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Approximation numbers of Sobolev embeddings of radial functions on isotropic manifolds

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Abstract. We regard the compact Sobolev embeddings between Besov and Sobolev spaces of radial functions on noncompact symmetric spaces of rank one. The asymptotic formula for the behaviour of approximation numbers of these embeddings is described.

1. Introduction

Approximation numbers measure the closeness by which a bounded operator may be approximated by linear maps of finite range, whereas entropy numbers measure compactness of the operator by means of finite coverings of an image of the unit ball. Both, approximation and entropy numbers, of compact Sobolev embeddings between function spaces of Sobolev and Besov type on the Euclidean case have been investigated by several authors: D. E. Edmunds and H. Triebel cf. [4], [5], D. Haroske and H. Triebel [10], [11] , D. Haroske [9], and A.M. Caetano [1]. Some of these results are described in [6], where one can find also the applications to the spectral theory of the pseudo-differential operators.

It was noticed in late seventies of the last century by W. A. Strauss [27] and S. Coleman, V. Glazer, and A. Martin [3] that radiality implies compactness of Sobolev embeddings, cf. also [19] and [13], [12] for corresponding results on Riemannian manifolds. W. Sickel and the first named author found the necessary and sufficient conditions for compactness of the embeddings of radial Besov and Triebel-Lizorkin spaces, cf. [21]. Later the corresponding theory was developed for more general symmetry conditions, cf. [24] as well as for function spaces defined on Riemannian manifolds, [22]. On manifolds the symmetry conditions are expressed in terms of invariance with respect to the action of a compact group of isometries.

If $\mathbf{X}$ is a connected d-dimensional Riemannian manifold and o is a fixed point of $\mathbf{X}$, then we call a function f radial if its value at x depends only on a distance of x to the point o. Isotropic Riemannian manifolds seems to be of special interest here since on such manifolds radial means invariant with respect to the action of isotropy group of the point o. A Riemannian manifold $\mathbf{X}$ is called isotropic if for each $y \in \mathbf{X}$ the linear isotropy group acts transitively on the unit sphere in the tangent space $T_y\mathbf{X}$. It is well known that a noncompact Riemannian manifold is isotropic if and only if it is either an Euclidean space or a noncompact Riemannian globally symmetric space of rank one.

The necessary and sufficient conditions for compactness of Sobolev embeddings for Besov and Sobolev function spaces defined on the Euclidean space $\mathbb{R}^d$ with $d \geq 2$, are the same as for the functions on the noncompact Riemannian symmetric space of rank one cf. [21], [22]. Namely the embeddings

$$(1) \qquad RH^{s_0}_{p_0}(\mathbf{X}) \hookrightarrow RH^{s_1}_{p_1}(\mathbf{X}) \quad \text{and} \quad RB^{s_0}_{p_0,q_0}(\mathbf{X}) \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbf{X})$$

$$(2) \qquad RH^{s_0}_{p_0}(\mathbb{R}^d) \hookrightarrow RH^{s_1}_{p_1}(\mathbb{R}^d) \quad \text{and} \quad RB^{s_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbb{R}^d)$$

are compact if and only if $p_0 < p_1$ and $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$.

Asymptotic behaviour of entropy e_n and approximation numbers a_n of the embeddings of radial functions on $\mathbb{R}^n$ was studied in [17] and [25] respectively. The behaviour of the entropy numbers in the case of symmetric spaces of rank one was described in [26].

Even though the necessary and sufficient conditions for compactness of Sobolev embedding are the same for $\mathbb{R}^d$ and $\mathbf{X}$ the asymptotic behaviour of corresponding approximation (as well as entropy) numbers is quite different.

We put $\frac{1}{p} = \frac{1}{p_0} - \frac{1}{p_1}$ and $t = \min\{p_0', p_1\}$. The main result of the paper describes the behaviour of the approximation sequence for embeddings (1)

as follows

$$a_k \sim \begin{cases} k^{-(s_0-s_1-\frac{1}{p})} & \text{if } \ 1 \le p_0 < p_1 \le 2 \ \text{ or } \ 2 \le p_0 < p_1 \le \infty\,, \\ k^{-(s_0-s_1-\frac{1}{p}+\frac{1}{2}-\frac{1}{t})} & \text{if } \ 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ s_0 - s_1 - \frac{1}{p} \ge \frac{1}{t}\,, \\ k^{-\left(s_0-s_1-\frac{1}{p}\right)\frac{t}{2}} & \text{if } \ 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ s_0 - s_1 - \frac{1}{p} < \frac{1}{t}. \end{cases}$$

Whereas, for the compact embeddings (2) we have

$$(3) \quad a_k \ \sim \ \begin{cases} k^{-\frac{d-1}{p}} & \text{if} \quad 1 \le p_0 < p_1 \le 2 \ \text{ or } \ 2 \le p_0 < p_1 \le \infty\,, \\ k^{-\frac{d-1}{p}-\frac{1}{2}+\frac{1}{t}} & \text{if} \quad 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ \frac{d-1}{p} > \frac{1}{t}\,, \\ k^{-\frac{d-1}{p}\frac{t}{2}} & \text{if} \quad 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ \frac{d-1}{p} \le \frac{1}{t}\,. \end{cases}$$

In both cases $(p_0, p_1) \neq (1, \infty)$, $d \ge 2$ and $a_n \sim b_n$ means that there are constants $c_1, c_2 > 0$ such that inequalities $c_1 a_n \le b_n \le c_2 a_n$ hold for any $n \in \mathbb{N}$.

2. Preliminaries

2.1. Approximation numbers. We recall the definition of approximation numbers and corresponding operator ideal quasi-norms, that will be used widely in the paper. We refer to books of B. Carl and I. Stephani [2] and A. Pietsch [20] for details, proofs and more information.

Let B_0 and B_1 be two complex Banach spaces and let $T : B_0 \to B_1$ be a bounded linear operator. The kth *approximation number* $a_k(T)$ of the operator $T : B_0 \to B_1$ is the infimum of all numbers $\|T - A\|$ where A runs over the collection of all continuous linear operators $A : B_0 \to B_1$ of rank smaller than k. So,

$$a_k(T) := \inf\{\|T - A\| \ : \ A \in L(B_0, B_1),\ \mathrm{rank}(A) < k\},$$

where $\mathrm{rank}(A)$ denote the dimension of the range $A(B_0)$.

Approximation numbers $a_k(T)$ form a decreasing sequence with $a_1(T) = \|T\|$. If this sequence converges to zero then the operator T is compact. The opposite implication is generally not true. It may happen that $\lim_{k\to\infty} a_k(T) > 0$ for some compact T if B_1 fails to have the approximation property. The approximation numbers have in particular the following properties:

- (additivity) $a_{n+k-1}(T_1 + T_2) \ \le \ a_k(T_1) + a_n(T_2)$,
- (multiplicativity) $a_{n+k-1}(T_1 T_2) \ \le \ a_k(T_1) a_n(T_2)$,
- (rank property) $a_n(T) = 0 \ \Leftrightarrow \ \mathrm{rank}\,(T) < n$.

Later on we use the notation of operator ideals, cf. [2] and [20] for details. Here we recall only just what we need for the proofs. Given a bounded linear operator T and a positive real number s we put

$$L^{(a)}_{s,\infty}(T) := \sup_{k\in\mathbb{N}} k^{1/s}\, a_k(T)\,.$$

The expression $L^{(a)}_{s,\infty}(T)$ is an example of an *operator ideal quasi-norm* This means in particular that there exists a number $0 < \varrho \le 1$ such that

$$L^{(a)}_{s,\infty}\Big(\sum_j T_j\Big)^{\varrho} \le \sum_j L^{(a)}_{s,\infty}(T_j)^{\varrho}\,, \tag{4}$$

cf. H. König, [16, 1.c.5].

