Strang-Fix theory for approximation order in weighted L^p -spaces and Herz spaces

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Abstract. In this paper, we study the Strang-Fix theory for approximation order in the weighted L^p -spaces and Herz spaces.

1. Introduction

In [10], Strang and Fix considered the relation between approximation order in $L^2(\mathbb{R}^n)$ of a given function and properties of its Fourier transform. In order to describe their results, we make the following definitions. Let $\varphi \in C_c(\mathbb{R}^n)$, where $C_c(\mathbb{R}^n)$ consists of all continuous functions on $\mathbb{R}^n$ with compact support. For a sequence c on $\mathbb{Z}^n$, the semi-discrete convolution product $\varphi *' c$ is defined by

$$\varphi *' c = \sum_{\nu \in \mathbb{Z}^n} \varphi(\cdot - \nu) c(\nu).$$

For $h > 0$, σ_h is the scaling operator defined by

$$\sigma_h f(x) = f(hx) \quad (x \in \mathbb{R}^n).$$

For a positive integer k , we denote the (unweighted) Sobolev space by $L_k^p(\mathbb{R}^n)$ (see section 2) and define the semi-norm by

$$|f|_{L_k^p(\mathbb{R}^n)} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} \quad (f \in L_k^p(\mathbb{R}^n)),$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. We say that the collection $\Phi = \{\varphi_1, \dots, \varphi_N\}$ of $C_c(\mathbb{R}^n)$ provides controlled L^p -approximation of order k if for each $f \in L_k^p(\mathbb{R}^n)$ there exist weights c_j^h ($h > 0$, $j = 1, \dots, N$) such that

$$(i) \quad \|f - \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right) h^{-n/p}\|_{L^p(\mathbb{R}^n)} \leq C_1 h^k |f|_{L_k^p(\mathbb{R}^n)}$$

and

$$(ii) \quad \sum_{j=1}^N \|c_j^h\|_{\ell^p(\mathbb{Z}^n)} \leq C_2 \|f\|_{L^p(\mathbb{R}^n)}$$

hold for some C_1 and C_2 independent of f and h . The Fourier transform $\hat{f}$ of a function $f \in L^1(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx.$$

The collection $\Phi = \{\varphi_1, \dots, \varphi_N\}$ of $C_c(\mathbb{R}^n)$ is said to satisfy the Strang-Fix condition of order k if there exist finitely supported sequences b_j ($j = 1, \dots, N$) on $\mathbb{Z}^n$ such that the function $\varphi = \sum_{j=1}^N \varphi_j *' b_j$ satisfies

$$\hat{\varphi}(0) \neq 0 \quad \text{and} \quad (\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \quad (|\alpha| < k, \quad \nu \in \mathbb{Z}^n \setminus \{0\}).$$

We note that the B-spline of degree k satisfies the Strang-Fix condition of order k (for the definition of the B-spline, see the proof of Theorem 1 below). Strang and Fix proved that, when Φ consists of only one function, Φ provides controlled L^2 -approximation of order k if and only if Φ satisfies the Strang-Fix condition of order k ([10]). Moreover, they conjectured that this equivalence holds when $\Phi = \{\varphi_1, \dots, \varphi_N\}$. However, a counterexample was found by Jia in [6].

We say that $\Phi = \{\varphi_1, \dots, \varphi_N\}$ provides local L^p -approximation of order k if for each $f \in L_k^p(\mathbb{R}^n)$ there exist weights c_j^h ($h > 0$, $j = 1, \dots, N$) such that

$$(iii) \quad \|f - \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right)\|_{L^p(\mathbb{R}^n)} \leq Ch^k |f|_{L_k^p(\mathbb{R}^n)}$$

and

$$(iv) \quad c_j^h(\nu) = 0 \text{ whenever } \text{dist}(h\nu, \text{supp } f) > r$$

hold for some C and r independent of f and h , where $\text{dist}(h\nu, \text{supp } f) = \inf_{x \in \text{supp } f} |h\nu - x|$. Note that in the definition of local L^p -approximation, the weights c_j^h depend on h . De Boor and Jia proved that Φ provides local L^p -approximation of order k if and only if Φ satisfies the Strang-Fix condition of order k ([2]).

Given a weight function w on $\mathbb{R}^n$, we denote the weighted L^p -space and the weighted Sobolev space by $L^p(w)$ and $L_k^p(w)$ (see section 2). We define the semi-norm on $L_k^p(w)$ by

$$|f|_{L_k^p(w)} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)}$$

and say that $\Phi = \{\varphi_1, \dots, \varphi_N\}$ provides local $L^p(w)$ -approximation of order k if for each $f \in L_k^p(w)$ there exist weights c_j^h ($h > 0$, $j = 1, \dots, N$) such that (iv) and

$$(v) \quad \|f - \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right)\|_{L^p(w)} \leq Ch^k |f|_{L_k^p(w)}$$

hold for some C and r independent of f and h .

We denote the (homogeneous) weighted Herz space and the (homogeneous) weighted Herz-Sobolev space by $\dot{K}_q^{\alpha,p}(w_1, w_2)$ and $\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$ (see section 2). Similarly, we define the semi-norm by

$$|f|_{\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)} = \sum_{|\alpha|=k} \|\partial^\alpha f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)}$$

and say that $\Phi = \{\varphi_1, \dots, \varphi_N\}$ provides local $\dot{K}_q^{\alpha,p}(w_1, w_2)$ -approximation of order k if for each $f \in \dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$ there exist weights c_j^h ($h > 0$, $j = 1, \dots, N$) such that (iv) and

$$(vi) \quad \|f - \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right)\|_{\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)} \leq Ch^k |f|_{\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)}$$

hold for some C and r independent of f and h .

