3.2.** Let $1 \leq p,q,q' \leq \infty$; $\frac{1}{q'} + \frac{1}{q} = 1$. We denote by $W_{\frac{2\alpha+2}{p}}$ the class of weights satisfying

(i) $w(r)r^{\frac{-(2\alpha+2)}{p}}$ belongs to $L^{q'}([1,+\infty[, \frac{dr}{r})$.

(ii) There exists $\varepsilon > 0$ such that $w(r)r^\varepsilon$ belongs to $L^{q'}([0,1], \frac{dr}{r})$.

The following proposition holds.

**Proposition 3.3.** *Let* $1 \leq p, q \leq \infty$ *and* $\omega \in W_{\frac{2\alpha+2}{p}}$ . *Then* $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ *is a Banach space.*

Remark that the weight $w(r) = r^{\gamma}$ belongs to $W_{\frac{2\alpha+2}{p}}$ for $\gamma < \frac{2\alpha+2}{p}$ which coincides with the classical case. To prove the above proposition we need the following lemmas.

**Lemma 3.2.** *Let* $h \in S_*(\mathbb{X})$ *and* $\psi \in S^1_{*,0}(\mathbb{X})$. *Then, for all* $k \in \mathbb{N}$, *there exists* $\psi_{[k]} \in S_{*,0}(\mathbb{X})$ *such that*

$$h_{\#}\psi_r = r^{2k}(\mathcal{L}^k_{\alpha}h)_{\#}(\psi_{[k]})_r \tag{3.2}$$

*where* $\mathcal{L}_{\alpha}$ *is the differential operator given in the introduction part. Furthermore there exists* $C > 0$ *such that*

$$\|h_{\#}\psi_r\|_1 \leq Cr^{2k} \quad \textit{for all } 0 \leq r \leq 1 \tag{3.3}$$

*and*

$$\|h_{\#}\psi_r\|_1 \leq C \quad \textit{for all } r \geq 1. \tag{3.4}$$

*Proof.* We obtain (3.2) by induction on $k$. The inequalities (3.3) and (3.4) follow from (3.2). □

**Lemma 3.3.** *Let* $1 \leq p, q \leq \infty$ *and* $\omega \in W_{\frac{2\alpha+2}{p}}$ . *For* $\phi$ *in* $S^1_{*,0}$, *put*

$$\Phi(g) = \int_0^{\infty} w(r)g(r)_{\#}\phi_r \frac{dr}{r}. \tag{3.5}$$

*Then* $\Phi$ *define a linear and continuous mapping from* $L^q(\mathbb{R}^*_+, L^p(\mathbb{X}), \frac{dr}{r})$ *to* $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$.

*Proof.* Let us first prove that, for $g \in L^q(\mathbb{R}^*_+, L^p(\mathbb{X}), \frac{dr}{r})$, $\Phi(g)$ define an element of $S'_*(\mathbb{X})$, that is for all $h \in S_*(\mathbb{X})$,

$$\int_0^{\infty} \Big| w(r)\langle g(r)_{\#}\phi_r, h\rangle \Big| \frac{dr}{r} < \infty.$$

Take $\psi \in S_*(\mathbb{X})$ such that $\mathcal{F}(\psi) = 1$ on $Supp\mathcal{F}\phi$. Then, using Hölder's and Young's inequalities, we obtain

$$\begin{aligned} |\langle g(r)_{\#}\phi_r, h > | &= | < g(r)_{\#}\phi_r, h_{\#}\psi_r\rangle| \\ &\leq \|g(r)_{\#}\phi_r\|_{\infty}\|h_{\#}\psi_r\|_1 \\ &\leq Cr^{-\frac{2\alpha+2}{p}}\|g(r)\|_p\|h_{\#}\psi_r\|_1. \end{aligned}$$

On the other hand, using Lemma 3.2, we get

$$\begin{aligned}
\int_0^\infty \Big| w(r) < g(r)_{\#}\phi_r, h > \Big| \frac{dr}{r} &\leq C\left\{ \int_0^1 w(r) r^{2k-\frac{2\alpha+2}{p}} \|g(r)\|_p \frac{dr}{r} \right. \\
&\quad \left. + \int_1^\infty w(r) r^{-\frac{2\alpha+2}{p}} \|g(r)\|_p \right\} \tfrac{dr}{r} \\
&\leq C\left( \int_0^1 \|g(r)\|_p^q \frac{dr}{r} \right)^{1/q} \\
&\quad \times \left( \int_0^1 \left( w(r) r^{2k-\frac{2\alpha+2}{p}} \right)^{\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}} \\
&\quad + C\left( \int_1^\infty \|g(r)\|_p^q \frac{dr}{r} \right)^{1/q} \\
&\quad \times \left( \int_1^\infty \left( w(r) r^{-\frac{2\alpha+2}{p}} \right)^{\bar{q}} \frac{dr}{r} \right)^{1/\bar{q}}.
\end{aligned}$$

where $\bar{q}$ is the conjugate exponent of $q$. Using the fact that $w$ belongs to $W_{\frac{2\alpha+2}{p}}$, we obtain for $k$ sufficiently large

$$\int_0^\infty \Big| w(r) < g(r)_{\#}\phi_r, h > \Big| \frac{dr}{r} \leq C\|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})} < \infty.$$