The following lemma concerning approximation numbers of embeddings of finite dimensional complex sequence spaces is essentially due to Gluskin [8] cf. also [6, Corollary 3.2.3] and [1]

Lemma 1. *Let $N, k \in \mathbb{N}$.*

(i) *If $1 \le p_0 \le p_1 \le 2$ or $2 \le p_0 \le p_1 \le \infty$, then there is a positive constant C independent of N and k such that*

$$a_k\Big(\mathrm{id}:\ \ell^N_{p_0} \mapsto \ell^N_{p_1}\Big) \ \le\ C\,.$$

(ii) *If $1 \le p_0 < 2 < p_1 \le \infty$, $(p_0, p_1) \ne (1,\infty)$, then there is a positive constant C independent of N and k such that*

$$a_k\Big(\mathrm{id}:\ \ell^N_{p_0} \mapsto \ell^N_{p_1}\Big) \ \le\ C \begin{cases} 1 & \text{if} \quad k \le N^{2/t}\,, \\ N^{1/t}k^{-1/2} & \text{if} \quad N^{2/t} < k \le N\,, \\ 0 & \text{if} \quad k > N\,, \end{cases}$$

where $\frac{1}{t} = \max\big(\frac{1}{p_1}, \frac{1}{p_0'}\big)$.

Moreover if $k \le \frac{N}{4}$ then in both cases we have an equivalence.

2.2. Function spaces, embeddings and traces. We recall the definition of Sobolev spaces and Besov spaces on Riemannian symmetric space. There are several possible approaches. Here we present the definition in terms of a heat semi-group. We assume that the reader is familiar with the definition and elementary properties of function from fractional Sobolev spaces H^s_p and Besov spaces $B^s_{p,q}$ on $\mathbb{R}^n$. All we need can be found in [29].

Our notation related to symmetric spaces is standard and can be found, for example, in [14] or [15]. Let $\mathbf{G}$ be noncompact connected semisimple Lie group with finite center of real rank one. Let $\mathbf{K}$ be its maximal compact

subgroup and let $\mathbf{X} = \mathbf{G}/\mathbf{K}$ be an associated symmetric spaces of dimension d. The Killing form of $\mathbf{G}$ induces a $\mathbf{G}$-invariant Riemannian metric on $\mathbf{X}$. So $\mathbf{X}$ is simply connected homogeneous Riemannian manifolds and $\mathbf{G}$ acts transitively on $\mathbf{X}$ as a group of isometries. If $o = e\mathbf{K}$ then $\mathbf{K}$ is an isotropy group of o. The assumption that $\mathbf{G}$ is a group of rank one implies that the group $\mathbf{K}$ acts transitively on any sphere $S(o,r) = \{x \in \mathbf{X} : \ |x| = r\}$, $|x|$ denotes the Riemannian distance from x to o.

Let Δ denote the Laplace operator on $\mathbf{X}$. The heat semi-group $\mathcal{H}_t = e^{t\Delta}$, $t \geq 0$, is a positive, symmetric, semigroup of contractions in $L_p(\mathbf{X})$, $1 \leq p \leq \infty$, such that $\mathcal{H}_t 1 = 1$. It is strongly continuous if $p < \infty$ and analytical if $1 < p < \infty$.

Let $\mathcal{S}_1(\mathbf{X})$ denote the L_1-Schwartz space on $\mathbf{X}$ and $\mathcal{S}_1'(\mathbf{X})$ be the corresponding space of distributions. For convenience we put

$$\mathcal{H}_{0,k} f \ = \ \sum_{\ell=0}^{k-1} \frac{1}{\ell!} (-\Delta)^{\ell} \mathcal{H}_1 f \,, \quad f \in \mathcal{S}_1'(\mathbf{X}) \,.$$

Definition 1. Let $s \in \mathbb{R}$ and $d \geq 2$.

(a) Let $1 < p < \infty$. A fractional Sobolev space $H_p^s(\mathbf{X})$ is defined by

$$H_p^s(\mathbf{X}) \ = \ \big\{ f \in \mathcal{S}_1'(\mathbf{X}) \ : \ \big\| f | H_p^s(\mathbf{X}) \big\| \ := \ \big\| (I - \Delta)^{s/2} f \big\|_p \ < \ \infty \big\} \,.$$

(b) Let $1 \leq p, q \leq \infty$ and $k > \frac{|s|}{2}$. A Besov space $B_{p,q}^s(\mathbf{X})$ is a space of distributions $f \in \mathcal{S}_1'(\mathbf{X})$ such that

$$\big\| f | B_{p,q}^s(\mathbf{X}) \big\| \ := \ \big\| \mathcal{H}_{0,k} f \big\|_p + \bigg(\int_0^1 t^{(k-s/2)q} \bigg\| \frac{d^k}{dt^k} \mathcal{H}_t f \bigg\|_p^q \frac{dt}{t} \bigg)^{1/q} \ < \ \infty \,.$$

Remark 1. (1) The definition of the Besov spaces is independent of k up to norm equivalence, cf. [22]. One can give an equivalent norm for the Sobolev spaces in term of heat semi- group, cf. ibidem.

(2) The spaces $H_p^s(\mathbf{X})$ and $B_{p,q}^s(\mathbf{X})$ are Banach spaces. If s is a positive integer then the space $H_p^s(\mathbf{X})$ coincides with the classical Sobolev space defined in term of gradient of order s.

(3) If $s > 0$ then $\|\mathcal{H}_{0,k} f\|_p$ can be replaced by $\|f\|_p$ in the definition of Besov spaces.

(4) The above definition coincides with the definition of H_p^s-$B_{p,q}^s$ spaces on a Riemannian manifold with bounded geometry by the uniform localization principle, cf. [22]. The last approach is due to H.Triebel, [30].

(5) The Besov spaces are real interpolation spaces of the Sobolev spaces. More precisely

$$\big(H_p^{s_0}(\mathbf{X})\,,\, H_p^{s_1}(\mathbf{X})\big)_{\theta,q} \;=\; B^s_{p,q}(\mathbf{X})\;,\; s=\theta s_0+(1-\theta)s_1\;,\; 0<\theta<1\;,\; s_0\neq s_1\;,$$

if $1<p<\infty$. In particular $H_2^s(\mathbf{X}) = B^s_{2,2}(\mathbf{X})$ and the norms are equivalent.

(6) Most of the results, we quote for Sobolev spaces, is also true, mutatis mutandis, for more general Triebel-Lizorkin function spaces $F^s_{p,q}$.

Since the group $\mathbf{K}$ acts transitively on spheres centered at o, a function is radial if and only if is invariant with respect to the action of $\mathbf{K}$. So the notation of radiality can be extended to distribution. The distribution f is called radial if

$$f(\varphi^k) \;=\; f(\varphi)\,,\ \text{where } \varphi^k(x)=\varphi(k\cdot x)\,,\ k\in\mathbf{K},\ x\in\mathbf{X}\,.$$

So for any possible s,p,q we can put

$$RB^s_{p,q}(\mathbf{X}) \;=\; \Big\{f\in B^s_{p,q}(\mathbf{X}) : f \text{ is radial}\Big\}.$$

The spaces $RH^s_p(\mathbf{X})$ are defined in the similar way. The both spaces are close subspaces of $B^s_{p,q}(\mathbf{X})$ and $H^s_p(\mathbf{X})$ respectively, so they are Banach spaces.