Using the notion of multiresolution approximation, we prove the following extension of [2] to weighted L^p -spaces. In the following two theorems, we denote Muckenhoupt's A_p -class by A_p (see section 2).

Theorem 1. *Suppose that $\Phi = \{\varphi_1, \dots, \varphi_N\}$ is a finite collection of $C_c(\mathbb{R}^n)$. Then the following are equivalent.*

- (i)' Φ satisfies the Strang-Fix condition of order k .
- (ii)' For all $p \in [1, \infty]$ and $w \in A_p$, Φ provides local $L^p(w)$ -approximation of order k .
- (iii)' For some $p \in [1, \infty]$ and $w \in A_p$, Φ provides local $L^p(w)$ -approximation of order k .

We also consider the result of [2] in weighted Herz spaces.

Theorem 2. Suppose that $\Phi = \{\varphi_1, \dots, \varphi_N\}$ is a finite collection of $C_c(\mathbb{R}^n)$. Let $w_1 \in A_{q_{w_1}}$, $w_2 \in A_{q_{w_2}}$, $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $1 < q < \infty$, where w_1 and w_2 satisfy the following

- (a) $1 \leq q_{w_1} \leq q$ and $-nq_{w_1}/q < \alpha q_{w_1} < n(1 - q_{w_1}/q)$ when $w_1 = w_2$,
- (b) $1 \leq q_{w_1} < \infty$, $1 \leq q_{w_2} \leq q$ and $0 < \alpha q_{w_1} < n(1 - q_{w_2}/q)$ when $w_1 \neq w_2$.

Then the following statements are equivalent.

- (i)'' Φ satisfies the Strang-Fix condition of order k .
- (ii)'' Φ provides local $\dot{K}_q^{\alpha, p}(w_1, w_2)$ -approximation of order k .

Lastly, we point out that the Strang-Fix theory for functions having non-compact support is given by, for example, Jia and Lei ([7]).

2. Preliminaries

This section is based on [3] and [7] (see also [5]). Given an appropriate function φ on $\mathbb{R}^n$, we define the multiresolution approximation $\{P_h f\}_{h>0}$ of a function f on $\mathbb{R}^n$ with respect to φ by

$$P_h f(x) = \sum_{\nu \in \mathbb{Z}^n} \langle f, \varphi_{h, \nu} \rangle \varphi_{h, \nu}(x),$$

where

$$\varphi_{h, \nu}(x) = h^{-n/2} \varphi(x/h - \nu)$$

and

$$\langle f, \varphi_{h, \nu} \rangle = \int_{\mathbb{R}^n} f(x) \overline{\varphi_{h, \nu}(x)} dx.$$

Let $B(0, r)$ be the Euclidean ball of radius r centered at the origin. The Hardy-Littlewood maximal function Mf of a locally integrable function f on $\mathbb{R}^n$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(0, r)|} \int_{B(0, r)} |f(x - y)| dy,$$

where $|B(0, r)|$ denotes the Lebesgue measure of $B(0, r)$.

A weight $w \geq 0$ on $\mathbb{R}^n$ is said to belong to A_p for $1 < p < \infty$ if

$$A_p(w) = \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \left(\frac{1}{|B|} \int_B w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ and p' is the conjugate exponent of p ($1/p + 1/p' = 1$). $A_p(w)$ is called the A_p -constant of w . The class A_1 is defined by

$$A_1(w) = \sup_B \left(\frac{1}{|B|} \int_B w(x) dx \right) \|w^{-1}\|_{L^\infty(B, dx)} < \infty,$$

where $\|w^{-1}\|_{L^\infty(B, dx)} = \text{ess sup}_{x \in B} w(x)^{-1}$. The class A_∞ is the union of A_p , $1 \leq p < \infty$. If $w \in A_\infty$, then $w > 0$ a.e. and w is locally integrable ([3, p. 134]). These classes were introduced by Muckenhoupt ([8]).

$C_c^\infty(\mathbb{R}^n)$ consists of all infinitely differentiable functions on $\mathbb{R}^n$ with compact support.

Let w be a weight function on $\mathbb{R}^n$ and $1 \leq p \leq \infty$. Then the weighted L^p -space $L^p(w)$ consists of all f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p} < \infty,$$

with the usual modification when $p = \infty$.

We give the definition of the weighted Sobolev space $L_k^p(w)$, where $1 \leq p \leq \infty$, k is a non-negative integer and w is a weight function on $\mathbb{R}^n$ ([9]). A function f on $\mathbb{R}^n$ belongs to $L_k^p(w)$ if $f \in L^p(w)$ and the partial derivatives $\partial^\alpha f$, taken in the sense of distributions, belong to $L^p(w)$, whenever $0 \leq |\alpha| \leq k$. The norm on $L_k^p(w)$ is given by

$$\|f\|_{L_k^p(w)} = \sum_{|\alpha| \leq k} \|\partial^\alpha f\|_{L^p(w)}.$$

In particular, we denote $L_k^p(w)$ by $L_k^p(\mathbb{R}^n)$ when $w \equiv 1$.

Let $B_k = B(0, 2^k)$ and $R_k = B_k \setminus B_{k-1}$, where $k \in \mathbb{Z}$. For a weight function w and a measurable set E , we write $w(E) = \int_E w(y) dy$. Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, w_1 and w_2 be weight functions on $\mathbb{R}^n$. The (homogeneous) weighted Herz space $\dot{K}_q^{\alpha, p}(w_1, w_2)$ consists of all f on $\mathbb{R}^n$ such that

$$\|f\|_{\dot{K}_q^{\alpha, p}(w_1, w_2)} = \left(\sum_{k \in \mathbb{Z}} w_1(B_k)^{\alpha p/n} \|f \chi_{R_k}\|_{L^q(w_2)}^p \right)^{1/p} < \infty.$$

Similarly, the weighted Herz-Sobolev space $\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$ is defined by using $\dot{K}_q^{\alpha,p}(w_1, w_2)$ instead of $L^p(w)$ ([9]).