Now, let $\psi$ in $S^1_{*,0}$. From Minkowski's inequality and Young's inequality we have

$$\begin{aligned}
\frac{\|\Phi(g)_{\#}\psi_\rho\|_p}{w(\rho)} &\leq C\int_\alpha^\beta w(r\rho) \frac{\|g(r\rho)\|_p}{w(\rho)} \frac{dr}{r} \\
&= C\int_0^\infty \mathbb{1}_{[\alpha,\beta]}(r) \|g(r\rho)\|_p \frac{dr}{r} \\
&= C(H \star G)(\rho)
\end{aligned}$$

with $H(s) = \mathbb{1}_{[\alpha,\beta]}(s)$, $G(s) = \|g(s)\|_p$ and $H \star G$ is the convolution of $H$ and $G$ on the group $(\mathbb{R}_+^*, \frac{dr}{r})$. Now, by Young's inequality, it holds

$$\begin{aligned}
\|\Phi(g)\|_{\Lambda^w_{p,q}(\mathbb{X})} &\leq C\Big\| H \star G \Big\|_{L^q(\mathbb{R}_+, \frac{d\rho}{\rho})} \\
&\leq C\|H\|_{L^1(\mathbb{R}_+, \frac{d\rho}{\rho})} \|G\|_{L^q(\mathbb{R}_+, \frac{d\rho}{\rho})} \\
&= C\|g\|_{L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})}.
\end{aligned}$$

The lemma is proved. □

*Proof of Proposition* 3.3. Let $\psi$ in $S^1_{*,0}$ and take $\phi = \psi$ in Lemma 3.3. Then $\Phi$ defined by (3.5) is a continuous linear mapping from $L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})$ to $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$. On the other hand the operator $\Psi$ associating to $f$ in $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$ the function $\Psi(f)$ defined on $\mathbb{R}_+^*$ by:

$$\Psi(f)(r) = \frac{f_{\#}\psi_r}{w(r)}$$

is obviously a linear isometry from $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$ to $L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})$ and using the decomposition (3.1), we obtain $\Phi \circ \Psi = Id_{\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})}$. This implies

$$(\Psi \circ \Phi) \circ \Psi = \Psi \quad \text{ on } \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X}).$$

So $\Psi\Big(\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})\Big) = Ker\Big(\Psi \circ \Phi - Id_{L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})}\Big)$ is a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})$. Since $\Psi$ is an isometry, then $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$ can be identified with a closed subspace of $L^q(\mathbb{R}_+^*, L^p(\mathbb{X}), \frac{dr}{r})$. The completeness of $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$ follows. □

**Proposition 3.4.** *Let $1 \le p \le \infty$, $1 \le q < \infty$ and $\omega$ a weight. Then the subspace $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X}) \cap \mathcal{C}^{\infty}_*(\mathbb{X})$ is dense in $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$.*

*Proof.* Let $\phi \in S^1_{*,0}(\mathbb{X})$ and $f \in \dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$. Then for $\varepsilon > 1$, the function

$$f_\varepsilon = \int_{1/\varepsilon}^{\varepsilon} f_{\#}\phi_{r\#}\phi_r \frac{dr}{r}$$

is obviously $\mathcal{C}^{\infty}$ and belongs to $\dot{\Lambda}^{\gamma}_{p,q}(\mathbb{X})$. Moreover the same reasoning given in Proposition 3.1 leads to

$$\|f_\varepsilon - f\|_{\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})} \le C \left\| \frac{\|f_{\#}\phi_r\|_p}{r^{\gamma}} \ \mathbb{1}_{\varepsilon}(r) \right\|_{L^q(\mathbb{R}_+, \frac{dr}{r})}$$

where $\mathbb{1}_\varepsilon$ is the characteristic function of the set $\mathbb{R} \setminus [1/\varepsilon, \varepsilon]$. And the right hand side of the above inequality tends to zero as $\varepsilon$ tends to $\infty$. □

**Proposition 3.5.** *Let $1 \le q \le \infty$. Then, for $1 \le p_1 \le p_2 \le \infty$ and $\omega_1, \omega_2$ two weights such that $r^{\frac{2\alpha+2}{p_1}}(\omega_1(r))^{-1} = r^{\frac{2\alpha+2}{p_2}}(\omega_2(r))^{-1}$, we have*

$$\dot{\Lambda}^{\omega_1}_{p_1,q}(\mathbb{X}) \ \subseteq \ \dot{\Lambda}^{\omega_2}_{p_2,q}(\mathbb{X}) \qquad (\textit{with continuous embedding}).$$

