Regularity results for singular elliptic problems

Loredana Caso

(Communicated by Miroslav Engliš)

2000 Mathematics Subject Classification. 35J25, 35B65, 35R05.

Keywords and phrases. Elliptic equations, regularity of solutions, discontinuous coefficients and data.

Abstract. Some local and global regularity results for solutions of linear elliptic equations in weighted spaces are proved. Here the leading coefficients are VMO functions, while the hypotheses on the other coefficients and the boundary conditions involve a suitable weight function.

1. Introduction

Consider the second order linear differential equation

$$(1.1) \quad L_o u = - \sum_{i,j=1}^n a_{ij} u_{x_i x_j} = f \quad \text{a. e. in } \Omega,$$

where L_o is a uniformly elliptic operator in a bounded open subset Ω of $\mathbb{R}^n$, $n \geq 3$, and $f \in L^p(\Omega)$, $p > 1$. A classical problem in the theory of linear elliptic equations in non-divergence form is the study of local and global regularity properties of solutions of (1.1).

It is well known that, when the a_{ij} 's are uniformly continuous in Ω , any solution u of (1.1) in $W_{\text{loc}}^{2,q}(\Omega)$, $1 < q \leq p$, belongs to $W_{\text{loc}}^{2,p}(\Omega)$ (see [11], Chapter 9). On the other hand, if the coefficients a_{ij} are required to be discontinuous, some other kind of assumptions are necessary to get regularity results for the solutions of (1.1). For instance, if either the a_{ij} 's belong to $W^{1,n}(\Omega)$ or they satisfy the "Cordes condition", then local $W^{2,p}$ -regularity results (with p belonging to a suitable neighborhood of $p = 2$) for the solutions of (1.1) have been obtained (see [10] and [2]).

More recently, some local regularity results in $W^{2,p}$, for any $p \in]1, +\infty[$, have been proved under the assumption that the coefficients a_{ij} are bounded and of class VMO (see [8]). Observe that this latter condition is always satisfied if the a_{ij} 's either are uniformly continuous or lie in $W^{1,n}$. The results of [8] have been extended to the case of operators whose lower order terms occur and belong to suitable spaces $L^r(\Omega)$ (see [16]).

If Ω is an arbitrary open subset of $\mathbb{R}^n$, $n \geq 3$, a $W^{2,p}(\Omega)$ -regularity result has been proved in [6] for the solutions in $W_{\text{loc}}^{2,q}(\bar{\Omega}) \cap \overset{\circ}{W}_{\text{loc}}^{1,q}(\bar{\Omega}) \cap L^p(\Omega)$, $1 < q \leq p$, of the linear elliptic equation

$$(1.2) \quad Lu = - \sum_{i,j=1}^n a_{ij} u_{x_i x_j} + \sum_{i=1}^n a_i u_{x_i} + a u = f,$$

$f \in L^p(\Omega)$, under the conditions that the coefficients a_{ij} are bounded and belong to the space $VMO(\Omega)$, while the coefficients a_i and a lie in suitable spaces of Morrey type. Moreover, in [4] this result has been improved, obtaining local and global regularity results under weaker assumptions; in particular, the a_{ij} 's are assumed locally VMO and satisfying a suitable condition at infinity.

The aim of this paper is to obtain some local and global regularity results for the solutions of the equation (1.2) in certain weighted Sobolev spaces. More precisely, let ρ be a suitable weight function and denote by S_ρ the subset of $\partial\Omega$ where ρ goes to zero. Suppose that there exist extensions a_{ij}^o of a_{ij} in $VMO(\Omega_o) \cap L^\infty(\Omega_o)$, where Ω_o is a regular open set containing Ω ; assume also that the coefficients a_i and a satisfy certain local summability hypotheses and are singular near S_ρ . If $f \in L_{\text{loc}}^p(\Omega)$, we will prove that all solutions of (1.2) in $W_{\text{loc}}^{2,q}(\bar{\Omega} \setminus S_\rho) \cap \overset{\circ}{W}_{\text{loc}}^{1,q}(\bar{\Omega} \setminus S_\rho)$, $1 < q \leq p$, belong to the Sobolev space $W_{\text{loc}}^{2,p}(\bar{\Omega} \setminus S_\rho)$. Moreover, if f and the solutions of (1.2) belong to certain weighted L^p -spaces, where the weight functions are suitable powers of ρ , then such solutions belong to the weighted Sobolev space $W_s^{2,p}(\Omega)$.

2. Notation and function spaces

In this paper we use the following basic notation: E , a generic Lebesgue measurable subset of $\mathbb{R}^n$; $\Sigma(E)$, the Lebesgue σ -algebra on E ; $|A|$, the Lebesgue measure of $A \in \Sigma(E)$; χ_A , the characteristic function of A ; $\mathfrak{D}(A)$, the class of restrictions to A of functions $\zeta \in C_o^\infty(\mathbb{R}^n)$ with $\bar{A} \cap \text{supp } \zeta \subseteq A$; $L_{\text{loc}}^p(A)$ ($p \in [1, +\infty]$), the class of functions g , defined on A , such that $\zeta g \in L^p(A)$ for all $\zeta \in \mathfrak{D}(A)$; $B(x, r)$, the open ball of radius r centered at x .

Let Ω be an open subset of $\mathbb{R}^n$. For each $E \in \Sigma(\Omega)$ we put

$$E(x, r) = E \cap B(x, r) \quad \forall x \in \mathbb{R}^n, \quad \forall r \in \mathbb{R}_+.$$

Denote by $\mathcal{A}(\Omega)$ the class of measurable functions $\rho : \Omega \rightarrow \mathbb{R}_+$ such that

$$(2.1) \quad \gamma^{-1} \rho(y) \leq \rho(x) \leq \gamma \rho(y) \quad \forall y \in \Omega, \quad \forall x \in \Omega(y, \rho(y)),$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y . For $\rho \in \mathcal{A}(\Omega)$, we put

$$S_\rho = \{z \in \partial\Omega : \lim_{x \rightarrow z} \rho(x) = 0\}.$$

It is known that

$$(2.2) \quad \rho \in L_{\text{loc}}^\infty(\bar{\Omega}), \quad \rho^{-1} \in L_{\text{loc}}^\infty(\bar{\Omega} \setminus S_\rho),$$

and, if $S_\rho \neq \emptyset$,

$$(2.3) \quad \rho(x) \leq \text{dist}(x, S_\rho) \quad \forall x \in \Omega$$

(see [5], [14]).