The following theorem describing the compactness of the Sobolev embeddings was proved in [22]

Theorem 1. *Let $-\infty<s_1,s_0<\infty$, $1\le p_0,p_1,q_0,q_1\le\infty$ and $d\ge 2$. The embeddings*

$$RB^{s_0}_{p_0,q_0}(\mathbf{X}) \;\hookrightarrow\; RB^{s_1}_{p_1,q_1}(\mathbf{X})\,,\qquad RH^{s_0}_{p_0}(\mathbf{X}) \;\hookrightarrow\; RH^{s_1}_{p_1}(\mathbf{X})$$

are compact if and only if $p_0<p_1$ and $s_0-\frac{d}{p_0} > s_1-\frac{d}{p_1}$ (with the restriction $1<p_0<p_1<\infty$ for Sobolev spaces.)

We will calculate the approximation numbers of the above embeddings.

For future use we need also some information on traces of radial Besov spaces on geodesic rays starting at origin o. The traces can be described in terms of weighted Besov spaces on the ray.

It can be proved that related trace and extension operators give us an isomorphisms of some radial and weighted Besov spaces, cf. [26].

To present this result in details we will need a positive weight function $v_p \in C^\infty(\mathbb{R})$ such that

$$v_p(t) \;=\; \exp\frac{\vartheta|t|}{p} \qquad \text{if} \quad |t| \geq 1 \tag{5}$$

(the exact behaviour near zero is not important). Here ϑ is a positive constant depending on $\mathbf{X}$. It is sufficient for us to regard the weighted Besov spaces $B^s_{p,q}(\mathbb{R}, v_p)$ with positive smoothness $s > 0$. They may be defined as follows. The function $f \in L_p(\mathbb{R})$ belongs to $B^s_{p,q}(\mathbb{R}, v_p)$ if and only if $v_p f \in B^s_{p,q}(\mathbb{R})$ moreover

$$\left\| f | B^s_{p,q}(\mathbb{R}, v_p) \right\| \;:=\; \left\| v_p f | B^s_{p,q}(\mathbb{R}) \right\|.$$

Let $\gamma : (-\infty, \infty) \to \mathbf{X}$ be a geodesic parametrized by the arc length such that $\gamma(0) = o$. It should be clear that we can define a weighted Besov space on the geodesic by pulling back to the weighted Besov space defined on $\mathbb{R}$, i.e.

$$f \in B^s_{p,q}(\gamma(-\infty, \infty), v_p) \quad \text{if and only if} \quad f \circ \gamma \in B^s_{p,q}(\mathbb{R}, v_p).$$

We are interested in a trace on $\gamma((0, \infty))$. For f continuous on $\mathbf{X}$ we put

$$\mathrm{tr} f(t) \;=\; f(\gamma(t)).$$

To describe the traces of radial function on geodesic ray out of the origin we need the following spaces.

Definition 2. Let $0 < \tau < \infty$, $1 \leq p, q \leq \infty$, and $s > 0$. Then we put

$$\begin{aligned} B^s_{p,q}(\gamma[\tau, \infty), v_p) \quad &:= \quad \{f \in B^s_{p,q}(\gamma(-\infty, \infty), v_p) : \ \mathrm{supp} f \subset \gamma[\tau, \infty)\}, \\ RB^s_{p,q}(\mathbf{X} \setminus B(o, \tau)) \quad &:= \quad \{f \in RB^s_{p,q}(\mathbf{X}) : \mathrm{supp} f \subset \{x \in \mathbf{X} : \ |x| \geq \tau\}\}. \end{aligned}$$

Theorem 2. *Let $1 \leq p, q \leq \infty$, $s > 0$ and $\tau > 0$. There exist continuous operators*

$$\begin{aligned} \mathrm{tr} \quad &: \quad RB^s_{p,q}(\mathbf{X} \setminus B(o, \tau)) \mapsto B^s_{p,q}(\gamma[\tau, \infty), v_p), \\ \mathrm{ext} \quad &: \quad B^s_{p,q}(\gamma[\tau; \infty), v_p) \mapsto RB^s_{p,q}(\mathbf{X} \setminus B(o, \tau)). \end{aligned}$$

such that $\mathrm{tr} \circ \mathrm{ext} = \mathrm{id}$ *and* $\mathrm{ext} \circ \mathrm{tr} = \mathrm{id}$.

The proof of this theorem can be found in [26], but it is similar to the proof of theorem about the traces on a straight line in the Euclidean case, cf. [17].

3. Approximation numbers of the Sobolev embeddings

Let us remind that $\frac{1}{p} = \frac{1}{p_0} - \frac{1}{p_1}$ and $t = \min\{p_0', p_1\}$, as before. The main result of the paper reads as follows

Theorem 3. *Let $s_0, s_1 \in \mathbb{R}$, $1 \le p_0 < p_1 \le \infty$ ($1 < p_0 < p_1 < \infty$ in the case of the Sobolev spaces), $(p_0, p_1) \ne (1, \infty)$, $1 \le q_0, q_1 \le \infty$, $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$ and $d \ge 2$. Then*

$$a_k\big(RB^{s_0}_{p_0,q_0}(\mathbf{X}) \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbf{X})\big) \sim k^{-\varkappa} \tag{6}$$

and

$$a_k\big(RH^{s_0}_{p_0}(\mathbf{X}) \hookrightarrow RH^{s_1}_{p_1}(\mathbf{X})\big) \sim k^{-\varkappa}\,, \tag{7}$$

where

(8)

$$\varkappa = \begin{cases} s_0 - s_1 - \frac{1}{p} & \text{if } 1 \le p_0 < p_1 \le 2 \text{ or } 2 \le p_0 < p_1 \le \infty \\ s_0 - s_1 - \frac{1}{p} + \frac{1}{2} - \frac{1}{t} & \text{if } 1 \le p_0 < 2 < p_1 \le \infty \text{ and } s_0 - s_1 - \frac{1}{p} \ge \frac{1}{t} \\ \big(s_0 - s_1 - \frac{1}{p}\big)\frac{t}{2} & \text{if } 1 \le p_0 < 2 < p_1 \le \infty \text{ and } s_0 - s_1 - \frac{1}{p} < \frac{1}{t}\,. \end{cases}$$

Remark 2. (1) The formula (7) follows from (6) by elementary embeddings and properties of approximation numbers. The proof of (6) is presented in Section 3.1.

(2) The estimates of the approximation numbers described in the above theorem are independent of the dimension d of the underlying space $\mathbf{X}$. This is in some contrast with the behaviour of the approximation numbers for radial function spaces defined on $\mathbb{R}^d$, where we have dependence on d in all cases as it was proved in [25], cf. (3). Moreover the asymptotic behaviour of the approximation numbers in (6) and (7) is the same as in the case of bounded domains in one dimensional Euclidean space $\mathbb{R}$, cf. [6].

The same phenomenon occurs for entropy numbers e_k of the embeddings (6) and (7). In this case we have

$$e_k \sim k^{-(s_0 - s_1)}\,.$$

as it was proved in [26].

(3) The theorem holds also for Triebel-Lizorkin spaces $F^s_{p,q}(\mathbf{X})$ with $1 \le p, q < \infty$.

3.1 The proof of the main theorem. The proof of the main theorem is based on the same idea as that for entropy numbers (see [26]). We divide the origin space into two parts: the local part near origin o and the global

part out of the origin. Since $\mathbf{X}$ is the symmetric space of real rank one the analysis of the local part can be reduced to corresponding embeddings of function spaces defined on the Euclidean ball. On the other hand the analysis of the global part can be reduced to embeddings of weighted function spaces on the real line due to the trace theorem cf. Theorem 2.