Lastly, we give the necessary lemmas.

Lemma 2.1 ([3, Proposition 2.7]). *Let φ be a function on $\mathbb{R}^n$ which is non-negative, radial, decreasing (as a function on $(0, \infty)$) and integrable. Then*

$$\sup_{\alpha > 0} |(\varphi_\alpha * f)(x)| \leq \|\varphi\|_{L^1(\mathbb{R}^n)} Mf(x),$$

where $\varphi_\alpha(x) = \alpha^{-n} \varphi(x/\alpha)$.

Lemma 2.2 ([3, Theorem 7.3]). *If $1 < p < \infty$, then the Hardy-Littlewood maximal operator M is bounded on $L^p(w)$ if and only if $w \in A_p$.*

Lemma 2.3. *Let $1 \leq p < \infty$ and $w \in A_p$. Then the following statements hold.*

(a) ([3, p. 134]) *There exists a constant $C > 0$ such that*

$$\frac{w(B)}{w(E)} \leq C \left(\frac{|B|}{|E|} \right)^p$$

for all balls B and measurable sets E such that $E \subset B$.

(b) ([3, Corollary 7.6]) *There exist constants C and $\delta > 0$ such that*

$$\frac{w(E)}{w(B)} \leq C \left(\frac{|E|}{|B|} \right)^\delta$$

for all balls B and measurable sets E such that $E \subset B$.

Lemma 2.4 ([9, Theorem 1.1]). *Let $1 \leq p < \infty$, $w \in A_p$ and k be a non-negative integer. Then $C_c^\infty(\mathbb{R}^n)$ is dense in the weighted Sobolev space $L_k^p(w)$.*

Lemma 2.5 ([9, Corollary 3.3]). *Let $w_1 \in A_{q_{w_1}}$, $w_2 \in A_{q_{w_2}}$, $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$ and k be a non-negative integer, where w_1 and w_2 satisfy either Theorem 2 (a) or Theorem 2 (b). Then $C_c^\infty(\mathbb{R}^n)$ is dense in the weighted Herz-Sobolev space $\dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$.*

Lemma 2.6 ([9, p. 126]). *Let $w_1 \in A_{q_{w_1}}$, $w_2 \in A_{q_{w_2}}$, $\alpha \in \mathbb{R}$, $0 < p < \infty$ and $1 < q < \infty$, where w_1 and w_2 satisfy either Theorem 2 (a) or Theorem 2 (b). Then the Hardy-Littlewood maximal operator M is bounded on $\dot{K}_q^{\alpha,p}(w_1, w_2)$.*

Corollary 2.3, Corollary 2.4 and Lemma 2.6 in [7] give the following lemma.

Lemma 2.7. *Let $\Phi = \{\varphi_1, \dots, \varphi_N\}$ be a finite collection in $C_c(\mathbb{R}^n)$. If Φ satisfies the Strang-Fix condition of order k , then there exist finitely supported sequences b_j ($j = 1, \dots, N$) on $\mathbb{Z}^n$ such that the function $\varphi = \sum_{j=1}^N \varphi_j *' b_j$ satisfies*

$$\hat{\varphi}(0) = 1, \quad \hat{\varphi}(2\pi\nu) = 0 \quad (\nu \in \mathbb{Z}^n \setminus \{0\})$$

and

$$(\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \quad (0 < |\alpha| < k, \quad \nu \in \mathbb{Z}^n).$$

Lemma 2.8 ([7, Lemma 5.2]). *Let k be a positive integer and Φ be a finite collection in $C_c(\mathbb{R}^n)$. For each $h > 0$, let $S_h(\Phi)$ denote the linear space spanned by $\varphi(\cdot/h - \nu)$, where $\varphi \in \Phi$ and $\nu \in \mathbb{Z}^n$. Suppose that there is a family $\{u_h\}_{0 < h < 1}$ of functions satisfying the following conditions*

- (a) $u_h \in S_h(\Phi)$ ($0 < h < 1$),
- (b) $\lim_{h \rightarrow 0} \hat{u}_h(0) = 1$,
- (c) $\lim_{h \rightarrow 0} (\partial^\alpha \hat{u}_h)(\xi/h)/h^{k-1} = 0$ ($\xi \in \mathbb{R}^n \setminus \{0\}$, $|\alpha| < k$).

Then Φ satisfies the Strang-Fix condition of order k .

3. Lemmas

Lemma 3.1 (a)' appears as [4, Chapter 5, Proposition 3.14].

Lemma 3.1. *Let k be a positive integer and $\epsilon > 0$. Suppose that φ is a function on $\mathbb{R}^n$ such that*

- (a) $|\varphi(x)| \leq \frac{C}{(1 + |x|)^{k-1+n+\epsilon}}$ ($x \in \mathbb{R}^n$),
- (b) $\hat{\varphi}(0) = 1, \quad \hat{\varphi}(2\pi\nu) = 0$ ($\nu \in \mathbb{Z}^n \setminus \{0\}$),
- (c) $(\partial^\alpha \hat{\varphi})(2\pi\nu) = 0$ ($0 < |\alpha| < k, \quad \nu \in \mathbb{Z}^n$),

where C is independent of $x \in \mathbb{R}^n$. Then for $h > 0$ we have that

- (a)' $\int_{\mathbb{R}^n} h^{-n} \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy = 1 \quad \text{a.e.}$
- (b)' $\int_{\mathbb{R}^n} (y - x)^\alpha \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy = 0 \quad \text{a.e.} \quad (0 < |\alpha| < k).$