If $r \in \mathbb{N}$, $1 \leq p \leq +\infty$, $s \in \mathbb{R}$ and $\rho \in \mathcal{A}(\Omega)$, we consider the space $W_s^{r,p}(\Omega)$ of distributions u on Ω such that $\rho^{s+|\alpha|-r} \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq r$, equipped with the norm

$$\|u\|_{W_s^{r,p}(\Omega)} = \sum_{|\alpha| \leq r} \|\rho^{s+|\alpha|-r} \partial^\alpha u\|_{L^p(\Omega)}.$$

Moreover, we denote by $\mathring{W}_s^{r,p}(\Omega)$ the closure of $C_o^\infty(\Omega)$ in $W_s^{r,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$. A more detailed account of properties of the above defined spaces can be found in [9], [1] and [15].

For $p \in [1, +\infty[$, denote by $K_s^p(\Omega)$ the set of all functions $g \in L_{\text{loc}}^p(\bar{\Omega} \setminus S_\rho)$ such that

$$(2.4) \quad \|g\|_{K_s^p(\Omega)} = \sup_{x \in \Omega} \left(\rho^{s-n/p}(x) \|g\|_{L^p(\Omega(x, \rho(x)))} \right) < +\infty,$$

endowed with the norm defined by (2.4). Moreover, $\tilde{K}_s^p(\Omega)$ denotes the closure of $L_s^\infty(\Omega)$ in $K_s^p(\Omega)$. It is known that a function g (in $K_s^p(\Omega)$) belongs to $\tilde{K}_s^p(\Omega)$ if and only if the function

$$\sigma_{p,g}^s(t) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} \frac{|E(x, \rho(x))|}{\rho^n(x)} \leq t}} \|g \chi_E\|_{K_s^p(\Omega)}, \quad t \in \mathbb{R}_+,$$

vanishes when t goes to zero. Thus a *modulus of continuity* of g in $\tilde{K}_s^p(\Omega)$ is a map $\tilde{\sigma}_p[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\sigma_{p,g}^s(t) \leq \tilde{\sigma}_p[g](t) \quad \forall t \in \mathbb{R}_+, \quad \lim_{t \rightarrow 0^+} \tilde{\sigma}_p[g](t) = 0.$$

For some properties of the spaces $K_s^p(\Omega)$ and $\tilde{K}_s^p(\Omega)$ we refer to [3] and [5].

If Ω has the property

$$(2.5) \quad |\Omega(x, r)| \geq A r^n \quad \forall x \in \Omega, \quad \forall r \in]0, 1],$$

where A is a positive constant independent of x and r , it is possible to consider the space $BMO(\Omega, t)$ ($t \in \mathbb{R}_+$) of functions $g \in L_{\text{loc}}^1(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega, t)} = \sup_{\substack{x \in \Omega \\ r \in]0, t]}} \int_{\Omega(x, r)} \left| g - \int_{\Omega(x, r)} g \right| dy < +\infty,$$

where $\int_{\Omega(x, r)} g dy = \frac{1}{|\Omega(x, r)|} \int_{\Omega(x, r)} g dy$.

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, where

$$t_A = \sup_{t \in \mathbb{R}_+} \left(\sup_{\substack{x \in \Omega \\ r \in]0, t]}} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right),$$

we shall say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega, t)} \rightarrow 0$ for $t \rightarrow 0^+$. A function $\eta[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega, t)} \leq \eta[g](t) \quad \forall t \in \mathbb{R}_+, \quad \lim_{t \rightarrow 0^+} \eta[g](t) = 0.$$

3. Preliminary results

Fix $\rho \in \mathcal{A}(\Omega)$. For all $x \in \Omega$ and for all $\lambda \in \mathbb{R}_+$ put

$$(3.1) \quad E_\lambda(x) = \{y \in \Omega : |x - y| < \lambda \rho(y)\}, \quad E(x) = E_1(x),$$

$$(3.2) \quad I_\lambda(x) = \Omega \cap B(x, \lambda \rho(x)), \quad I(x) = I_1(x).$$

Obviously

$$(3.3) \quad y \in I_\lambda(x) \Leftrightarrow x \in E_\lambda(y) \quad \forall x, y \in \Omega.$$

It is easy to prove that

$$(3.4) \quad E(x) \subset I_\gamma(x) \quad \forall x \in \Omega,$$

and

$$(3.5) \quad I_{\lambda_o \gamma^{-1}}(x) \subset E_\lambda(x) \quad \forall x \in \Omega,$$

for all $\lambda \in \mathbb{R}_+$, where γ is the constant in (2.1) and $\lambda_o = \min\{\lambda, \gamma\}$.

We consider the following conditions on ρ :

(α) there exists an extension $\tilde{\rho} \in \mathcal{A}(\mathbb{R}^n \setminus S_\rho)$ of ρ to $\mathbb{R}^n \setminus S_\rho$,

(β) $H = \inf_{\Omega} \rho^{-n}(x) |I(x)| \in \mathbb{R}_+$.

Remark 3.1. If condition (α) holds and $S_\rho \neq \emptyset$, then

$$(3.6) \quad S_{\tilde{\rho}} = S_\rho$$

and there exists $a \in]0, 1[$ such that

$$(3.7) \quad \rho(x) \leq a \operatorname{dist}(x, S_\rho) \quad \forall x \in \Omega$$

(see [3] and [5]).