So to prove Theorem 3 we divide the identity operator

$$\mathrm{id} \ : \ RB^{s_0}_{p_0,q_0}(\mathbf{X}) \ \hookrightarrow RB^{s_1}_{p_1,q_1}(\mathbf{X})$$

into two parts in the following way. Let $\phi \in C_0^\infty(\mathbf{X})$, $\phi(x) = 1$ for $x \in B(o,\frac{1}{2})$, $\operatorname{supp}\phi \subset \overline{B(o,1)}$, we put $\mathrm{id}_1 f = \phi f$, $\mathrm{id}_2 f = (1-\phi)f$. Then

$$\mathrm{id} \ = \ \mathrm{id}_1 \ + \ \mathrm{id}_2$$

and in consequence by additivity of approximation numbers

$$a_{2k}(\mathrm{id}) \le a_k(\mathrm{id}_1) + a_k(\mathrm{id}_2). \tag{9}$$

Moreover, it should be clear that

$$a_k(\mathrm{id}_1) \ \sim \ a_k(\mathrm{Id}_1) \quad \text{and} \quad a_k(\mathrm{id}_2) \ \sim \ a_k(\mathrm{Id}_2)\,,$$

where

$$\mathrm{Id}_1 \ : \ \widetilde{RB}^{s_0}_{p_0,q_0}\big(\overline{B(o,1)}\big) \ \hookrightarrow \widetilde{RB}^{s_1}_{p_1,q_1}\big(\overline{B(o,1)}\big)$$

and

$$\mathrm{Id}_2 \ : \ RB^{s_0}_{p_0,q_0}\big(\mathbf{X}\setminus B(o,1)\big) \ \hookrightarrow RB^{s_1}_{p_1,q_1}\big(\mathbf{X}\setminus B(o,1)\big).$$

Here the spaces $\widetilde{RB}^{s}_{p,q}\big(\overline{B(o,1)}\big)$ are defined by

$$\widetilde{RB}^{s}_{p,q}\Big(\overline{B(o,1)}\Big) \ = \ \big\{f \in RB^s_{p,q}(\mathbf{X}) : \ \operatorname{supp} f \subset \overline{B(o,1)}\big\}. \tag{10}$$

whereas the spaces $RB^s_{p,q}\big(\mathbf{X}\setminus B(o,1)\big)$ were introduced in Definition 2. Thus to estimate $a_k(\mathrm{id})$ it is sufficient to estimate $a_k(\mathrm{Id}_1)$ and $a_k(\mathrm{Id}_2)$.

But the space described in (10) is isomorphic to the corresponding space $\widetilde{RB}^{s}_{p,q}\Big(\overline{B_e(0,1)}\Big)$ defined on the unit ball $\overline{B_e(0,1)} \subset \mathbb{R}^d$ of the Euclidean space $\mathbb{R}^d$, in the way similar to (10). Thus

$$a_k(\mathrm{Id}_1) \sim a_k\Big(\widetilde{RB}^{s_0}_{p_0,q_0}\big(\overline{B_e(0,1)}\big) \hookrightarrow \widetilde{RB}^{s_1}_{p_1,q_1}\big(\overline{B_e(0,1)}\big)\Big). \tag{11}$$

The last approximation numbers are estimated from above in Subsection 3.1.2

On the other hand, the space $RB^{s_i}_{p_i,q_i}\big(\mathbf{X}\setminus B(o,1)\big)$ is isomorphic to the space $B^{s_i}_{p_i,q_i}\big(\gamma[1,\infty),v_{p_i}\big)$, cf. (5) and Theorem 2.

By standard arguments

$$
\begin{aligned}
(12)\qquad a_k\big(\mathrm{Id}_2\big) &\sim a_k\Big(B^{s_0}_{p_0,q_0}\big(\gamma[1,\infty),v_p\big)\hookrightarrow B^{s_1}_{p_1,q_1}\big(\gamma[1,\infty)\big)\Big)\\
&\sim a_k\Big(B^{s_0}_{p_0,q_0}\big(\mathbb{R},v_p\big)\hookrightarrow B^{s_1}_{p_1,q_1}\big(\mathbb{R}\big)\Big)
\end{aligned}
$$

where $\frac{1}{p}=\frac{1}{p_0}-\frac{1}{p_1}$. Both the lower and upper estimates of the last approximation numbers are known at least for specialist. Indeed, it follows easily from the estimates for polynomial weights, cf. Lemma 3 below and Proposition 2. This estimates will finish the estimates of $a_k(\mathrm{id})$ from above as well as give us the estimates from below since we have the following commutative diagram:

$$
\begin{array}{ccc}
RB^{s_0}_{p_0,q_0}(\mathbf{X}\setminus B(o,1)) & \xrightarrow{\tilde{\mathrm{id}}} & RB^{s_0}_{p_0,q_0}(\mathbf{X})\\
\mathrm{Id}_2\big\downarrow & & \big\downarrow\mathrm{id}\\
RB^{s_1}_{p_1,q_1}(\mathbf{X}\setminus B(o,1)) & \xleftarrow{\mathrm{T}_\phi} & RB^{s_1}_{p_1,q_1}(\mathbf{X})\,.
\end{array}
$$

Here T_ϕ denotes an operator $T_\phi : f\mapsto(1-\phi)f$.

The approximation numbers $a_n\Big(B^{s_0}_{p_0,q_0}\big(\mathbb{R},v_p\big)\hookrightarrow B^{s_1}_{p_1,q_1}\big(\mathbb{R}\big)\Big)$ are described in Subsection 3.1.3. But first we need information about approximation numbers of some weighted sequence spaces.

3.1.1. Approximation numbers of some sequence spaces. We will need weighted spaces of double indexed sequences. Let $1\le p,q\le\infty$. Let $\delta\ge 0$ and let $w:\mathbb{N}_0\times\mathbb{N}_0\to\mathbb{R}_+$ denote a weight function. In particular for the positive $\alpha>0$ we put

(13)

$$
w_\alpha(j,k)=(1+k)^\alpha \text{ and } \widetilde{w_\alpha}(j,k)=(1+2^{-j}k)^\alpha \text{ and } v_\alpha(j,k)=\exp(\alpha 2^{-j}k)
$$

We will work with the following weighted sequence spaces

$$
(14)\quad \ell_q(2^{j\delta}\ell_p(w))=\Bigg\{(s_{j,k})_{j,k}\ :\ \|s_{j,k}|\ell_q(2^{j\delta}\ell_p(w))\|\ =\ \left(\sum_{j=0}^\infty\left(\sum_{k=0}^\infty 2^{j\delta p}w(j,k)|s_{j,k}|^p\right)^{q/p}\right)^{1/q}<\infty\Bigg\}
$$

(with the usual modification if $p=\infty$ or $q=\infty$). If $\delta=0$ we will write $\ell_q(\ell_p(w))$.

In future considerations it will be useful to regard the following subspaces of the above sequence spaces:

$$(15)\quad \ell_q(2^{j\delta}\ell_p^{\gamma 2^j}(w)) = \left\{ (s_{j,k})_{j,k} \in \ell_q(2^{j\delta}\ell_p(w)) : s_{j,k} = 0 \text{ if } k > \gamma 2^j \right\}$$

$$(16)\quad \ell_q(2^{j\delta}\widetilde{\ell}_p^{\gamma 2^j}(w)) = \left\{ (s_{j,k})_{j,k} \in \ell_q(2^{j\delta}\ell_p(w)) : s_{j,k} = 0 \text{ if } k \le \gamma 2^j \right\},$$

$\gamma \in \mathbb{N}$. Now we formulate the main result of this subsection.