Proof. We prove only (b)'. We note that $\hat{\varphi}$ is in $C^{k-1}(\mathbb{R}^n)$ and there exists a constant C such that

$$(3.1) \quad \sum_{\nu \in \mathbb{Z}^n} |\varphi(x - \nu) \varphi(y - \nu)| \leq \frac{C}{(1 + |x - y|)^{k-1+n+\epsilon}} \quad (x, y \in \mathbb{R}^n).$$

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ be such that $0 < |\alpha| < k$. Condition (c) gives

$$(3.2) \quad \int_{\mathbb{R}^n} y^\beta \varphi(y) dy = 0 \quad (0 < |\beta| < k).$$

By (3.2) and $\hat{\varphi}(0) = 1$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^n} (y - x)^\alpha \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\ &= \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \int_{\mathbb{R}^n} (y - x)^\alpha \overline{\varphi(y/h - \nu)} dy \\ &= h^n \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \int_{\mathbb{R}^n} (h(y + \nu) - x)^\alpha \overline{\varphi(y)} dy \\ &= h^n \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \int_{\mathbb{R}^n} (h\nu - x)^\alpha \overline{\varphi(y)} dy \\ &= (-1)^{|\alpha|} h^{|\alpha|+n} \sum_{\nu \in \mathbb{Z}^n} (x/h - \nu)^\alpha \varphi(x/h - \nu). \end{aligned}$$

Hence, it suffices to prove that

$$\sum_{\nu \in \mathbb{Z}^n} (x - \nu)^\alpha \varphi(x - \nu) = 0 \quad a.e. \ x \in \mathbb{R}^n.$$

Since $x^\alpha \varphi(x)$ is in $L^1(\mathbb{R}^n)$, if we set $A(x) = \sum_{k \in \mathbb{Z}^n} (x - k)^\alpha \varphi(x - k)$, then $A(x/2\pi)$ is a $2\pi\mathbb{Z}^n$ -periodic function which is integrable on $[-\pi, \pi)^n$. We show that $A(x) = 0$ a.e. To do this, it is enough to prove that all the Fourier coefficients are zero. By (c), we see that

$$\begin{aligned} & \frac{1}{(2\pi)^n} \int_{[-\pi, \pi)^n} A(x/2\pi) e^{-i\ell \cdot x} dx \\ &= \int_{[-\frac{1}{2}, \frac{1}{2})^n} \left(\sum_{\nu \in \mathbb{Z}^n} (x - \nu)^\alpha \varphi(x - \nu) \right) e^{-2\pi i \ell \cdot x} dx \\ &= \int_{\mathbb{R}^n} x^\alpha \varphi(x) e^{-2\pi i \ell \cdot x} dx = i^{|\alpha|} (\partial^\alpha \hat{\varphi})(2\pi \ell) = 0 \quad \text{for all } \ell \in \mathbb{Z}^n. \end{aligned}$$

The proof is complete. $\square$

When we consider local $L^1(w)$ -approximation, the following lemma plays an important role.

Lemma 3.2. *Let $1 \leq p < \infty$ and $w \in A_p$. If φ is a function on $\mathbb{R}^n$ which is non-negative, radial, decreasing and integrable, then there exists a constant C such that*

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x - \alpha y)| \varphi(y) dy \right)^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx$$

for all $f \in L^p(w)$ and $\alpha \in \mathbb{R}$.

Proof. Since φ is radial, by a change of variable, we may assume $\alpha > 0$. We first consider the case $p = 1$. We note that $w \in A_1$ if and only if there exists a constant C such that $Mw(x) \leq Cw(x)$ a.e. ([3, p.134]). By a change of variable, Fubini's theorem and $\varphi(y) = \varphi(-y)$, we have that

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - \alpha y)| \varphi(y) dy w(x) dx \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(y')| \varphi\left(\frac{x - y'}{\alpha}\right) \left(\frac{dy'}{\alpha^n}\right) w(x) dx \\ &= \int_{\mathbb{R}^n} |f(y)| \left(\int_{\mathbb{R}^n} \frac{1}{\alpha^n} \varphi\left(\frac{y - x}{\alpha}\right) w(x) dx \right) dy \\ &= \int_{\mathbb{R}^n} |f(y)| (\varphi_\alpha * w)(y) dy. \end{aligned}$$

From Lemma 2.1, we see that

$$\begin{aligned} \int_{\mathbb{R}^n} |f(y)| (\varphi_\alpha * w)(y) dy &\leq \int_{\mathbb{R}^n} |f(y)| \sup_{\alpha > 0} (\varphi_\alpha * w)(y) dy \\ &\leq \int_{\mathbb{R}^n} |f(y)| \|\varphi\|_{L^1(\mathbb{R}^n)} Mw(y) dy. \end{aligned}$$

Hence, since $w \in A_1$, we get that

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x - \alpha y)| \varphi(y) dy w(x) dx &\leq \|\varphi\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)| Mw(y) dy \\ &\leq C \|\varphi\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n} |f(y)| w(y) dy. \end{aligned}$$

Since $\int_{\mathbb{R}^n} |f(x - \alpha y)| \varphi(y) dy = \varphi_\alpha * |f|(x)$, by Lemma 2.1 and Lemma 2.2, we can prove Lemma 3.2 when $p > 1$. The proof is complete. $\square$

4. Strang-Fix theory in weighted L^p -spaces

In this section, we prove Theorem 1. The boundedness of P_h on $L^p(w)$ is given in, for example, [1]. Since the proof of the boundedness of P_h on $L^1(w)$ is omitted in [1], we give the proof.