If (α) holds, we consider the sets $\tilde{E}_\lambda(x)$ and $\tilde{I}_\lambda(x)$, $x \in \mathbb{R}^n \setminus S_\rho$, defined by (3.1), (3.2), respectively, corresponding to $\rho = \tilde{\rho}$. From Remark 3.1 it follows that

$$(3.8) \quad \tilde{I}_\lambda(x) = B(x, \lambda \tilde{\rho}(x)) \quad \forall x \in \mathbb{R}^n \setminus S_\rho, \quad \forall \lambda \in]0, 1].$$

Lemma 3.1. *If condition (α) holds, then*

$$(3.9) \quad \omega_n \gamma^{-n} \tilde{\rho}^n(x) \leq |\tilde{E}(x)| \leq \omega_n \gamma^n \tilde{\rho}^n(x) \quad \forall x \in \mathbb{R}^n \setminus S_\rho,$$

while if (β) holds, then

$$(3.10) \quad H \rho^n(x) \leq |E(x)| \leq \omega_n \gamma^n \rho^n(x) \quad \forall x \in \Omega,$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. The result easily follows from (3.4), (3.5) and (3.8). $\square$

Now we define the function $\rho_o : \mathbb{R}^n \setminus S_\rho \rightarrow \mathbb{R}_+$ by

$$\begin{aligned} \rho_o(x) &= \tilde{\rho}(x) \quad \text{if } (\alpha) \text{ holds,} \\ \rho_o(x) &= \begin{cases} \rho(x) & x \in \Omega \text{ if } (\beta) \text{ holds} \\ 0 & x \in \mathbb{R}^n \setminus (\Omega \cup S_\rho) \end{cases} \end{aligned}$$

For any $x \in \mathbb{R}^n \setminus S_\rho$, put

$$\begin{aligned} I^o(x) &= (\mathbb{R}^n \setminus S_\rho) \cap B(x, \rho_o(x)), \\ E^o(x) &= \{y \in \mathbb{R}^n \setminus S_\rho : |y - x| < \rho_o(y)\}, \end{aligned}$$

and for $x, y \in \mathbb{R}^n \setminus S_\rho$, let

$$\Phi^o(x, y) = \begin{cases} 1 & y \in I^o(x) \\ 0 & y \notin I^o(x) \end{cases}.$$

Lemma 3.2. *Assume that either (α) or (β) holds. Then for any $p, q \in [1, +\infty[$, $q \geq p$, and for all $s \in \mathbb{R}$ there exist $c_1, c_2 \in \mathbb{R}_+$ such that*

$$(3.11) \quad \int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sp-n}(x) \|u_o\|_{L^p(I^o(x))}^p dx \geq c_1 \int_{\Omega} \rho^{sp}(x) |u(x)|^p dx,$$

$$(3.12) \quad \int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sq-n}(x) \|u_o\|_{L^q(I^o(x))}^q dx \leq c_2 \left(\int_{\Omega} \rho^{sp}(x) |u(x)|^p dx \right)^{\frac{q}{p}},$$

for all $u \in L_{\text{loc}}^p(\bar{\Omega} \setminus S_\rho)$, where u_o is the zero extension of u outside Ω , c_1 depends on γ, n, s, p and also on H if (β) holds, while c_2 depends on γ, n, s, p, q .

Proof. Fix $p \in [1, +\infty[$, $s \in \mathbb{R}$ and $u \in L_{\text{loc}}^p(\bar{\Omega} \setminus S_\rho)$. Then

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sp-n}(x) \|u_o\|_{L^p(I^o(x))}^p dx \\ &= \int_{\mathbb{R}^n \setminus S_\rho} \left(\rho_o^{sp-n}(x) \int_{I^o(x)} |u_o(y)|^p \Phi^o(x, y) dy \right) dx \end{aligned}$$

$$= \int_{\mathbb{R}^n \setminus S_\rho} \left(|u_o(y)|^p \int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sp-n}(x) \Phi^o(x, y) dx \right) dy.$$

So by (2.1), (3.3) and Lemma 3.1, we deduce (3.11).

Fix now $q \geq p$ and put

$$h(x) = \rho_o^{s(q-p)}(x) \left(\int_{I^o(x)} |u_o(y)|^p dy \right)^{q/p-1}, \quad x \in \mathbb{R}^n \setminus S_\rho.$$

Thus from (2.1) we have

$$\begin{aligned} & \int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sq-n}(x) \|u_o\|_{L^p(I^o(x))}^q dx \\ &= \int_{\mathbb{R}^n \setminus S_\rho} \left(\rho_o^{sp-n}(x) h(x) \int_{I^o(x)} |u_o(y)|^p \Phi^o(x, y) dy \right) dx \\ &\leq c_3 \int_{\mathbb{R}^n \setminus S_\rho} \left(\rho_o^{(s-\frac{n}{q})p}(y) |u_o(y)|^p \left(\int_{E^o(y)} dx \right)^{p/q} \right) dy \\ &\quad \cdot \left(\int_{\mathbb{R}^n \setminus S_\rho} \rho_o^{sq-n}(x) \|u_o\|_{L^p(I^o(x))}^q dx \right)^{1-p/q}, \end{aligned}$$

where c_3 depends on γ, s, p, q . It follows from Lemma 3.1 that (3.12) holds. $\square$

Consider the following hypothesis on Ω :

(i_1) there exist $b \in]0, 1]$ and $\theta \in]0, \frac{\pi}{2}[$ such that

$$\forall x \in \Omega \quad \exists C_\theta(x) : \overline{C_\theta(x, b\rho(x))} \subset \Omega,$$

where $C_\theta(x)$ is an open indefinite cone with vertex in x and opening θ and $C_\theta(x, b\rho(x)) = C_\theta(x) \cap B(x, b\rho(x))$.

Remark 3.2. If condition (i_1) holds, then also (β) holds (see [14]).

Fix $k, r \in \mathbb{N}$, $p, p_o \in [1, +\infty[$ and consider the following condition

$$(i_2) \quad \frac{1}{rp} + \frac{r-1}{p_o} \leq 1, \quad 0 \leq k < r, \quad \frac{k}{r} \leq a \leq \vartheta, \quad \vartheta \in \left] \frac{r-1}{r}, 1 \right[, \quad q \geq \max\{p, p_o\}$$

where $\frac{1}{q} = \frac{k}{n} + a \left(\frac{1}{p} - \frac{r}{n} \right) + \frac{1-a}{p_o}$.