*Proof.* Let $\psi$ in $S^1_{*,0}$ and let $1 \le p_3 \le \infty$ such that $\frac{1}{p_1} + \frac{1}{p_3} = 1 + \frac{1}{p_2}$. We consider $\phi \in S_*(\mathbb{X})$ satisfying $\mathcal{F}\phi = 1$ on $supp(\mathcal{F}\psi)$. Then we obtain

for all $r > 0$ and $f \in \dot{\Lambda}^{\omega_1}_{p_1,q}(\mathbb{X})$

$$\begin{aligned}\|f_{\#}\psi_r\|_{p_2} &= \|f_{\#}\psi_{r\#}\phi_r\|_{p_2}\\ &\leq \|f_{\#}\psi_r\|_{p_1}\|\phi_r\|_{p_3}\\ &= C\|f_{\#}\psi_r\|_{p_1} r^{(2\alpha+2)(\frac{1}{p_3}-1)}.\end{aligned}$$

So, it holds

$$\frac{\|f_{\#}\psi_r\|_{p_2}}{\omega_2(r)} \leq C\frac{\|f_{\#}\psi_r\|_{p_1}}{\omega_1(r)}.$$

Which leads to $\mathbf{A}^{\omega_2}_{p_2,q}(f) \leq C\mathbf{A}^{\omega_1}_{p_1,q}(f)$. □

To obtain more general properties we introduce an equivalent discrete norm on $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ replacing the group $\mathbb{R}^*_+$ by the 2-powers group $\mathbb{D}_2 = \{2^j; j \in \mathbb{Z}\}$ equipped with the invariant counting measure.

**Theorem 3.1.** *Let $1 \leq p, q \leq \infty$, $\omega$ aweight and $\theta \in S_{*,0}(\mathbb{X})$ such that $\mathcal{F}\theta \in \mathcal{D}_*(]0,\infty[)$ and, for fixed $\lambda_2 > 2\lambda_1 > 0$, $\mathcal{F}\theta(\lambda) \neq 0$ on $[\lambda_1, \lambda_2]$. For $f$ in $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ put*

$$\mathbf{D}^{\omega,\theta}_{p,q}(f) = \begin{cases} \Big(\sum\limits_{j\in\mathbb{Z}}\Big(\dfrac{\|f_{\#}\theta_{2^j}\|_p}{\omega(2^j)}\Big)^q\Big)^{\frac{1}{q}}, & \text{if} \quad 1 \leq q < \infty,\\ \sup\limits_{j\in\mathbb{Z}}\Big(\dfrac{\|f_{\#}\theta_{2^j}\|_p}{\omega(2^j)}\Big), & \text{if} \quad q = \infty.\end{cases}$$

*Then $\mathbf{D}^{\omega,\theta}_{p,q}$ is a norm on $\dot{\Lambda}^{\omega}_{p,q}(\mathbb{X})$ equivalent to the norm $\mathbf{A}^{\omega}_{p,q}$.*

**Remark 3.3.** For $\omega$ a weight, two different functions $\theta_1$ and $\theta_2$ satisfying the hypothesis of the above theorem give two equivalent norms.

*Proof of the Theorem 3.1.* To obtain the desired result we will compare the norms $\mathbf{A}^{\omega}_{p,q}$ and $\mathbf{D}^{\omega}_{p,q}$. Using the fact that $\mathcal{F}\theta \neq 0$ on $[\lambda_1, \lambda_2]$ for $\lambda_2 > 2\lambda_1 > 0$, there exists $\mathcal{F}\sigma$ belonging to $\mathcal{D}_*(]0,\infty[)$ such that $(\mathcal{F}\theta)(\mathcal{F}\sigma) = 1$ on $[\lambda_1, \lambda_2]$. Let $\psi \in S^1_{*,0}$ satisfying $supp(\mathcal{F}\psi) \subset [2\lambda_1, \lambda_2]$. This gives, for all $1 \leq r \leq 2$ and $\lambda \in [0,\infty[$

$$\mathcal{F}\psi(2^j r\lambda) = \mathcal{F}\psi(2^j r\lambda)\mathcal{F}\theta(2^j\lambda)\mathcal{F}\sigma(2^j\lambda). \tag{3.6}$$

Since $\omega$ satisfies (2.4), it holds for $1 \leq q < \infty$ that

$$\begin{aligned}\mathbf{A}^{\omega}_{p,q}(f) &= \Big(\sum_{j\in\mathbb{Z}}\int_1^2\Big(\frac{\|f_{\#}\psi_{2^j r}\|_p}{\omega(r2^j)}\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\\ &\leq C\Big(\sum_{j\in\mathbb{Z}}\int_1^2\Big(\frac{\|f_{\#}\theta_{2^j}\|_p}{\omega(2^j)}\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}} = CLog(2)\,\mathbf{D}^{\omega}_{p,q}(f).\end{aligned}$$