Theorem 4.1. *Let $1 \leq p < \infty$, $w \in A_p$, $\epsilon > 0$ and k be a positive integer. Suppose that φ is a function on $\mathbb{R}^n$ such that*

$$\begin{aligned} \text{(a)} \quad & |\varphi(x)| \leq \frac{C}{(1+|x|)^{k+n+\epsilon}} \quad (x \in \mathbb{R}^n), \\ \text{(b)} \quad & \hat{\varphi}(0) = 1, \quad \hat{\varphi}(2\pi\nu) = 0 \quad (\nu \in \mathbb{Z}^n \setminus \{0\}), \\ \text{(c)} \quad & (\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \quad (0 < |\alpha| < k, \quad \nu \in \mathbb{Z}^n), \end{aligned}$$

where C is independent of $x \in \mathbb{R}^n$. Then there exists a constant C such that

$$\|P_h f - f\|_{L^p(w)} \leq Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)}$$

for all $f \in L_k^p(w)$ and $h > 0$.

Proof. We first show that, for $h > 0$, P_h is bounded on $L^p(w)$. Let $f \in L^p(w)$. From (3.1), we have that

$$\begin{aligned} |P_h f(x)| &= \left| \sum_{\nu \in \mathbb{Z}^n} \langle f, \varphi_{h,\nu} \rangle \varphi_{h,\nu}(x) \right| \\ &= \left| \int_{\mathbb{R}^n} h^{-n} f(y) \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \right| \\ &\leq C \int_{\mathbb{R}^n} h^{-n} |f(y)| \frac{1}{(1+h^{-1}|x-y|)^{k+n+\epsilon}} dy \\ (4.1) \quad &= C \int_{\mathbb{R}^n} \frac{1}{(1+|y'|)^{k+n+\epsilon}} |f(x-hy')| dy'. \end{aligned}$$

Thus, from Lemma 3.2, we can see that

$$\|P_h f\|_{L^p(w)} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}.$$

Hence, by the boundedness of P_h on $L^p(w)$ and Lemma 2.4, it suffices to prove that

$$\|P_h f - f\|_{L^p(w)} \leq Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)} \quad (f \in C_c^\infty(\mathbb{R}^n)).$$

Let $f \in C_c^\infty(\mathbb{R}^n)$. By Lemma 3.1, for almost all $x \in \mathbb{R}^n$ we have that

$$\begin{aligned}
& P_h f(x) - f(x) \\
&= \int_{\mathbb{R}^n} h^{-n} f(y) \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\
&\quad - f(x) \int_{\mathbb{R}^n} h^{-n} \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\
&= \int_{\mathbb{R}^n} h^{-n} f(y) \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\
&\quad - f(x) \int_{\mathbb{R}^n} h^{-n} \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\
&\quad - \sum_{0 < |\alpha| < k} \frac{(\partial^\alpha f)(x)}{\alpha!} \int_{\mathbb{R}^n} h^{-n} (y-x)^\alpha \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy \\
&= \int_{\mathbb{R}^n} h^{-n} \left(f(y) - \sum_{|\alpha| < k} \frac{(\partial^\alpha f)(x)}{\alpha!} (y-x)^\alpha \right) \sum_{\nu \in \mathbb{Z}^n} \varphi(x/h - \nu) \overline{\varphi(y/h - \nu)} dy.
\end{aligned}$$

By Taylor's theorem, (3.1) and a change of variable, we see that

$$\begin{aligned}
& |P_h f(x) - f(x)| \\
&\leq \int_{\mathbb{R}^n} h^{-n} \frac{C}{(1 + h^{-1}|y-x|)^{k+n+\epsilon}} \\
&\quad \times \left| k \sum_{|\alpha|=k} \frac{(y-x)^\alpha}{\alpha!} \int_0^1 (1-t)^{k-1} (\partial^\alpha f)(x + t(y-x)) dt \right| dy \\
&\leq C \sum_{|\alpha|=k} \int_{\mathbb{R}^n} h^{-n} |y-x|^{|\alpha|} \\
&\quad \times \frac{1}{(1 + h^{-1}|y-x|)^{k+n+\epsilon}} \int_0^1 |(\partial^\alpha f)(x + t(y-x))| dt dy \\
&= C \sum_{|\alpha|=k} \int_0^1 \int_{\mathbb{R}^n} h^{-n} |y-x|^k \\
&\quad \times \frac{1}{(1 + h^{-1}|y-x|)^{k+n+\epsilon}} |(\partial^\alpha f)(x + t(y-x))| dy dt \\
&= Ch^k \sum_{|\alpha|=k} \int_0^1 \int_{\mathbb{R}^n} |y'|^k \frac{1}{(1 + |y'|)^{k+n+\epsilon}} |(\partial^\alpha f)(x + thy')| dy' dt
\end{aligned}$$

$$(4.2) \quad \leq Ch^k \sum_{|\alpha|=k} \int_0^1 \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{n+\epsilon}} |(\partial^\alpha f)(x - thy)| dy dt.$$

Thus, by Minkowski's inequality for integrals and Lemma 3.2, we get that

$$\begin{aligned} & \|P_h f - f\|_{L^p(w)} \\ & \leq Ch^k \sum_{|\alpha|=k} \int_0^1 \left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{n+\epsilon}} |(\partial^\alpha f)(x - thy)| dy \right)^p w(x) dx \right\}^{1/p} dt \\ & \leq Ch^k \sum_{|\alpha|=k} \int_0^1 \|\partial^\alpha f\|_{L^p(w)} dt = Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^p(w)} \end{aligned}$$

The proof is complete. $\square$

We are now ready to prove Theorem 1. If $w > 0$ a.e., then $\|\cdot\|_{L^\infty(w)} = \|\cdot\|_{L^\infty(\mathbb{R}^n)}$. On the other hand, if $w \in A_\infty$ then $w > 0$ a.e. (see section 2). Thus, since $L_k^\infty(w) = L_k^\infty(\mathbb{R}^n)$ when $w \in A_\infty$, Theorem 1 for $p = \infty$ was proved by de Boor and Jia ([2]). Hence, for the proof of Theorem 1, we consider only the case $1 \leq p < \infty$.