Lemma 3.3. Assume that conditions (i_1) and (i_2) hold. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\alpha_1 - \alpha_2 = r + \frac{n}{p_o} - \frac{n}{p}.$$

Then for any function u such that $\partial^r u \in L_{\text{loc}}^p(\Omega)$, $\partial^r u \in L_{\alpha_1}^p(\Omega)$, $u \in L_{\alpha_2}^{p_o}(\Omega)$, we have

$$\|\partial^k u\|_{L_{\alpha_o}^q(\Omega)} \leq c \left(\|\partial^r u\|_{L_{\alpha_1}^p(\Omega)}^a \cdot \|u\|_{L_{\alpha_2}^{p_o}(\Omega)}^{1-a} + \|u\|_{L_{\alpha_2}^{p_o}(\Omega)} \right),$$

where $\alpha_o = a\alpha_1 + (1-a)\alpha_2$ and $c \in \mathbb{R}_+$ depends on $\Omega, n, r, k, \alpha_1, \alpha_2, p, p_o, q, \rho$.

Proof. The proof is similar to that of Theorem 2.1 in [13], using Lemma 3.2 instead of Lemmas 1.1 and 1.2 of [13]. $\square$

We recall now two imbedding results proved in [5] that will be useful for our purposes. Consider real numbers r, p, q, s such that

$$(i_3) \quad r \in \mathbb{N}, \quad s \in \mathbb{R}, \quad 1 \leq p \leq q < +\infty, \quad q \geq \frac{n}{r}, \quad q > \frac{n}{r} \text{ if } \frac{n}{r} = p > 1.$$

Lemma 3.4. Suppose that conditions (i_1) and (i_3) hold. Then for all $g \in K_{-s+r}^q(\Omega)$ and for all $u \in W_s^{r,p}(\Omega)$, we have $gu \in L^p(\Omega)$ and

$$(3.13) \quad \|gu\|_{L^p(\Omega)} \leq c_o \|g\|_{K_{-s+r}^q(\Omega)} \|u\|_{W_s^{r,p}(\Omega)},$$

where c_o depends on $\Omega, n, p, q, r, s, \rho$.

Lemma 3.5. Suppose that the hypotheses of Lemma 3.4 are satisfied and let $g \in \tilde{K}_{-s+r}^q(\Omega)$. Then for any $\varepsilon \in \mathbb{R}_+$ there exists $c(\varepsilon) \in \mathbb{R}_+$ such that

$$(3.14) \quad \|gu\|_{L^p(\Omega)} \leq \varepsilon \|u\|_{W_s^{r,p}(\Omega)} + c(\varepsilon) \|u\|_{L_{s-r}^p(\Omega)},$$

where $c(\varepsilon)$ depends on $\varepsilon, \Omega, n, p, q, r, s, \rho, \|g\|_{K_{-s+r}^q(\Omega)}$ and $\tilde{\sigma}_q[g]$.

Observe that in the original statements of the above lemmas given in [5], the dependence of the constants was not explicated.

4. An a priori bound

From now on we suppose that $n \geq 3$. Fix $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ and consider the following condition on Ω :

(h_1) there exists an open subset Ω_o of $\mathbb{R}^n$ with the uniform $C^{1,1}$ -regularity property, such that

$$\Omega \subset \Omega_o, \quad \partial\Omega \setminus S_\rho \subset \partial\Omega_o.$$

Remark 4.1. If condition (h_1) holds and $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$, then Ω satisfies (i_1) (see [5]).

Let $p \in]1, +\infty[$, $s \in \mathbb{R}$. Consider in Ω the differential operator

$$L = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a,$$

with the following conditions on the coefficients:

(h_2) there exist extensions a_{ij}^o of a_{ij} to Ω_o such that

$$(h_3) \quad \begin{cases} a_{ij}^o = a_{ji}^o \in L^\infty(\Omega_o) \cap VMO(\Omega_o), & i, j = 1, \dots, n, \\ \exists \nu \in \mathbb{R}_+ : \sum_{i,j=1}^n a_{ij}^o \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } \Omega_o, \quad \forall \xi \in \mathbb{R}^n, \\ a_i \in \tilde{K}_1^{t_1}(\Omega), \quad i = 1, \dots, n, \text{ where } t_1 \geq p, \quad t_1 \geq n, \\ \quad \text{and } t_1 > p \quad \text{if } p = n, \\ a \in \tilde{K}_2^{t_2}(\Omega), \quad \text{where } t_2 \geq p, \quad t_2 \geq \frac{n}{2}, \\ \quad \text{and } t_2 > p \quad \text{if } p = \frac{n}{2}. \end{cases}$$

We put

$$L_o = - \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Theorem 4.1. Assume that conditions (h_1) – (h_3) hold. Then there exists a positive real number c such that

$$(4.1) \quad \|u\|_{W_s^{2,p}(\Omega)} \leq c \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)} \right) \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_{s-1}^{1,p}(\Omega),$$

where c depends on $\Omega, n, p, t_1, t_2, s, \rho, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \|a_i\|_{K_1^{t_1}(\Omega)}, \|a\|_{K_2^{t_2}(\Omega)}, \eta[a_{ij}^o], \tilde{\sigma}_{t_1}[a_i]$ and $\tilde{\sigma}_{t_2}[a]$.