Conversely, let $\mathcal{F}\theta$ supported by $[\lambda_1, \lambda_2]$; $0 < \lambda_1 < \lambda_2$ and let $\psi \in S^1_{*,0}(\mathbb{X})$ satisfying $\mathcal{F}\psi = 1$ on $[\lambda_1/2, \lambda_2]$. Then it holds, for $1/2 \le r \le 1$

$$\mathcal{F}\theta(2^j\lambda) = \mathcal{F}\psi(2^j r\lambda)\mathcal{F}\theta(2^j\lambda) \quad \textit{for all } \lambda \in [0, \infty[. \tag{3.7}$$

The same reasoning as above leads to

$$\mathbf{D}^{\omega}_{p,q}(f) \le C\mathbf{A}^{\omega}_{p,q}(f).$$

For the case $q = \infty$, assume $f \in \dot{\Lambda}^{\omega}_{p,\infty}(\mathbb{X})$ such that $\mathbf{D}^{\omega}_{p,\infty}(f) < \infty$ and let $r > 0$ and $j \in \mathbb{Z}$ such that $2^j \le r \le 2^{j+1}$, then, from (3.6), we get

$$\|f_{\#}\psi_r\|_p \le C\|f_{\#}\theta_{2^j}\|_p \le C\,\mathbf{D}^{\omega}_{p,\infty}(f)\omega(2^j) \le C\,\mathbf{D}^{\omega}_{p,\infty}(f)\omega(r)$$

which implies that $\mathbf{A}^{\omega}_{p,\infty}(f) \le C\,\mathbf{D}^{\omega}_{p,\infty}(f)$.
Conversely let us take $f \in \dot{\Lambda}^{\omega}_{p,\infty}(\mathbb{X})$. Then, from (3.7) we obtain for $1/2 \le r \le 1$ the following estimation

$$\|f_{\#}\theta_{2^j}\|_p \le C\|f_{\#}\psi_{2^j r}\|_p \le C\mathbf{A}^{\omega}_{p,\infty}(f)\omega(2^j r) \le C\mathbf{A}^{\omega}_{p,\infty}(f)\omega(2^j).$$

This completes the proof of the theorem. □

**Proposition 3.6.** *Let $1 \le p \le \infty$ and $\omega$ a weight. Then, for $1 \le q_1 \le q_2 < \infty$ we have*

$$\dot{\Lambda}^{\omega}_{p,q_1}(\mathbb{X}) \subseteq \dot{\Lambda}^{\omega}_{p,q_2}(\mathbb{X}) \quad (\textit{with continuous embedding}). \tag{3.8}$$

*Proof.* The result holds using the discrete norm and the fact that $l^{q_1} \subset l^{q_2}$ for all $1 \le q_1 \le q_2 \le \infty$ and we have

$$\Big(\sum |u_j|^{q_2}\Big)^{1/q_2} \le \Big(\sum |u_j|^{q_1}\Big)^{1/q_1}$$

for all $(u_j) \in l^{q_1}$. □

## 4. Characterization of the Spaces $\Lambda^{\omega}_{p,q}(\mathbb{X}) = \dot{\Lambda}^{\omega}_{p,q}(\mathbb{X}) \cap L^p$

In this section we study the spaces $\Lambda^{\omega}_{p,q}(\mathbb{X}) = \dot{\Lambda}^{\omega}_{p,q}(\mathbb{X}) \cap L^p(d\mu_\alpha)$ endowed with the norm $\mathbf{A}^{\omega}_{p,q}$ and we give some characterizations using equivalent norms for some classes of weights $\omega$. Let us first start with the following definition.

**Definition 4.1.** Let $0 \le \varepsilon, \delta < \infty$, $1 \le q, q' \le \infty$; $\frac{1}{q'} + \frac{1}{q} = 1$ and $\omega$ be a weight.

(i) $\omega$ is said to be a $d_\varepsilon$-weight if there exists $C > 0$ such that

$$\int_0^s r^\varepsilon \omega(r) \frac{dr}{r} \leq C s^\varepsilon \omega(s), \qquad \text{for all } s > 0. \tag{4.1}$$

(ii) $\omega$ is said to be a $b_\delta$-weight if there exists $C > 0$ such that

$$\int_s^\infty \frac{\omega(r)}{r^\delta} \frac{dr}{r} \leq C \frac{\omega(s)}{s^\delta}, \qquad \text{for all } s > 0. \tag{4.2}$$

If $(d_\varepsilon)$ (resp. $(b_\delta)$) denotes the class of $d_\varepsilon$-weight (resp. $b_\delta$-weight) we write

$$\mathcal{W}_{\varepsilon,\delta} = (d_\varepsilon) \cap (b_\delta)$$

and

$$\mathcal{W}^q_{\varepsilon,\delta} = \Big\{ \omega : weight \ / \ \omega(r) = \chi^{\frac{1}{q'}}(r) \nu^{-\frac{1}{q}}(r^{-1}); \ \chi, \nu \in \mathcal{W}_{\varepsilon,\delta} \Big\}.$$

The following properties hold

**P1.** If $\omega \in \mathcal{W}^q_{\varepsilon,\delta}$ then $\omega \in \mathcal{W}^q_{\varepsilon',\delta'}$, for any $\varepsilon' > \varepsilon$ and $\delta' > \delta$.