Lemma 2. *Let* $1 \le p_0 < p_1 \le \infty$, $(p_0, p_1) \ne (1, \infty)$, $1 \le q_0, q_1 \le \infty$, $\alpha > 0$, $\delta > 0$ *and* $\gamma > 0$. *Then for all* $k \in \mathbb{N}$ *the following estimate*

$$a_k(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0}^{\gamma 2^j}(w_\alpha)) \to \ell_{q_1}(\ell_{p_1}^{\gamma 2^j}(w_\alpha))) \ \sim\ k^{-\beta}$$

holds, where

$$\beta = \begin{cases} \delta + \frac{\alpha}{p} & \text{for} \quad 1 \le p_0 < p_1 \le 2 \quad \text{or} \quad 2 \le p_0 < p_1 \le \infty\,, \\ \delta + \frac{\alpha}{p} + \frac{1}{2} - \frac{1}{t} & \text{for} \quad 1 \le p_0 < 2 < p_1 \le \infty \quad \text{and} \quad \delta + \frac{\alpha}{p} \ge \frac{1}{t}\,, \\ (\delta + \frac{\alpha}{p})\frac{t}{2} & \text{for} \quad 1 \le p_0 < 2 < p_1 \le \infty \quad \text{and} \quad \delta + \frac{\alpha}{p} < \frac{1}{t}\,, \end{cases}$$

and $\frac{1}{p} = \frac{1}{p_0} - \frac{1}{p_1}$, $t = \min\{p_1, p_0'\}$.

Proof. Step 1. Preparations. Let

$$\Lambda := \{\lambda = (\lambda_{j,\ell})_{j,\ell} : \quad \lambda_{j,\ell} \in \mathbb{C}, \quad j \in \mathbb{N}_0,\, 0 \le \ell \le \gamma 2^j\}$$

and

$$B_0 \ = \ \ell_{q_0}(2^{j\delta}\ell_{p_0}^{\gamma 2^j}(w_\alpha))\,, \quad B_1 \ = \ \ell_{q_1}(\ell_{p_1}^{\gamma 2^j}(w_\alpha))\,.$$

Let $P_j : \Lambda \mapsto \Lambda$ be the canonical projection onto j-level, i.e. for $\lambda = \big(\lambda_{k,\ell}\big)$ we put

$$(P_j\lambda)_\ell := \begin{cases} \lambda_{k,\ell} & \text{if} \quad k = j, \\ 0 & \text{otherwise}, \end{cases} \qquad \ell \in \mathbb{N}_0\,.$$

Monotonicity arguments and elementary properties of the approximation numbers yield

$$(17)\qquad a_k\Big(P_j : B_0 \mapsto B_1\Big) \ \le\ 2^{-j\delta}\, a_k\Big(\mathrm{Id} : \ell_{p_0}^{\gamma 2^j}(w_\alpha) \mapsto \ell_{p_1}^{\gamma 2^j}(w_\alpha)\Big)\,,$$

and

$$a_k\Big(\mathrm{Id}:\ \ell_{p_0}^{\gamma 2^j}(w_\alpha)\mapsto \ell_{p_1}^{\gamma 2^j}(w_\alpha)\Big) \quad = \quad a_k\Big(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}\Big) \tag{18}$$

where D_σ is a diagonal operator defined by the sequence $\sigma_\ell = (1+\ell)^{-\alpha/p}$.

Step 2. The estimate from above.

Substep 2.1. First we regard the case $1 \le p_0 < p_1 \le 2$ or $2 \le p_0 < p_1 \le \infty$. Using (17) and (18) we find

$$L_{r,\infty}^{(a)}(P_j) \le c\, 2^{-j\delta}\, L_{r,\infty}^{(a)}(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j})\,. \tag{19}$$

We have

$$a_k\Big(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_0}^{\gamma 2^j}\Big) \ = \ \sigma_k \ = \ (1+k)^{-\alpha/p}, \tag{20}$$

cf. [20, p. 108]. Now using (20) and the elementary properties of approximation numbers we get

$$k^{\frac{1}{r}} a_{2k-1}\Big(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}\Big) \ \le\ C\ k^{\frac{1}{r}-\frac{\alpha}{p}} a_k\Big(\mathrm{id}:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}\Big)\,.$$

In consequence

$$L_{r,\infty}^{(a)}\Big(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}\Big) \ \le\ C\, L_{s,\infty}^{(a)}\Big(\mathrm{id}:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}\Big) \tag{21}$$

if $\frac{1}{s} = \frac{1}{r} - \frac{\alpha}{p} > 0$. But Lemma 1 implies

$$L_{s,\infty}^{(a)}(\mathrm{id}:\ \ell_{p_0}^{N}\mapsto \ell_{p_1}^{N}) \ \le\ C\ N^{\frac{1}{s}} \text{ if } 1 \le p_0 < p_1 \le 2 \text{ or } 2 \le p_0 < p_1 \le \infty\,. \tag{22}$$

Under the assumption $1/r > \alpha/p$ we conclude from (21) and (22) that

$$L_{r,\infty}^{(a)}(D_\sigma:\ \ell_{p_0}^{\gamma 2^j}\mapsto \ell_{p_1}^{\gamma 2^j}) \ \le\ C\, 2^{-j(\frac{\alpha}{p}-\frac{1}{r})}\,, \tag{23}$$

where the constant C depends on γ but is independent of j.

Now, for given $M \in \mathbb{N}_0$ let

$$P := \sum_{j=0}^{M} P_j \qquad \text{and} \qquad Q := \sum_{j=M+1}^{\infty} P_j\,. \tag{24}$$

Hence (4), (19), (23) and (24) yield:

$$L_{r,\infty}^{(a)}(P)^\varrho \le \sum_{j=0}^{M} L_{r,\infty}^{(a)}(P_j)^\varrho \le c_1 \sum_{j=0}^{M} 2^{-j\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p})} \le c_2\, 2^{-M\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p})}\,, \tag{25}$$

with the constant c_2 independent of M, if $\frac{1}{r} > \frac{\alpha}{p} + \delta$. Hence for every $\delta > 0$ we have

$$a_{2^M}(P : B_0 \mapsto B_1) \le c_3\, 2^{-M(\delta+\frac{\alpha}{p})}\,.$$

In a similar way to (25) we obtain

$$L^{(a)}_{r,\infty}(Q)^\varrho \;\le\; c_1 \sum_{j=M+1}^{\infty} 2^{-j\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p})} \;\le\; c_2 2^{-M\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p})}\,,$$

if $r > 0$ such that $\frac{\alpha}{p} < \frac{1}{r} < \delta + \frac{\alpha}{p}$. Hence, for r s.t. $\frac{\alpha}{p} < \frac{1}{r} < \delta + \frac{\alpha}{p}$ and $k = 2^M$ we have

$$a_{2^M}(Q : B_0 \mapsto B_1) \le c_3\, 2^{-M(\delta+\frac{\alpha}{p})}\,.$$

This implies the estimate from above for $1 \le p_0 < p_1 \le 2$ or $2 \le p_0 < p_1 \le \infty$.