Proof. Fix $u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}_{s-1}^{1,p}(\Omega)$. Consider a function $\phi \in C_o^\infty(\mathbb{R}^n)$ such that

$$(4.2) \quad 0 \leq \phi \leq 1, \quad \phi|_{B(0, \frac{1}{2})} = 1, \quad \text{supp } \phi \subset B(0, 1).$$

For each $x \in \Omega$, let

$$(4.3) \quad \psi = \psi^x : y \in \mathbb{R} \rightarrow \phi\left(\frac{y-x}{\rho(x)}\right);$$

then

$$(4.4) \quad \begin{cases} 0 \leq \psi \leq 1, \quad \psi|_{I_{\frac{1}{2}}(x)} = 1, \quad \text{supp } \psi \cap \Omega \subset I(x), \\ |\partial^\alpha \psi| \leq c_\alpha \rho^{-|\alpha|}(x) \quad \forall \alpha \in \mathbb{N}_o^n. \end{cases}$$

If u_o denotes the zero extension of u outside Ω , it follows from (2.2), (2.3) and (3.2) that

$$\psi u_o \in W^{2,p}(\Omega_o) \cap \overset{\circ}{W}^{1,p}(\Omega_o).$$

We consider in Ω_o the differential operator

$$\mathfrak{L}_o = - \sum_{i,j=1}^n a_{ij}^o \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then from hypotheses (h_1) , (h_2) and from Theorem 5.1 in [6], we have

$$(4.5) \quad \|(\psi u_o)_{xx}\|_{L^p(\Omega_o)} \leq c_1 \left(\|\mathfrak{L}_o(\psi u_o)\|_{L^p(\Omega_o)} + \|\psi u_o\|_{L^p(\Omega_o)} \right),$$

where c_1 depends on $\Omega, n, p, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \eta[a_{ij}^o]$.

Observe that

$$(4.6) \quad \|\mathfrak{L}_o(\psi u_o)\|_{L^p(\Omega_o)} = \|L_o(\psi u)\|_{L^p(\Omega)}$$

and

$$(4.7) \quad L_o(\psi u) = \psi L_o u - \sum_{i,j=1}^n a_{ij} \psi_{x_i x_j} u - 2 \sum_{i,j=1}^n a_{ij} \psi_{x_i} u_{x_j}.$$

Thus, from (4.6) and (4.7) we deduce

$$(4.8) \quad \begin{aligned} & \|\mathfrak{L}_o(\psi u_o)\|_{L^p(\Omega_o)} + \|\psi u_o\|_{L^p(\Omega_o)} \\ & \leq c_2 \left(\|\psi L_o u\|_{L^p(\Omega)} + \|\psi_{xx} u\|_{L^p(\Omega)} + \|\psi_x u_x\|_{L^p(\Omega)} + \|\psi u\|_{L^p(\Omega)} \right), \end{aligned}$$

where c_2 depends on $n, p, \|a_{ij}^o\|_{L^\infty(\Omega_o)}$.

It follows from (4.4), (4.5) and (4.8) that

$$(4.9) \quad \begin{aligned} & \rho^{-2}(x) \|u\|_{L^p(I_{\frac{1}{2}}(x))} + \rho^{-1}(x) \|u_x\|_{L^p(I_{\frac{1}{2}}(x))} + \|u_{xx}\|_{L^p(I_{\frac{1}{2}}(x))} \\ & \leq c_3 \left(\|L_o u\|_{L^p(I(x))} + \rho^{-2}(x) \|u\|_{L^p(I(x))} + \rho^{-1}(x) \|u_x\|_{L^p(I(x))} \right), \end{aligned}$$

where c_3 depends on $\Omega, n, p, \rho, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \eta[a_{ij}^o]$.

Then by (4.9) and by Lemma 3.2 we have

$$(4.10) \quad \int_{\Omega} \rho^{(s-2)p}(x) |u|^p dx + \int_{\Omega} \rho^{(s-1)p}(x) |u_x|^p dx + \int_{\Omega} \rho^{sp}(x) |u_{xx}|^p dx \\ \leq c_4 \left(\int_{\Omega} \rho^{sp}(x) |L_o u|^p dx + \int_{\Omega} \rho^{(s-2)p}(x) |u|^p dx \right. \\ \left. + \int_{\Omega} \rho^{(s-1)p}(x) |u_x|^p dx \right),$$

where c_4 depends on the same parameters as c_3 .

Therefore

$$(4.11) \quad \|u\|_{W_s^{2,p}(\Omega)} \leq c_5 \left(\|L_o u\|_{L_s^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)} + \|u_x\|_{L_{s-1}^p(\Omega)} \right),$$

where c_5 depends on the same parameters as c_4 .

Moreover, by (h_3) and Lemma 3.5, for each $\varepsilon \in \mathbb{R}_+$ there exists $c(\varepsilon) \in \mathbb{R}_+$ such that

$$(4.12) \quad \left\| \sum_{i=1}^n a_i u_{x_i} + a u \right\|_{L_s^p(\Omega)} \\ \leq \varepsilon \|u\|_{W_s^{2,p}(\Omega)} + c(\varepsilon) \left(\|u\|_{L_{s-2}^p(\Omega)} + \|u_x\|_{L_{s-1}^p(\Omega)} \right),$$

where $c(\varepsilon)$ depends on $\varepsilon, \Omega, n, p, t_1, t_2, s, \rho, \|a_i\|_{K_1^{t_1}(\Omega)}, \|a\|_{K_2^{t_2}(\Omega)}, \tilde{\sigma}_{t_1}[a_i]$ and $\tilde{\sigma}_{t_2}[a]$.

Using (4.11) and (4.12), we have

$$(4.13) \quad \|u\|_{W_s^{2,p}(\Omega)} \leq c_6 \left(\|L u\|_{L_s^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)} + \|u_x\|_{L_{s-1}^p(\Omega)} \right),$$

with $c_6 \in \mathbb{R}_+$ depending on $\Omega, n, p, t_1, t_2, s, \rho, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \|a_i\|_{K_1^{t_1}(\Omega)}, \|a\|_{K_2^{t_2}(\Omega)}, \eta[a_{ij}^o], \tilde{\sigma}_{t_1}[a_i]$ and $\tilde{\sigma}_{t_2}[a]$.