**P2.** Let $\bar{\omega}(r) = \omega(r^{-1})$. Then $\omega \in (b_\varepsilon)$ if and only if $\bar{\omega} \in (d_\varepsilon)$.

Note that, $w(r) = r^\gamma$ belongs to $\mathcal{W}^q_{0,2}$ for $0 < \gamma < 2$.

**Theorem 4.1** (First characterization theorem)**.** *Let $1 \leq p, \ q \leq \infty$ and $\omega$ a weight belonging to $W^q_{0,2}$. For $f$ in $\Lambda^\omega_{p,q}(\mathbb{X})$ put*

$$\mathbf{B}^\omega_{p,q}(f) = \begin{cases} \Big( \int_{\mathbb{X}} \Big( \frac{\|\Delta_x f\|_p}{\omega(x)} \Big)^q \frac{dx}{x} \Big)^{\frac{1}{q}}, & \text{if} \quad 1 \leq q < \infty, \\ \operatorname*{ess\,sup}_{x \in \mathbb{X}} \Big( \frac{\|\Delta_x f\|_p}{\omega(x)} \Big), & \text{if} \quad q = \infty \end{cases}$$

*where $\Delta_x f = T^{(\alpha)}_x f - f$, for all $x \in \mathbb{X}$. Then $\mathbf{B}^\omega_{p,q}$ is a norm on $\Lambda^\omega_{p,q}(\mathbb{X})$ equivalent to $\mathbf{A}^\omega_{p,q}$.*

**Remark 4.1.** Note that the case $1 < q < \infty$ could have been shown by interpolation with the extreme cases ($q = 1$ and $q = \infty$), but a direct proof is presented in this paper.

Let us first start with some basic lemmas that will be useful for our purposes.

**Lemma A** (Schur lemma) (see [2])**.** *Let $1 < q < \infty$ and $q'$ its conjugate exponent. Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be a pair of $\sigma$-finite measure spaces and let $F : \Omega_1 \times \Omega_2 \to \mathbb{R}_+$ be a measurable function. Define $T_F f$*

*for all measurable positive function $f$ on $\Omega_1$ by*

$$T_F f(\omega_2) = \int_{\Omega_1} F(\omega_1, \omega_2) f(\omega_1) d\mu_1(\omega_1), \quad \text{for all } \omega_2 \in \Omega_2.$$

*If there exist $C > 0$ and measurable functions $h_i : \Omega_i \to ]0, +\infty[$ ; $(i = 1, 2)$ such that*

$$\int_{\Omega_1} F(\omega_1, \omega_2) h_1^{q'}(\omega_1) d\mu_1(\omega_1) \leq C h_2^{q'}(\omega_2) \quad \mu_2 - a.e.$$

$$\int_{\Omega_2} F(\omega_1, \omega_2) h_2^{q}(\omega_2) d\mu_2(\omega_2) \leq C h_1^{q}(\omega_1) \quad \mu_1 - a.e.$$

*Then $T_F$ can be extended as a bounded operator from $L^q(\Omega_1, \mu_1)$ into $L^q(\Omega_2, \mu_2)$.*

**Lemma B.** *Let $1 < q, q' < \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$ and $\omega \in \mathcal{W}^q_{\varepsilon,\delta}$; $\omega(r) = \chi^{\frac{1}{q'}}(r)\nu^{-\frac{1}{q}}(r^{-1})$. For $0 \leq \varepsilon < \delta < \infty$ and $a, b \in \mathbb{R}^*_+$ let us set*

$$R^{\varepsilon,\delta}(a, b) = \frac{\omega(a)}{\omega(b)} \min\left(\left(\frac{a}{b}\right)^{\varepsilon}, \left(\frac{b}{a}\right)^{\delta}\right).$$

*Then there exist $C > 0$ and a measurable function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\int_0^\infty R^{\varepsilon,\delta}(a, b) g^{q'}(a) \frac{da}{a} \leq C g^{q'}(b) \quad a.e.\ b > 0$$

*and*

$$\int_0^\infty R^{\varepsilon,\delta}(a, b) g^{q}(b) \frac{db}{b} \leq C g^{q}(a) \quad a.e.\ a > 0.$$

*Proof.* It suffices to take $g(r) = \chi^{\frac{1}{qq'}}(r)\nu^{-\frac{1}{qq'}}(r^{-1})$. □

The main idea in the proof of the above Theorem is based on the following lemmas.