Substep 2.2. Now let $1 \le p_0 < 2 < p_1 \le \infty$ and let $(p_0, p_1) \ne (1, \infty)$. First we prove that

$$(26)\quad L^{(a)}_{r,\infty}(D_\sigma : \ell^{\gamma 2^j}_{p_0} \to \ell^{\gamma 2^j}_{p_1}) \le C\, 2^{j(\frac{1}{r}-\frac{1}{2}+\frac{1}{t}-\frac{\alpha}{p})} \quad \text{if } \frac{1}{r} > \max\big(\frac{1}{2}, \frac{\alpha}{p} + \frac{1}{2} - \frac{1}{t}\big),$$

$$(27)\quad L^{(a)}_{r,\infty}(D_\sigma : \ell^{\gamma 2^j}_{p_0} \to \ell^{\gamma 2^j}_{p_1}) \le C\, 2^{j(\frac{2}{rt}-\frac{\alpha}{p})} \quad \text{if } \frac{t\alpha}{2p} < \frac{1}{r} < \frac{1}{2} \text{ (that means } \frac{\alpha}{p} < \frac{1}{t}).$$

We put $N = \gamma 2^j$. Let us choose $K \in \mathbb{N}$ such that $2^{K-1} \le N < 2^K$ and let $\Pi_i : \ell^N_{p_0} \to \ell^N_{p_0}$ be a projection defined in the following way

$$(\Pi_i \lambda)_\ell := \begin{cases} \lambda_\ell & \text{if } \quad 2^{i-1} \le \ell < \min(2^i, N), \\ 0 & \text{otherwise} \end{cases}\,,$$

for $\lambda = \big(\lambda_\ell\big)$, $i = 1, 2, \ldots, K$. Then

$$D_\sigma \;=\; \sum_{i=1}^{K} D_\sigma \circ \Pi_i\,.$$

Multiplicativity of the approximation numbers yields

$$a_k\Big(D_\sigma \circ \Pi_i : \ell^N_{p_0} \to \ell^N_{p_1}\Big) \;\le\; 2^{-(i-1)\frac{\alpha}{p}}\, a_k\Big(\mathrm{id} : \ell^{2^{i-1}}_{p_0} \to \ell^{2^{i-1}}_{p_1}\Big), \qquad i \in \mathbb{N}.$$

Let r be real positive number. Lemma 1 implies

$$L^{(a)}_{r,\infty}(D_\sigma \circ \Pi_i : \ell^N_{p_0} \to \ell^N_{p_1}) \le C\, 2^{(i-1)(\frac{1}{t}+\frac{1}{r}-\frac{1}{2}-\frac{\alpha}{p})} \quad \text{if} \quad \frac{1}{2} < \frac{1}{r}\,, \tag{28}$$

$$L^{(a)}_{r,\infty}(D_\sigma \circ \Pi_i : \ell^N_{p_0} \to \ell^N_{p_1}) \le C\, 2^{(i-1)(\frac{2}{rt}-\frac{\alpha}{p})} \quad \text{if} \quad 0 < \frac{1}{r} < \frac{1}{2}\,. \tag{29}$$

We choose r such that $\frac{1}{r} > \frac{1}{2}$ and that $\frac{1}{r} - \frac{1}{2} + \frac{1}{t} - \frac{\alpha}{p} > 0$. The formulas (4) and (28) yield

$$\begin{aligned} L^{(a)}_{r,\infty}(D_\sigma)^\varrho &\le \sum_{i=1}^{K} L^{(a)}_{r,\infty}(D_\sigma \circ \Pi_i)^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho i(\frac{\alpha}{p}-\frac{1}{t}-\frac{1}{r}+\frac{1}{2})} \le c_2\, 2^{\varrho K(\frac{1}{r}-\frac{1}{2}+\frac{1}{t}-\frac{\alpha}{p})}. \end{aligned}$$

The constant c_2 is independent of K. This implies (26). In the similar way if $\frac{t\alpha}{2p} < \frac{1}{r} < \frac{1}{2}$, then the formulas (4) and (29) yield

$$\begin{aligned} L^{(a)}_{r,\infty}(D_\sigma)^\varrho &\le \sum_{i=1}^{K} L^{(a)}_{r,\infty}(D_\sigma \circ \Pi_i)^\varrho \\ &\le c_1 \sum_{i=0}^{K-1} 2^{-\varrho i(\frac{\alpha}{p}-\frac{2}{rt})} \le c_2\, 2^{\varrho K(\frac{2}{rt}-\frac{\alpha}{p})} \end{aligned}$$

and the constant c_2 is independent of K. This implies (27).

Substep 2.3. Once more let $1 \le p_0 < 2 < p_1 \le \infty$, $(p_0, p_1) \ne (1, \infty)$. Using (26) instead of (23) we get

$$L^{(a)}_{r,\infty}(P)^\varrho \le \sum_{j=0}^{M} L^{(a)}_{r,\infty}(P_j)^\varrho \le c_3 2^{-M\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t})} \quad \text{if} \quad \frac{1}{r} > \max\left\{\delta + \frac{\alpha}{p} + \frac{1}{2} - \frac{1}{t}, \frac{1}{2}\right\},$$

and in consequence

$$a_{2^M}(P : B_0 \mapsto B_1) \le c_3\, 2^{-M(\delta+\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t})} \quad \text{for} \quad 1 \le p_0 < 2 < p_1 \le \infty\,, \quad (p_0, p_1) \ne (1, \infty). \tag{30}$$

In the similar way

$$(31)\quad L^{(a)}_{r,\infty}(Q)^{\varrho} \le c_4 \begin{cases} 2^{-M\varrho(\delta-\frac{1}{r}+\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t})} \\ \quad \text{if } \max\left(\frac{1}{2},\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t}\right) < \frac{1}{r} < \delta+\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t} \\ 2^{-M\varrho(\delta+\frac{\alpha}{p}-\frac{2}{tr})} \\ \quad \text{if } \frac{t\alpha}{2p} < \frac{1}{r} < \min\left(\frac{1}{2},\frac{t}{2}(\delta+\frac{\alpha}{p})\right). \end{cases}$$

Now, (30) and the first case of (31) imply the estimates from above for $\delta+\frac{\alpha}{p} > \frac{1}{t}$.

For $\delta+\frac{\alpha}{p} \le \frac{1}{t}$ we take $k=[2^{M\frac{2}{t}}]$. Then $\frac{\alpha}{p} < \frac{1}{t}$ so by the second case of (31) we have

$$(32)\qquad a_k(Q: B_0 \mapsto B_1) \le c\, k^{-\frac{1}{r}} 2^{-(\delta+\frac{\alpha}{p}-\frac{2}{tr})} \;\le\; c\, k^{-\frac{t}{2}(\delta+\frac{\alpha}{p})}.$$

By standard argument the above estimates hold for any positive integer k. Moreover, (30) implies

$$(33)\qquad a_k(P: B_0 \mapsto B_1) \le C\, k^{-(\delta+\frac{\alpha}{p}+\frac{1}{2}-\frac{1}{t})} \;\le\; C\, k^{-\frac{t}{2}(\delta+\frac{\alpha}{p})}$$

since $\delta+\frac{\alpha}{p} \le \frac{1}{t}$. Now (32) and (33) give us the estimates from above in the remaining case.