On the other hand, from Lemma 3.3 there exists $c_7 \in \mathbb{R}_+$, depending on Ω, n, p, s, ρ , such that

$$(4.14) \quad \|u_x\|_{L_{s-1}^p(\Omega)} \leq c_7 \left(\|u_{xx}\|_{L_s^p(\Omega)}^{\frac{1}{2}} \cdot \|u\|_{L_{s-2}^p(\Omega)}^{\frac{1}{2}} + \|u\|_{L_{s-2}^p(\Omega)} \right).$$

Finally, (4.1) follows from (4.13) and (4.14). $\square$

5. Main results

In this section we will prove two regularity results, the first of local and the other of global type. In the case of local regularity, we require that the coefficients of lower order satisfy suitable local summability conditions:

$$(h'_3) \quad \begin{cases} a_i \in L_{\text{loc}}^{t_1}(\bar{\Omega} \setminus S_\rho), \quad i = 1, \dots, n, \quad \text{where } t_1 \geq p, \quad t_1 > n, \\ a \in L_{\text{loc}}^{t_2}(\bar{\Omega} \setminus S_\rho), \quad \text{where } t_2 \geq p, \quad t_2 > \frac{n}{2}. \end{cases}$$

For $p \in]1, +\infty[$, we denote by $W_{\text{loc}}^{2,p}(\bar{\Omega} \setminus S_\rho)$ and $\mathring{W}_{\text{loc}}^{1,p}(\bar{\Omega} \setminus S_\rho)$ the spaces of all functions $u : \Omega \rightarrow \mathbb{R}$ such that $\zeta u \in W^{2,p}(\Omega)$ or $\zeta u \in \mathring{W}^{1,p}(\Omega)$, respectively, for each $\zeta \in \mathfrak{D}(\bar{\Omega} \setminus S_\rho)$.

Theorem 5.1. *Assume that conditions (h_1) , (h_2) and (h'_3) hold and let u be a solution of the problem*

$$(5.1) \quad \begin{cases} u \in W_{\text{loc}}^{2,q}(\bar{\Omega} \setminus S_\rho) \cap \mathring{W}_{\text{loc}}^{1,q}(\bar{\Omega} \setminus S_\rho), \\ Lu \in L_{\text{loc}}^p(\bar{\Omega} \setminus S_\rho), \end{cases}$$

where $q \in]1, p[$. Then $u \in W_{\text{loc}}^{2,p}(\bar{\Omega} \setminus S_\rho)$.

Proof. Assume that $q < p$ and fix $\psi \in \mathfrak{D}(\bar{\Omega} \setminus S_\rho)$. We claim that $\psi u \in W^{2,p}(\Omega)$. To this aim, let $k \in \mathbb{N}$ such that

$$k - 1 < n \left(\frac{1}{q} - \frac{1}{p} \right) \leq k,$$

so that

$$(5.2) \quad \frac{1}{q_h} = \frac{1}{q} - \frac{h}{n} > \frac{1}{p}, \quad h = 0, \dots, k-1, \quad \frac{1}{q} - \frac{k}{n} \leq \frac{1}{p} = \frac{1}{q_k}.$$

Thus it is enough to prove that

$$(5.3) \quad \psi u \in W^{2,q_h}(\Omega) \Rightarrow \psi u \in W^{2,q_{h+1}}(\Omega)$$

for all $h = 0, \dots, k-1$.

Observe that

$$(5.4) \quad \psi u_o \in W^{2,q_h}(\Omega_o) \cap \mathring{W}^{1,q_h}(\Omega_o),$$

where u_o is the zero extension of u outside Ω . Let A be an open subset of $\mathbb{R}^n$ such that

$$\text{supp } \psi \subset A, \quad A \cap \bar{\Omega} \subset \bar{\Omega} \setminus S_\rho,$$

and fix $\chi \in C_o^\infty(A)$ such that

$$\chi = 1 \quad \text{on} \quad \text{supp } \psi.$$

Consider in Ω_o the differential operator

$$\mathfrak{L} = - \sum_{i,j=1}^n a_{ij}^o \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i^o \frac{\partial}{\partial x_i} + a^o,$$

where a_i^o and a^o are the zero extensions of a_i and a , respectively, outside Ω . Observe that

$$(5.5) \quad \|\mathfrak{L}(\psi u_o)\|_{L^{q_{h+1}}(\Omega_o)} = \|L(\psi u)\|_{L^{q_{h+1}}(\Omega)}$$

and

$$(5.6) \quad L(\psi u) = \psi L u + \sum_{i=1}^n \alpha_i (\chi u)_{x_i} + \alpha (\chi u),$$

where

$$\begin{aligned} \alpha_i &= -2 \sum_{j=1}^n a_{ij} \psi_{x_j}, \quad i = 1, \dots, n, \\ \alpha &= - \sum_{i,j=1}^n a_{ij} \psi_{x_i x_j} + \sum_{i=1}^n a_i \psi_{x_i}. \end{aligned}$$

Using now the same argument of the proof of Lemma 4.1 in [4], it can be proved that $L(\psi u) \in L^{q_{h+1}}(\Omega)$ and so, by (5.5), also that $\mathfrak{L}(\psi u_o) \in L^{q_{h+1}}(\Omega_o)$. Application of Lemma 4.2 of [7] yields that $\psi u_o \in W^{2,q_{h+1}}(\Omega_o)$ and hence (5.3) holds. $\square$

Theorem 5.2. *Suppose that conditions (h_1) – (h_3) hold with $t_1 > n$ and $t_2 > n/2$. Then any solution u of the problem*

$$(5.7) \quad \begin{cases} u \in W_{\text{loc}}^{2,q}(\bar{\Omega} \setminus S_\rho) \cap \overset{\circ}{W}_{\text{loc}}^{1,q}(\bar{\Omega} \setminus S_\rho) \cap L_{s-2}^p(\Omega), \\ Lu \in L_s^p(\Omega), \end{cases}$$

(with $q \in]1, p[$) belongs to $W_s^{2,p}(\Omega)$.