**Lemma 4.1.** *Let $\psi, \phi \in S_{*,0}(\mathbb{X})$. Then*

$$\|\Delta_x \phi_\rho\|_1 \leq C \min\left(1, \left(\frac{x}{\rho}\right)^2\right). \tag{4.3}$$

*Proof.* Using the fact that $\|\Delta_x \phi\|_1 \leq C x^2$ (see [11]) and the contraction property of the generalized translation operators we obtain

$$\|\Delta_x \phi\|_1 \leq C \min(1, x^2).$$

On the other hand, from the expression of the kernel $W_\alpha$, one can see easily that

$$\|\Delta_x \phi_\rho\|_1 = \|\Delta_{\frac{x}{\rho}} \phi\|_1.$$

So it holds

$$\|\Delta_x \phi_\rho\|_1 \leq C \min\Big(1, \Big(\frac{x}{\rho}\Big)^2\Big). \tag{4.4}$$

The lemma is proved. □

**Lemma 4.2.** *Let* $\psi, \phi \in S_{*,0}(\mathbb{X})$. *Then*

$$\|\phi_{\rho\#}\, \psi_r\|_1 \leq C \min\Big(\Big(\frac{r}{\rho}\Big)^2, \Big(\frac{\rho}{r}\Big)^2\Big), \qquad \textit{for all } r, \rho > 0. \tag{4.5}$$

*Proof.* Let us take $r \leq \rho$. Since $\int_{\mathbb{X}} \phi_\rho(y) d\mu_\alpha(y) = 0$ we have

$$\phi_{\rho\#}\, \psi_r(y) = \int_{\mathbb{X}} \psi_r(x) \Delta_x \phi_\rho(y) d\mu_\alpha(x).$$

From Minkowski's inequality one gets

$$\begin{aligned} \|\psi_{r\#}\, \phi_\rho\|_1 &\leq \int_{\mathbb{X}} |\psi_r(x)|\, \|\Delta_x \phi_\rho\|_1 d\mu_\alpha(x) \\ &= C \int_{\mathbb{X}} \Big(\frac{x}{r}\Big)^{2\alpha+2} \Big|\psi\Big(\frac{x}{r}\Big)\Big|\, \|\Delta_x \phi_\rho\|_1 \frac{dx}{x}. \end{aligned} \tag{4.6}$$

Taking into account that $\psi$ belongs to $S_*(\mathbb{X})$, then there exists $C > 0$ such that

$$\Big(\frac{x}{r}\Big)^{2\alpha+2} \Big|\psi\Big(\frac{x}{r}\Big)\Big| \leq C\Big(\frac{x}{r}\Big)^{-3} \quad \textit{and} \quad \Big|\psi\Big(\frac{x}{r}\Big)\Big| \leq C.$$

So we get

$$\Big(\frac{x}{r}\Big)^{2\alpha+2} \Big|\psi\Big(\frac{x}{r}\Big)\Big| \leq C \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^3\Big). \tag{4.7}$$

Hence from (4.6), (4.7) and (4.4) we obtain

$$\|\psi_{r\#}\, \phi_\rho\|_1 \leq C \int_0^\infty \Big(\frac{x}{\rho}\Big)^2 \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^3\Big) \frac{dx}{x} = C\Big(\frac{r}{\rho}\Big)^2.$$

The lemma is proved. □

One of the main tool used in this paper is the Calderón reproducing formula that we recall in the following lemma (see [17]).

**Lemma 4.3.** *Let* $\psi \in S^1_{*,0}$, *and* $1 < \varepsilon < \infty$. *For* $f \in L^p(d\mu_\alpha)\,(1 \le p < \infty)$ *define*

$$f_\varepsilon = \int_{1/\varepsilon}^{\varepsilon} f_\# \psi_{r\#} \psi_r \frac{dr}{r}.$$

*Then* $f_\varepsilon$ *converges to* $f$ *in* $L^p(d\mu_\alpha)$ *as* $\varepsilon \to \infty$.

**Lemma 4.4.** **(i)** *Let* $1 \le p \le \infty$ *and* $\psi \in S_{*,0}$. *Then we have for all* $\beta > 0$ *and* $f$ *such that* $\mathbf{B}^\omega_{p,q}(f) < \infty$

$$\|f_\# \psi_r\|_p \le C \int_{\mathbb{X}} \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big) \|\Delta_x f\|_p \frac{dx}{x}, \quad a.e. \ \ r > 0. \tag{4.8}$$

**(ii)** *For all* $f \in \Lambda^\omega_{p,q}(\mathbb{X})$ *it holds*

$$\|\Delta_x f\|_p \le C \int_0^\infty \min\Big(1, \Big(\frac{x}{r}\Big)^2\Big) \|f_\# \psi_r\|_p \frac{dr}{r}, \quad a.e.\ x \in \mathbb{X}. \tag{4.9}$$

*Proof.* The result follows from Lemmas 4.1, 4.2 and 4.3. □

*Proof of Theorem* 4.1. **(i)** Let us start with the case $q = \infty$ which follows immediately from (4.10) and (4.11). Assume $f$ such that $\mathbf{B}^\omega_{p,\infty}(f) < \infty$. Using (4.10) with $\beta \ge 2$, we get

$$\begin{aligned}
\|f_\# \psi_r\|_p &\le C \int_{\mathbb{X}} \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big) \|\Delta_x f\|_p \frac{dx}{x} \\
&\le C\, \mathbf{B}^\omega_{p,\infty}(f) \int_{\mathbb{X}} \omega(x) \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big) \frac{dx}{x} \\
&\le C\, \mathbf{B}^\omega_{p,\infty}(f) \omega(r).
\end{aligned}$$

That is $\mathbf{A}^\omega_{p,\infty}(f) \le C\, \mathbf{B}^\omega_{p,\infty}(f)$.