Proof of Theorem 1. We first prove (i)' $\Rightarrow$ (ii)'. Suppose that $1 \leq p < \infty$, $w \in A_p$ and $\Phi = \{\varphi_1, \dots, \varphi_N\} \subset C_c(\mathbb{R}^n)$ satisfies the Strang-Fix condition of order k . By Lemma 2.7, there exist finitely supported sequences b_j ($j = 1, \dots, N$) such that the function $\varphi = \sum_{j=1}^N \varphi_j *' b_j$ satisfies

$$\hat{\varphi}(0) = 1, \quad \hat{\varphi}(2\pi\nu) = 0 \quad (\nu \in \mathbb{Z}^n \setminus \{0\})$$

and

$$(\partial^\alpha \hat{\varphi})(2\pi\nu) = 0 \quad (0 < |\alpha| < k, \quad \nu \in \mathbb{Z}^n).$$

Since $\varphi \in C_c(\mathbb{R}^n)$, by Theorem 4.1, we have that

$$(4.3) \quad \|P_h f - f\|_{L^p(w)} \leq Ch^k \|f\|_{L_k^p(w)} \quad (f \in L_k^p(w), \quad h > 0),$$

where $P_h f = \sum_{\ell \in \mathbb{Z}^n} \langle f, \varphi_{h,\ell} \rangle \varphi_{h,\ell}$. Since $\varphi_{h,\ell} = \sum_{j=1}^N (\varphi_j *' b_j)_{h,\ell}$, we have that

$$\begin{aligned} P_h f &= \sum_{\ell \in \mathbb{Z}^n} \langle f, \varphi_{h,\ell} \rangle \left(\sum_{j=1}^N (\varphi_j *' b_j)_{h,\ell} \right) \\ &= \sum_{j=1}^N \sum_{\ell \in \mathbb{Z}^n} \langle f, \varphi_{h,\ell} \rangle \left(h^{-n/2} \sum_{\nu \in \mathbb{Z}^n} \varphi_j(\cdot/h - \ell - \nu) b_j(\nu) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^N \sum_{\ell \in \mathbb{Z}^n} \langle f, \varphi_{h,\ell} \rangle \left(h^{-n/2} \sum_{\nu \in \mathbb{Z}^n} \varphi_j(\cdot/h - \nu) b_j(\nu - \ell) \right) \\
&= \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} \left(\sum_{\ell \in \mathbb{Z}^n} h^{-n/2} \langle f, \varphi_{h,\ell} \rangle b_j(\nu - \ell) \right) \varphi_j(\cdot/h - \nu).
\end{aligned}$$

If $c_j^h(\nu) = \sum_{\ell \in \mathbb{Z}^n} h^{-n/2} \langle f, \varphi_{h,\ell} \rangle b_j(\nu - \ell)$, then we can write that

$$(4.4) \quad P_h f = \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right).$$

Hence, by (4.3) and (4.4), it suffices to prove that there exists a constant $r > 0$ such that $c_j^h(\nu) = 0$ whenever $\text{dist}(h\nu, \text{supp } f) > r$ (that is, (iv)). Let $R > 0$ and $M > 0$ be such that $\text{supp } \varphi \subset B(0, R)$ and $b_j(\nu) = 0$ ($j = 1, \dots, N$, $|\nu| > M$). We also suppose $0 < h < 1$. Since

$$c_j^h(\nu) = \sum_{\ell \in \mathbb{Z}^n} h^{-n/2} \langle f, \varphi_{h,\nu-\ell} \rangle b_j(\ell) = \sum_{|\ell| \leq M} h^{-n/2} \langle f, \varphi_{h,\nu-\ell} \rangle b_j(\ell),$$

if we take $r = R + M$, then $c_j^h(\nu) = 0$ whenever $\text{dist}(h\nu, \text{supp } f) > r$. This shows (i)' $\Rightarrow$ (ii)'.

(ii)' $\Rightarrow$ (iii)' is clear. We next prove (iii)' $\Rightarrow$ (i)'. The proof is similar to one of [7, Theorem 5.1]. Let $1 \leq p < \infty$, $w \in A_p$ and $0 < h < 1$. For a positive integer m , B_m denotes the B-spline of degree m defined by $B_1 = \chi_{[-1/2, 1/2]}$ and for $m = 2, 3, \dots$

$$B_m(x) = B_{m-1} * B_1(x) \quad (x \in \mathbb{R}),$$

where $\chi_{[-1/2, 1/2]}$ is the characteristic function of $[-1/2, 1/2]$. We approximate the following tensor product

$$u(x) = \prod_{j=1}^n B_{k+1}(x_j) \quad (x = (x_1, \dots, x_n) \in \mathbb{R}^n).$$