Step 3. We estimate the approximation numbers from below. The space $\ell_{q_1}\big(\ell^{\gamma 2^j}_{p_1}\big)$ is isomorphic to $\ell_{q_1}\big(\ell^{\gamma 2^j}_{p_1}(w_\alpha)\big)$ and the isomorphism is given by the diagonal operator $a_{j,i} \mapsto (1+i)^{-\alpha/p_1} a_{j,i}$. Moreover the inverse operator $a_{j,i} \mapsto (1+i)^{\alpha/p_1} a_{j,i}$ is an isomorphism of the space $\ell_{q_0} 2^{j\delta}\big(\ell^{\gamma 2^j}_{p_0}(w_\alpha)\big)$ onto $\ell_{q_0} 2^{j\delta}\big(\ell^{\gamma 2^j}_{p_0}(w_\beta)\big)$ with $\beta = \frac{\alpha p_0}{p}$. This implies the equivalence of approximation numbers

$$a_k\Big(\ell_{q_0} 2^{j\delta}\big(\ell^{\gamma 2^j}_{p_0}(w_\alpha)\big) \to \ell_{q_1}\big(\ell^{\gamma 2^j}_{p_1}(w_\alpha)\big)\Big) \sim a_k\Big(\ell_{q_0} 2^{j\delta}\big(\ell^{\gamma 2^j}_{p_0}(w_\beta)\big) \to \ell_{q_1}\big(\ell^{\gamma 2^j}_{p_1}\big)\Big).$$

Thus we can regard the following commutative diagram

$$\begin{array}{ccc} \ell^{\gamma 2^j}_{p_0} & \xrightarrow{\;R\;} & \ell_{q_0}\big(2^{j\delta}\ell^{\gamma 2^j}_{p_0}(w_\beta)\big) \\ {\scriptstyle \mathrm{Id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \ell^{\gamma 2^j}_{p_1} & \xleftarrow[\;P\;]{} & \ell_{q_1}\big(\ell^{\gamma 2^j}_{p_1}\big) \end{array}$$

where an operator $R: \ell^{\gamma 2^j}_{p_0} \to \ell_{q_0}\big(2^{j\delta}\ell^{\gamma 2^j}_{p_0}(w_\beta)\big)$ is given by

$$(R\lambda)_{u,i} := \begin{cases} \lambda_i & u=j, \\ 0 & \text{otherwise.} \end{cases} \qquad u \in \mathbb{N}_0, \quad 0 \le i \le \gamma 2^j.$$

and P is a projection. It should be clear that $\|R\| \le c\, 2^{j(\delta+\frac{\alpha}{p})}$ and $\|P\| = 1$. So, by Lemma 1,

(i) if $1 \le p_0 < p_1 \le 2$ or $2 \le p_0 < p_1 \le \infty$ then taking $k = \gamma 2^{j-2}$ we get

$$\begin{aligned} 1 &\le C\, a_k\Big(\mathrm{Id} : \ell_{p_0}^{\gamma 2^j} \mapsto \ell_{p_1}^{\gamma 2^j}\Big) \\ &\le C\, 2^{j(\delta+\frac{\alpha}{p})} a_k\Big(\mathrm{id} : \ell_{q_0} 2^{j\delta}\big(\ell_{p_0}^{\gamma 2^j}(w_\alpha)\big) \mapsto \ell_{q_1}\big(\ell_{p_1}^{\gamma 2^j}(w_\alpha)\big)\Big). \end{aligned}$$

(ii) If $1 \le p_0 < 2 < p_1 \le \infty$ and $\delta + \frac{\alpha}{p} \ge \frac{1}{t}$ then taking one more $k = \gamma 2^{j-2}$ we have

$$\begin{aligned} 2^{-j(\frac{1}{2}-\frac{1}{t})} &\le C\, a_k\Big(\mathrm{Id} : \ell_{p_0}^{\gamma 2^j} \mapsto \ell_{p_1}^{\gamma 2^j}\Big) \\ &\le C\, 2^{j(\delta+\frac{\alpha}{p})} a_k\Big(\mathrm{id} : \ell_{q_0} 2^{j\delta}\big(\ell_{p_0}^{\gamma 2^j}(w_\alpha)\big) \mapsto \ell_{q_1}\big(\ell_{p_1}^{\gamma 2^j}(w_\alpha)\big)\Big). \end{aligned}$$

(iii) If $1 \le p_0 < 2 < p_1 \le \infty$ and $\delta + \frac{\alpha}{p} < \frac{1}{t}$ then we take $k = \big[2^{j\frac{2}{t}}\big]$. So,

$$\begin{aligned} 1 &\le C\, a_k\Big(\mathrm{Id} : \ell_{p_0}^{\gamma 2^j} \mapsto \ell_{p_1}^{\gamma 2^j}\Big) \\ &\le C\, 2^{j(\delta+\frac{\alpha}{p})} a_k\Big(\mathrm{id} : \ell_{q_0}\big(\ell_{p_0}^{\gamma 2^j}(w_\alpha)\big) \mapsto \ell_{q_1} 2^{j\delta}\big(\ell_{p_1}^{\gamma 2^j}(w_\alpha)\big)\Big). \end{aligned}$$

This finishes the proof. □

Lemma 3. *Let* $1 \le p_0 < p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, $\alpha, \delta > 0$. *Then the following estimate*

$$a_k\big(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \to \ell_{q_1}(\ell_{p_1})\big) \ \sim\ k^{-\varkappa},$$

holds, where

$$\varkappa = \begin{cases} \delta & \text{if} \quad 1 \le p_0 < p_1 \le 2 \text{ or } 2 \le p_0 < p_1 \le \infty, \\ \delta + \frac{1}{2} - \frac{1}{t} & \text{if} \quad 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ \delta > \frac{1}{t}, \\ \delta\frac{t}{2} & \text{if} \quad 1 \le p_0 < 2 < p_1 \le \infty \ \text{ and } \ \delta \le \frac{1}{t}. \end{cases}$$

Proof. The estimates from above follows immediately from the corresponding results for polynomial weights cf. [23]. Let us choose $\beta > \delta$. Then

$$\ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \hookrightarrow \ell_{q_0}(2^{j\delta}\ell_{p_0}(\widetilde{w_\beta})) \hookrightarrow \ell_{q_1}(\ell_{p_1})$$

So, the results from in [23] implies

$$\begin{aligned} a_k\big(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \to \ell_{q_1}(\ell_{p_1})\big) &\le C a_k\big(\mathrm{id} : \ell_{q_0}(2^{j\delta}\ell_{p_0}(\widetilde{w_\beta})) \to \ell_{q_1}(\ell_{p_1})\big) \\ &\le C\, k^{-\varkappa}, \end{aligned}$$

where $\varkappa$ is defined as before.

The estimates from below can be proved in the way similar to *Step* 3 in the proof of the last lemma. We regard the following commutative diagram

$$\begin{array}{ccc} \ell_{p_0}^N & \xrightarrow{\ S\ } & \ell_{q_0}\big(2^{j\delta}\ell_{p_0}(v_\alpha)\big) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \ell_{p_1}^N & \xleftarrow{\ T\ } & \ell_{q_1}(\ell_{p_1})\,, \end{array}$$

with

$$(T\lambda)_i \;=\; \lambda_{j,i}\;, \quad i=1,\dots,N \quad \text{and} \quad (S\eta)_{k,i} \;=\; \begin{cases} \eta_i & k=j \ \text{ and } \ i\le N, \\ 0 & \text{otherwise}\,; \end{cases}$$

Then

$$(34)\; a_k\big(\,\mathrm{id}:\ell_{p_0}^N \longrightarrow \ell_{p_1}^N\big) \;\le\; C\,2^{j\delta} a_k\big(\,\mathrm{id}:\ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \longrightarrow \ell_{q_1}(\ell_{p_1})\big)\,.$$