Take now $f \in \Lambda^\omega_{p,\infty}(\mathbb{X})$. Then, from (4.11) it holds

$$\begin{aligned}
\|\Delta_x f\|_p &\le C \int_0^\infty \min\Big(1, \Big(\frac{x}{r}\Big)^2\Big) \|f_\# \psi_r\|_p \frac{dr}{r} \\
&\le C \mathbf{A}^\omega_{p,\infty}(f) \int_0^\infty \omega(r) \min\Big(1, \Big(\frac{x}{r}\Big)^2\Big) \frac{dr}{r} \\
&\le C \mathbf{A}^\omega_{p,\infty}(f) \omega(x).
\end{aligned}$$

**(ii)** Let us prove the case $q = 1$. Assume $f$ such that $\mathbf{B}^\omega_{p,1}(f) < \infty$. We shall prove that $\mathbf{A}^\omega_{p,1}(f) \le C\, \mathbf{B}^\omega_{p,1}(f)$. From (4.10), with $\beta \ge 2$ we obtain

$$\mathbf{A}^\omega_{p,1}(f) \quad = \quad \int_0^\infty \frac{\|f_\# \psi_r\|_p}{\omega(r)} \frac{dr}{r}$$

$$\begin{aligned}
&\leq C\int_0^\infty \frac{1}{\omega(r)}\int_{\mathbb{X}} \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big)\|\Delta_x f\|_p \frac{dx}{x}\frac{dr}{r}\\
&= C\int_{\mathbb{X}} \|\Delta_x f\|_p \int_0^\infty \frac{1}{\omega(r)} \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big)\frac{dr}{r}\frac{dx}{x}\\
&\leq C\int_{\mathbb{X}} \frac{\|\Delta_x f\|_p}{\omega(x)}\frac{dx}{x}\\
&= C\,\mathbf{B}^{\omega}_{p,1}(f).
\end{aligned}$$

Conversely, let us take $f$ in $\Lambda^{\omega}_{p,1}(\mathbb{X})$. Then using (4.11), we get

$$\begin{aligned}
\mathbf{B}^{\omega}_{p,1}(f) &= \int_{\mathbb{X}} \frac{\|\Delta_x f\|_p}{\omega(x)}\frac{dx}{x}\\
&\leq C\int_{\mathbb{X}}\int_0^\infty \frac{1}{\omega(x)} \min\Big(1, \Big(\frac{x}{r}\Big)^{2}\Big)\|f_{\#}\psi_r\|_p \frac{dr}{r}\frac{dx}{x}\\
&= C\int_0^\infty \|f_{\#}\psi_r\|_p \int_{\mathbb{X}} \frac{1}{\omega(x)} \min\Big(1, \Big(\frac{x}{r}\Big)^{2}\Big)\frac{dx}{x}\frac{dr}{r}\\
&\leq C\int_0^\infty \frac{\|f_{\#}\psi_r\|_p}{\omega(r)}\frac{dr}{r}\\
&= C\mathbf{A}^{\omega}_{p,1}(f).
\end{aligned}$$

**(iii)** Let us now prove the case $1 < q < \infty$. Using the inequality (4.10) we obtain, for $\beta \geq 2$

$$\begin{aligned}
\frac{\|f_{\#}\psi_r\|_p}{\omega(r)} &\leq C\int_{\mathbb{X}} \frac{\omega(x)}{\omega(r)} \min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2}, \Big(\frac{r}{x}\Big)^{\beta}\Big)\frac{\|\Delta_x f\|_p}{\omega(x)}\frac{dx}{x}\\
&= C\int_{\mathbb{X}} R^{2\alpha+2,\beta}(x,r)\frac{\|\Delta_x f\|_p}{\omega(x)}\frac{dx}{x}\\
&= CT_{R^{2\alpha+2,\beta}}\Big(\frac{\|\Delta_x f\|_p}{\omega(x)}\Big)(r)
\end{aligned}$$

where $R^{2\alpha+2,\beta}(x,r) = \frac{\omega(x)}{\omega(r)}\min((\frac{x}{r})^{2\alpha+2}, (\frac{r}{x})^{\beta})$. By Lemma A and Lemma B, we obtain the boundedness of the operator $T_{R^{2\alpha+2,\beta}}$ from $L^q(\frac{dx}{x}) := L^q(\mathbb{X},, \frac{dx}{x})$ into $L^q(\frac{dr}{r}) := L^q(\mathbb{R}^*_+, \frac{dr}{r})$. Moreover the hypothesis $\mathbf{B}^{\omega}_{p,q}(f) < \infty$ means that the function $x \longmapsto \frac{\|\Delta_x f\|_p}{\omega(x)}$ belongs to $L^q(\frac{dx}{x})$. Hence