Proof. Fix a solution u of problem (5.7). By Theorem 5.1 we have that $u \in W_{\text{loc}}^{2,p}(\bar{\Omega} \setminus S_\rho)$. Let $r, r' \in]0, 1[$ with $r < r'$ and $\phi \in C_o^\infty(\mathbb{R}^n)$ such that

$$\phi|_{B(0,r)} = 1, \quad \text{supp } \phi \subset B(0, r'), \quad \sup_{\mathbb{R}^n} |\partial^\alpha \phi| \leq c_\alpha (r' - r)^{-|\alpha|} \quad \forall \alpha \in \mathbb{N}_o^n.$$

For each $x \in \Omega$, consider the function ψ defined by (4.3). In our situation we obtain

$$(5.8) \quad \begin{cases} \psi|_{B(x, r'\rho(x))} = 1, & \text{supp } \psi \subset B(x, r'\rho(x)), \\ \sup_{\mathbb{R}^n} |\partial^\alpha \psi| \leq c_\alpha (r' - r)^{-|\alpha|} \rho^{-|\alpha|}(x) & \forall \alpha \in \mathbb{N}_o^n. \end{cases}$$

It is easy to prove that $\psi u \in W_s^{2,p}(\Omega) \cap \overset{o}{W}_{s-1}^{1,p}(\Omega)$ and so it follows from Theorem 4.1 that

$$(5.9) \quad \|\psi u\|_{W_s^{2,p}(\Omega)} \leq c_1 \left(\|L(\psi u)\|_{L_s^p(\Omega)} + \|\psi u\|_{L_{s-2}^p(\Omega)} \right),$$

where c_1 depends on $\Omega, n, p, t_1, t_2, s, \rho, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \|a_i\|_{K_1^{t_1}(\Omega)}, \|a\|_{K_2^{t_2}(\Omega)}, \eta[a_{ij}^o], \tilde{\sigma}_{t_1}[a_i]$ and $\tilde{\sigma}_{t_2}[a]$.

Note that

$$L(\psi u) = \psi L u + \sum_{i,j=1}^n a_{ij} \psi_{x_i x_j} u - 2 \sum_{i,j=1}^n a_{ij} (\psi_{x_i} u)_{x_j} + \sum_{i=1}^n a_i \psi_{x_i} u,$$

and so, by (5.9),

$$(5.10) \quad \begin{aligned} \|\psi u\|_{W_s^{2,p}(\Omega)} &\leq c_2 \left(\|\psi L u\|_{L_s^p(\Omega)} + \left\| \sum_{i=1}^n (\psi_{x_i} u)_x \right\|_{L_s^p(\Omega)} \right. \\ &\quad \left. + \|\psi_{xx} u\|_{L_s^p(\Omega)} + \left\| \sum_{i=1}^n a_i \psi_{x_i} u \right\|_{L_s^p(\Omega)} + \|\psi u\|_{L_{s-2}^p(\Omega)} \right), \end{aligned}$$

where c_2 depends on the same parameters as c_1 .

From Lemma 3.3 we have

$$(5.11) \quad \begin{aligned} &\|(\psi_{x_i} u)_x\|_{L_s^p(\Omega)} \\ &\leq c_3 \left(\|(\psi_{x_i} u)_{xx}\|_{L_{s+1}^p(\Omega)}^{\frac{1}{2}} \cdot \|\psi_{x_i} u\|_{L_{s-1}^p(\Omega)}^{\frac{1}{2}} + \|\psi_{x_i} u\|_{L_{s-1}^p(\Omega)} \right), \end{aligned}$$

where c_3 depends on Ω, n, p, s, ρ .

Moreover, (5.8) gives

$$\begin{aligned}
 (5.12) \quad & \|(\psi_{x_i} u)_{xx}\|_{L_{s+1}^p(\Omega)} \\
 & \leq c_4 \left(\|\psi_{xxx} u\|_{L_{s+1}^p(I_{r'}(x))} + \|\psi_{xx} u_x\|_{L_{s+1}^p(I_{r'}(x))} + \|\psi_x u_{xx}\|_{L_{s+1}^p(I_{r'}(x))} \right) \\
 & \leq c_5 \left((r' - r)^{-3} \rho^{-3}(x) \|u\|_{L_{s+1}^p(I_{r'}(x))} + (r' - r)^{-2} \rho^{-2}(x) \|u_x\|_{L_{s+1}^p(I_{r'}(x))} \right. \\
 & \quad \left. + (r' - r)^{-1} \rho^{-1}(x) \|u_{xx}\|_{L_{s+1}^p(I_{r'}(x))} \right),
 \end{aligned}$$

where c_4 depends on n, p , and c_5 depends on n, p, ρ .

Since $(r' - r)^{-1} \leq (r' - r)^{-2}$, it follows from (5.11), (5.12) and (5.8) that

$$\begin{aligned}
 (5.13) \quad & \|(\psi_{x_i} u)_x\|_{L_s^p(\Omega)} \leq c_6 \left((r' - r)^{-2} \rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} + (r' - r)^{-1} \right. \\
 & \quad \cdot \rho^{-1}(x) \|u_x\|_{L_s^p(I_{r'}(x))} + \|u_{xx}\|_{L_s^p(I_{r'}(x))} \Big)^{\frac{1}{2}} \\
 & \quad \cdot \left((r' - r)^{-2} \rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} \right)^{\frac{1}{2}},
 \end{aligned}$$

where c_6 depends on Ω, n, p, s, ρ .

On the other hand, by Lemma 3.4, (5.8) and (5.13) we have

$$\begin{aligned}
 (5.14) \quad & \|a_i \psi_{x_i} u\|_{L_s^p(\Omega)} \leq c_7 \|\psi_{x_i} u\|_{W_{s-1}^{2,p}(\Omega)} \\
 & \leq c_8 \left(\rho^{-1}(x) \|\psi_{x_i} u\|_{L_s^p(I_{r'}(x))} + \|(\psi_{x_i} u)_x\|_{L_s^p(I_{r'}(x))} \right) \\
 & \leq c_9 \left((r' - r)^{-2} \rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} + (r' - r)^{-1} \right. \\
 & \quad \cdot \rho^{-1}(x) \|u_x\|_{L_s^p(I_{r'}(x))} + \|u_{xx}\|_{L_s^p(I_{r'}(x))} \Big)^{\frac{1}{2}} \\
 & \quad \cdot \left((r' - r)^{-2} \rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} \right)^{\frac{1}{2}},
 \end{aligned}$$

where $c_7 - c_9$ depend on $\Omega, n, p, t_1, s, \rho$ and $\|a_i\|_{K_1^{t_1}(\Omega)}$.