Since $u \in L_k^p(w)$, we can find weights c_j^h ($0 < h < 1$, $j = 1, \dots, N$) so that conditions (iv) and (v) are satisfied. Let $u_h = \sigma_{1/h} \left(\sum_{j=1}^N \varphi_j *' c_j^h \right)$ and $g_h = u - u_h$. Then conditions (iv) and (v) imply $c_j^h(\nu) = 0$ whenever $\text{dist}(h\nu, \text{supp } f) > r$ and

$$(4.5) \quad \|g_h\|_{L^p(w)} \leq Ch^k \quad (0 < h < 1).$$

Since $\text{supp } u \subset [-k, k]^n$, if $R = r + \sqrt{n}k + 1$, then $c_j^h(\nu) = 0$ ($|h\nu| > R$). Hence, we have that

$$(4.6) \quad u_h = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} \varphi_j(\cdot/h - \nu) c_j^h(\nu) = \sum_{j=1}^N \sum_{|h\nu| \leq R} \varphi_j(\cdot/h - \nu) c_j^h(\nu).$$

By (4.6) and $\varphi_j \in C_c(\mathbb{R}^n)$ ($j = 1, \dots, N$), there exists a constant R' such that $\text{supp } g_h \subset B(0, R')$ ($0 < h < 1$). Thus, by Hölder's inequality and (4.5), we see that

$$\begin{aligned} |\partial^\alpha \hat{g}_h(\xi)| &= \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} x^\alpha g_h(x) dx \right| \\ &\leq \int_{B(0, R')} |x|^{|\alpha|} |g_h(x)| w(x)^{1/p} w(x)^{-1/p} dx \\ &\leq C_{R'} \|g_h\|_{L^p(w)} \left(w^{1-p'}(B_{R'}) \right)^{1/p'} \\ &\leq C_\alpha h^k \quad (0 < h < 1). \end{aligned}$$

Therefore, we get that

$$(4.7) \quad \|\partial^\alpha \hat{g}_h\|_{L^\infty(\mathbb{R}^n)} \leq Ch^k \quad (|\alpha| < k, \quad 0 < h < 1).$$

On the other hand, since

$$\hat{u}(\xi) = \prod_{j=1}^n \left\{ \frac{\sin(\xi_j/2)}{\xi_j/2} \right\}^{k+1} \quad (\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n),$$

we have that

$$(4.8) \quad \hat{u}(0) = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{(\partial^\alpha \hat{u})(\xi/h)}{h^{k-1}} = 0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| < k).$$

Hence, by (4.7) and (4.8), we get that

$$(4.9) \quad \lim_{h \rightarrow 0} \hat{u}_h(0) = 1 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{(\partial^\alpha \hat{u}_h)(\xi/h)}{h^{k-1}} = 0 \quad (\xi \in \mathbb{R}^n \setminus \{0\}, \quad |\alpha| < k).$$

From (4.6) and (4.9), we see that $\{u_h\}_{0 < h < 1}$ satisfies the hypothesis of Lemma 2.8. Thus, by Lemma 2.8, Φ satisfies the Strang-Fix condition of order k . The proof is complete.

5. Strang-Fix theory in weighted Herz spaces

In this section, we prove Theorem 2. We first prove the weighted Herz spaces version of Theorem 4.1.

Theorem 5.1. *Let $w_1 \in A_{q_{w_1}}$, $w_2 \in A_{q_{w_2}}$, $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$, $\epsilon > 0$ and k be a positive integer, where w_1 and w_2 satisfy either Theorem 2 (a) or Theorem 2 (b). Suppose that a function φ on $\mathbb{R}^n$ satisfies the hypothesis of Theorem 4.1. Then there exists a constant C such that*

$$\|P_h f - f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} \leq Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}$$

for all $f \in \dot{K}_{q,k}^{\alpha,p}(w_1,w_2)$ and $h > 0$.

Proof. By (4.1) and Lemma 2.1, we see that

$$|P_h f(x)| \leq C \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{k+n+\epsilon}} |f(x-hy)| dy \leq CMf(x).$$

Hence, from Lemma 2.6, we have that P_h is bounded on $\dot{K}_q^{\alpha,p}$. By the boundedness of P_h on $\dot{K}_q^{\alpha,p}(w_1,w_2)$ and Lemma 2.5, it is enough to prove that

$$(5.1) \quad \|P_h f - f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} \leq Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} \quad (f \in C_c^\infty(\mathbb{R}^n)).$$

Let $f \in C_c^\infty(\mathbb{R}^n)$. From (4.2) and Lemma 2.1, we have that

$$\begin{aligned} |P_h f(x) - f(x)| &\leq Ch^k \sum_{|\alpha|=k} \int_0^1 \int_{\mathbb{R}^n} \frac{1}{(1+|y|)^{n+\epsilon}} |(\partial^\alpha f)(x-thy)| dy dt \\ &\leq Ch^k \sum_{|\alpha|=k} M[\partial^\alpha f](x). \end{aligned}$$

Hence, by Lemma 2.6, we see that

$$\begin{aligned} \|P_h f - f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} &\leq Ch^k \sum_{|\alpha|=k} \|M[\partial^\alpha f]\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)} \\ &\leq Ch^k \sum_{|\alpha|=k} \|\partial^\alpha f\|_{\dot{K}_q^{\alpha,p}(w_1,w_2)}. \end{aligned}$$

Hence, we get (5.1). The proof is complete. $\square$

We are now ready to prove Theorem 2.