$$\begin{aligned}
\mathbf{A}^{\omega}_{p,q}(f) &= \Big\|\frac{\|f_{\#}\psi_r\|_p}{\omega(r)}\Big\|_{L^q(\frac{dr}{r})} \leq C\Big\|T_F\Big(\frac{\|\Delta_x f\|_p}{\omega(x)}\Big)\Big\|_{L^q(\frac{dr}{r})}\\
&\leq C\Big\|\frac{\|\Delta_x f\|_p}{\omega(x)}\Big\|_{L^q(\frac{dx}{x})} = C\,\mathbf{B}^{\omega}_{p,q}(f).
\end{aligned}$$

Conversely, let us take $f \in \Lambda^{\omega}_{p,q}(\mathbb{X})$. From (4.11) we get

$$\begin{aligned}\frac{\|\Delta_x f\|_p}{\omega(x)} &\leq C\int_0^\infty \frac{\omega(r)}{\omega(x)}\min\Big(1,\Big(\frac{x}{r}\Big)^2\Big)\frac{\|f_\# \psi_r\|_p}{\omega(r)}\frac{dr}{r}\\ &= C\int_0^\infty R^{0,2}(r,x)\frac{\|f_\# \psi_r\|_p}{\omega(r)}\frac{dr}{r}\\ &= C\,T_{R^{0,2}}\Big(\frac{\|f_\# \psi_r\|_p}{\omega(r)}\Big)(x)\end{aligned}$$

where $R^{0,2}(r,x) = \frac{\omega(r)}{\omega(x)}\min(1,(\frac{x}{r})^2)$. We proceed as above with adequate changes to obtain

$$\begin{aligned}\mathbf{B}^{\omega}_{p,q}(f) &\leq C\Big\|T_{R^{0,2}}\Big(\frac{\|f_\# \psi_r\|_p}{\omega(r)}\Big)\Big\|_{L^q(\frac{dx}{x})}\\ &\leq C\Big\|\frac{\|f_\# \psi_r\|_p}{\omega(r)}\Big\|_{L^q(\frac{dr}{r})}\\ &= C\mathbf{A}^{\omega}_{p,q}(f)\end{aligned}$$

which gives the desired result.

**Theorem 4.2** (Second characterization theorem)**.** *Let $1 \leq p, q \leq \infty$ and $\omega \in \mathcal{W}^q_{0,2}$. For $f$ in $\Lambda^{\omega}_{p,q}(\mathbb{X})$ put*

$$\mathbf{C}^{\omega}_{p,q}(f) = \begin{cases} \Big(\displaystyle\int_{\mathbb{X}}\Big(\frac{m_p(f,x)}{\omega(x)}\Big)^q\frac{dx}{x}\Big)^{\frac{1}{q}}, & \text{if} \quad 1 \leq q < \infty,\\ \operatorname*{ess\,sup}_{x\in\mathbb{X}}\Big(\frac{m_p(f,x)}{\omega(x)}\Big), & \text{if} \quad q = \infty\end{cases}$$

*where $m_p(f,x) = \sup\limits_{0\leq y\leq x}\|\Delta_y f\|_p$ is the generalized modulus of continuity associated with the generalized translation operators $T^{(\alpha)}_x$. Then $\mathbf{C}^{\omega}_{p,q}$ is a norm on $\Lambda^{\omega}_{p,q}(\mathbb{X})$ equivalent to $\mathbf{A}^{\omega}_{p,q}$.*

*Proof.* To compare $\mathbf{A}^{\omega}_{p,q}(f)$ and $\mathbf{C}^{\omega}_{p,q}(f)$ we proceed as in the proof of the Theorem 4.1 using the following lemma instead of Lemma 4.4. □

**Lemma 4.5.** *(i) Let $1 \leq p \leq \infty$ and $\psi \in S_{*,0}$. Then we have for all $\beta > 0$ and $f$ such that $\mathbf{B}^{\omega}_{p,q}(f) < \infty$*

$$\|f_\# \psi_r\|_p \leq C\int_{\mathbb{X}}\min\Big(\Big(\frac{x}{r}\Big)^{2\alpha+2},\Big(\frac{r}{x}\Big)^{\beta}\Big)m_p(f,x)\frac{dx}{x}, \quad a.e. \ \ r > 0. \tag{4.10}$$

*(ii) For all* $f \in \Lambda^{\omega}_{p,q}(\mathbb{X})$ *it holds*

$$m_p(f,x) \leq C \int_0^{\infty} \min\left(1, \left(\frac{x}{r}\right)^2\right) \|f_{\#}\psi_r\|_p \frac{dr}{r}, \quad a.e.\ x \in \mathbb{X}. \tag{4.11}$$

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