Therefore by (5.8), (5.10), (5.13) and (5.14) we obtain

$$\begin{aligned}
 (5.15) \quad & \rho^{-2}(x) \|u\|_{L_s^p(I_r(x))} + \rho^{-1}(x) \|u_x\|_{L_s^p(I_r(x))} + \|u_{xx}\|_{L_s^p(I_r(x))} \\
 & \leq c_{10} \|\psi u\|_{W_s^{2,p}(\Omega)}
 \end{aligned}$$

$$\leq c_{11}(r' - r)^{-2} \left[\|Lu\|_{L_s^p(I_{r'}(x))} + \left(\rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} + \rho^{-1}(x) \right. \right. \\ \left. \left. \cdot \|u_x\|_{L_s^p(I_{r'}(x))} + \|u_{xx}\|_{L_s^p(I_{r'}(x))} \right)^{\frac{1}{2}} \cdot \left(\rho^{-2}(x) \|u\|_{L_s^p(I_{r'}(x))} \right)^{\frac{1}{2}} \right],$$

where c_{10} depends on p, ρ , and c_{11} depends on $\Omega, n, p, t_1, t_2, s, \rho, \nu, \|a_{ij}^o\|_{L^\infty(\Omega_o)}, \|a_i\|_{K_1^{t_1}(\Omega)}, \|a\|_{K_2^{t_2}(\Omega)}, \eta[a_{ij}^o], \tilde{\sigma}_{t_1}[a_i]$ and $\tilde{\sigma}_{t_2}[a]$.

Applying a well-known lemma of monotonicity due to C. Miranda (see [12], Lemma 3.1) it follows that

$$(5.16) \quad \rho^{-2}(x) \|u\|_{L_s^p(I_{\frac{1}{2}}(x))} + \rho^{-1}(x) \|u_x\|_{L_s^p(I_{\frac{1}{2}}(x))} + \|u_{xx}\|_{L_s^p(I_{\frac{1}{2}}(x))} \\ \leq c_{12} \left(\|Lu\|_{L_s^p(I(x))} + \rho^{-2}(x) \|u\|_{L_s^p(I(x))} \right),$$

with c_{12} dependent on the same parameters as c_{11} .

By Lemma 3.2 we deduce

$$(5.17) \quad \|u\|_{W_s^{2,p}(\Omega)} \leq c_{13} \left(\|Lu\|_{L_s^p(\Omega)} + \|u\|_{L_{s-2}^p(\Omega)} \right),$$

where c_{13} depends on the same parameters as c_{11} . Finally, (5.17) proves the result. $\square$

References

- [1] V. Benci and D. Fortunato, *Weighted Sobolev spaces and the nonlinear Dirichlet problem in unbounded domains*, Ann. Mat. Pura Appl., 121 (1979), 319–336.
- [2] S. Campanato, *Un risultato relativo alle equazioni ellittiche del secondo ordine di tipo non variazionale*, Ann. Scuola Norm. Sup. Pisa, 21 (1967), 701–707.
- [3] A. Canale, L. Caso, and P. Di Gironimo, *Weighted norm inequalities on irregular domains*, Rend. Accad. Naz. Sci. XL Mem. Mat., 16 (1992), 193–209.
- [4] L. Caso, P. Cavaliere and M. Transirico, *Uniqueness results for elliptic equations with VMO-coefficients*, Int. J. Pure Appl. Math., 13 (2004), 499–512.
- [5] L. Caso and M. Transirico, *Some remarks on a class of weight functions*, Comment. Math. Univ. Carolin., 37 (1996), 469–477.

- [6] P. Cavaliere, M. Longobardi and A. Vitolo, *Imbedding estimates and elliptic equations with discontinuous coefficients in unbounded domains*, *Matematiche (Catania)*, 51 (1996), 87–104.
- [7] P. Cavaliere, M. Transirico and M. Troisi, *Uniqueness result for elliptic equations in unbounded domains*, *Matematiche (Catania)*, 54 (1999), 139–146.
- [8] F. Chiarenza, F. Frasca and P. Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, *Ricerche Mat.*, 15 (1991), 149–168.
- [9] D. E. Edmunds and W. D. Evans, *Elliptic and degenerate-elliptic operators in unbounded domains*, *Ann. Sc. Norm. Sup. di Pisa*, 27 (1973), 591–640.
- [10] M. Franciosi and N. Fusco, *$W^{2,p}$ regularity for the solutions of elliptic non divergence form equations with rough coefficients*, *Ricerche Mat.*, 38 (1989), 93–106.
- [11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd edition, Springer-Verlag, Berlin (1983).
- [12] C. Miranda, *Teoremi di unicità in domini non limitati e teoremi di Liouville per le soluzioni dei problemi al contorno relativi alle equazioni ellittiche*, *Ann. Mat. Pura Appl.*, 59 (1962), 189–212.
- [13] M. Troisi, *Teoremi di inclusione negli spazi di Sobolev con peso*, *Ricerche Mat.*, 18 (1969), 49–74.
- [14] M. Troisi, *Su una classe di funzoni peso*, *Rend. Accad. Naz. Sci. XL Mem. Mat.*, 10 (1986), 141–152.
- [15] M. Troisi, *Su una classe di spazi di Sobolev con peso*, *Rend. Accad. Naz. Sci. XL Mem. Mat.*, 10 (1986), 177–189.
- [16] C. Vitanza, *$W^{2,p}$ -regularity for a class of elliptic second order equations with discontinuous coefficients*, *Matematiche (Catania)*, 17 (1992), 177–186.

Dipartimento di Matematica e Informatica
Facoltà di Scienze MM.FF.NN.
Università di Salerno
Via Ponte don Melillo
I 84084 Fisciano (SA)
Italy
(E-mail : lorcaso@unisa.it)

(Received : May 2005)