Proof of Theorem 2. Since we have proved Theorem 5.1, the proof of (i)" $\Rightarrow$ (ii)" is similar to one of Theorem 1 (i)' $\Rightarrow$ (ii)'. We prove only (ii)" $\Rightarrow$ (i)". From the proof of Theorem 1 (iii)' $\Rightarrow$ (i)', to prove (ii)" $\Rightarrow$ (i)", it is enough to show that $\otimes_{j=1}^n B_{k+1} \in \dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$ and for each $R > 0$ there exists a constant $C_R > 0$ such that

$$(5.2) \quad \int_{B(0,R)} |f(y)| dy \leq C_R \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \quad (f \in \dot{K}_q^{\alpha,p}(w_1, w_2)),$$

where $\otimes_{j=1}^n B_{k+1}$ denotes the tensor product of B-splines of degree $k+1$. Since $\partial^\alpha (\otimes_{j=1}^n B_{k+1}) \in L_c^\infty(\mathbb{R}^n)$ ($|\alpha| \leq k$), to prove $\otimes_{j=1}^n B_{k+1} \in \dot{K}_{q,k}^{\alpha,p}(w_1, w_2)$, it is enough to show $L_c^\infty(\mathbb{R}^n) \subset \dot{K}_q^{\alpha,p}(w_1, w_2)$, where $L_c^\infty(\mathbb{R}^n)$ consists of all $L^\infty(\mathbb{R}^n)$ -functions with compact support.

We first prove $L_c^\infty(\mathbb{R}^n) \subset \dot{K}_q^{\alpha,p}(w_1, w_2)$. Let $f \in L_c^\infty(\mathbb{R}^n)$. If we take $k_0 \in \mathbb{Z}$ such that $\text{supp } f \subset B_{k_0} = B(0, 2^{k_0})$, then

$$\begin{aligned} & \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} \\ &= \left(\sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} \|f \chi_{R_k}\|_{L^q(w_2)}^p \right)^{1/p} \\ &\leq \|f\|_{L^\infty(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha p/n} w_2(B_k)^{p/q} \right)^{1/p} \\ &= C_{k_0, w_1, w_2} \|f\|_{L^\infty(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{k_0} \left[\frac{w_1(B_k)}{w_1(B_{k_0})} \right]^{\alpha p/n} \left[\frac{w_2(B_k)}{w_2(B_{k_0})} \right]^{p/q} \right)^{1/p}. \end{aligned}$$

Since $\alpha/n + 1/q > 0$ when $w_1 = w_2$ or $\alpha > 0$ when $w_1 \neq w_2$, from Lemma 2.3 (b), we have that

$$\begin{aligned} \left[\frac{w_1(B_k)}{w_1(B_{k_0})} \right]^{\alpha p/n} \left[\frac{w_2(B_k)}{w_2(B_{k_0})} \right]^{p/q} &\leq C_{w_1, w_2} \left[\frac{|B_k|}{|B_{k_0}|} \right]^{\alpha p \delta_{w_1}/n} \left[\frac{|B_k|}{|B_{k_0}|} \right]^{p \delta_{w_2}/q} \\ &= C_{n, w_1, w_2} 2^{np(k-k_0)(\alpha \delta_{w_1}/n + \delta_{w_2}/q)}. \end{aligned}$$

Thus, since $\alpha \delta_{w_1}/n + \delta_{w_2}/q > 0$, we get that

$$\begin{aligned} \|f\|_{\dot{K}_q^{\alpha,p}(w_1, w_2)} &\leq C_{k_0, w_1, w_2} \|f\|_{L^\infty(\mathbb{R}^n)} \left(\sum_{k=-\infty}^{k_0} 2^{np(k-k_0)(\alpha \delta_{w_1}/n + \delta_{w_2}/q)} \right)^{1/p} \\ &< \infty. \end{aligned}$$

We next prove (5.2). We take $k_0 \in \mathbb{Z}$ such that $B(0, R) \subset B_{k_0} = B(0, 2^{k_0})$. By $w \in A_q$ and the definition of A_p -weights, we see that

$$\begin{aligned}
\int_{B(0, R)} |f(y)| dy &\leq \int_{B_{k_0}} |f(y)| dy = \sum_{k=-\infty}^{k_0} \int_{B_k \setminus B_{k-1}} |f(y)| dy \\
&\leq \sum_{k=-\infty}^{k_0} \|f \chi_{R_k}\|_{L^q(w_2)} \left(w_2^{1-q'}(B_k) \right)^{1/q'} \\
&\leq A_q(w_2)^{1/q} \sum_{k=-\infty}^{k_0} \|f \chi_{R_k}\|_{L^q(w_2)} |B_k| w_2(B_k)^{-1/q} \\
&= C_{k_0, w_1, w_2} \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha/n} \|f \chi_{R_k}\|_{L^q(w_2)} \\
&\quad \times |B_k| \left[\frac{w_1(B_{k_0})}{w_1(B_k)} \right]^{\alpha/n} \left[\frac{w_2(B_{k_0})}{w_2(B_k)} \right]^{1/q}.
\end{aligned}$$

Since $\alpha/n + 1/q > 0$ when $w_1 = w_2$ or $\alpha > 0$ when $w_1 \neq w_2$, from Lemma 2.3 (a), we have that

$$\begin{aligned}
\left[\frac{w_1(B_{k_0})}{w_1(B_k)} \right]^{\alpha/n} \left[\frac{w_2(B_{k_0})}{w_2(B_k)} \right]^{1/q} &\leq C_{w_1, w_2} \left[\frac{|B_{k_0}|}{|B_k|} \right]^{\alpha q w_1/n} \left[\frac{|B_{k_0}|}{|B_k|} \right]^{q w_2/q} \\
&= C_{n, w_1, w_2} 2^{n(k_0-k)(\alpha q w_1/n + q w_2/q)}.
\end{aligned}$$

Thus, we see that

$$\int_{B(0, R)} |f(y)| dy \leq C \sum_{k=-\infty}^{k_0} w_1(B_k)^{\alpha/n} \|f \chi_{R_k}\|_{L^q(w_2)} 2^{n(k_0-k)(\alpha q w_1/n + q w_2/q - 1)},$$

where C depends only on R , w_1 and w_2 . Since $\alpha q w_1/n + q w_2/q - 1 < 0$, we get (5.2). The proof is complete.

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