(i) For $1\le p_0<p_1\le 2$ or $2\le p_0<p_1\le\infty$, and for $1\le p_0<2<p_1\le\infty$ and $\delta>\frac{1}{t}$ we take $k=2^{j-2}$ and $N=2^j$. So by Lemma 1 and (34) we get

$$\begin{aligned} C2^{-j\delta} \;&\le\; a_k\big(\,\mathrm{id}:\ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \longrightarrow \ell_{q_1}(\ell_{p_1})\big), \\ C2^{-j(\frac12-\frac1t+\delta)} \;&\le\; a_k\big(\,\mathrm{id}:\ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \longrightarrow \ell_{q_1}(\ell_{p_1})\big), \end{aligned}$$

respectively.
(ii) For $1\le p_0<2<p_1\le\infty$ and $\delta\le\frac1t$ we take $k=[2^{j\frac{2}{t}}]$ and N as above. Using once again (34) and Lemma 1 we get

$$Ck^{-\frac{\delta t}{2}} \;\le\; C2^{-j\delta} \le a_k\big(\,\mathrm{id}:\ell_{q_0}(2^{j\delta}\ell_{p_0}(v_\alpha)) \longrightarrow \ell_{q_1}(\ell_{p_1})\big). \qquad \square$$

3.1.2. Approximation numbers for function spaces: local analysis. Now we apply the estimates for approximation numbers of embeddings of sequence spaces to function spaces, first we regard the local part near origin. We use the Epperson-Frazier approach to radial Besov spaces and their construction of the radial φ-transform, cf. [7]. We apply the construction with normalization described [17] that is different to the original one. For any possible s and p Epperson and Frazier constructed two families of radial functions $\varphi_{j,k}^{(s,p)}\in\mathcal{S}(\mathbb{R}^d)$ and $\eta_{j,k}^{(s,p)}\in\mathcal{S}(\mathbb{R}^d)$, $j=0,1,\dots$, $k=1,2,\dots$, such that any radial distribution $f\in\mathcal{S}'(\mathbb{R}^n)$ can be decomposed into

$$f \;=\; \sum_{j=0}^{\infty}\sum_{k=1}^{\infty} < f,\varphi_{j,k}^{(s,p)} > \eta_{j,k}^{(s,p)}$$

(convergence in $\mathcal{S}'(\mathbb{R}^n)$), cf. [7]. Moreover the following theorem holds for radial Besov spaces.

Theorem 4 (Epperson-Frazier)**.** *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. The operators*

$$S_{(s,p)} \,:\, RB^s_{p,q}(\mathbb{R}^n) \longrightarrow \ell_q(\ell_p(w_{d-1}))$$

and

$$T_{(s,p)} \,:\, \ell_q(\ell_p(w_{d-1})) \longrightarrow RB^s_{p,q}(\mathbb{R}^n)$$

defined by

$$S_{(s,p)}(f) \;=\; (< f, \varphi^{(s,p)}_{j,k} >)_{j,k}\,, \quad T_{(s,p)}((s_{j,k+1})) \;=\; \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} s_{j,k}\,\eta^{(s,p)}_{j,k+1}\,.$$

are bounded. The operator T is a retraction, i.e. $T \circ S \;=\; \mathrm{id}$.

The last theorem and the next proposition reduce the estimates from above of approximation numbers of Sobolev embeddings of the function spaces to the estimates of approximation numbers of embeddings of the sequence spaces defined in (14)-(16). The following lemma was proved in [26]

Lemma 4. *Let $1 \le p_0 < p_1 \le \infty$, and $1 \le q_0, q_1 \le \infty$. Let $s_0 - \frac{d}{p_0} > s_1 - \frac{d}{p_1}$, $s_0 > 0$ and $\alpha \ge d-1$. We put $\delta = s_0 - s_1 + d(\frac{1}{p_1} - \frac{1}{p_0})$ and $\tilde{s} = s_1 + d(\frac{1}{p_0} - \frac{1}{p_1})$. Then*

$$S_{(\tilde{s},p_0)} \;=\; S_{(s_1,p_1)} \;=\; S\,, \qquad T_{(\tilde{s},p_0)} \;=\; T_{(s_1,p_1)} \;=\; T$$

and the following diagram is commutative

$$\begin{array}{ccc}
\widetilde{RB}^{s_0}_{p_0,q_0}\Big(\overline{B_e(0,1)}\Big) & \xrightarrow{\;S\;} & \ell_{q_0}(2^{j\delta}\ell^{2^{j+2}}_{p_0}(w_{d-1})) \;\oplus\; \ell_{q_0}(2^{j\delta}\widetilde{\ell}^{2^{j+2}}_{p_0}(w_\alpha)) \\
\mathrm{id}\Big\downarrow & & \Big\downarrow\mathrm{id} \\
\widetilde{RB}^{s_1}_{p_1,q_1}\Big(\overline{B_e(0,1)}\Big) & \xleftarrow{\;T\;} & \ell_{q_1}(\ell^{2^{j+2}}_{p_1}(w_{d-1})) \;\oplus\; \ell_{q_1}(\widetilde{\ell}^{2^{j+2}}_{p_1}(w_\alpha))\,.
\end{array}$$

The last lemma has an important consequence. Since α is at our disposal and can be as large as we want, approximation numbers of the embeddings of the above function spaces can be estimated from above by approximation numbers of sequence spaces of the type $\ell_q(2^{j\delta}\ell^{2^{j+2}}_p(w_{d-1}))$.

Proposition 1. *Suppose $1 \le p_0 < p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$ and $s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1}) > 0$. Then there exist a positive constant C such that*

$$a_k\Big(\widetilde{RB}^{s_0}_{p_0,q_0}\Big(\overline{B_e(0,1)}\Big) \to \widetilde{RB}^{s_1}_{p_1,q_1}\Big(\overline{B_e(0,1)}\Big)\Big) \;\le\; Ck^{-\varkappa}\,,$$

where $\varkappa$ *is defined by* (8).

Proof. By Lemma 4 we have

$$
\begin{aligned}
(35)\quad a_k\Big(\,\mathrm{id} : \widetilde{RB}^{s_0}_{p_0,q_0}\Big(\overline{B_e(0,1)}\Big) \to \widetilde{RB}^{s_1}_{p_1,q_1}\Big(\overline{B_e(0,1)}\Big)\Big) \\
\le C\, a_k\Big(\,\mathrm{Id}\,:\, \ell_{q_0}(2^{j\delta}\ell^{2^{j+2}}_{p_0}(w_{d-1})) \oplus \ell_{q_0}(2^{j\delta}\widetilde{\ell}^{2^{j+2}}_{p_0}(w_\alpha)) \\
\to \ell_{q_1}(\ell^{2^{j+2}}_{p_1}(w_{d-1}))\ \oplus \ell_{q_1}(\widetilde{\ell}^{2^{j+2}}_{p_1}(w_\alpha))\Big).
\end{aligned}
$$

We divide the operator Id into two parts

$$
(\mathrm{id}_1(\lambda))_{j,\ell} \;=\; \begin{cases}\lambda_{j,\ell} & \ell \le 2^{j+2}\\ 0 & \text{otherwise}\,,\end{cases} \qquad (\mathrm{id}_2(\lambda))_{j,\ell} \;=\; \begin{cases}\lambda_{j,\ell} & \ell \ge 2^{j+2}+1\\ 0 & \text{otherwise}\,.\end{cases}
$$

Then $\mathrm{Id} = \mathrm{id}_1 + \mathrm{id}_2$ and

$$
(36)\qquad a_{2k-1}\big(\,\mathrm{Id}\,\big) \;\le\; a_k\big(\,\mathrm{id}_1\,\big) \;+\; a_k\big(\,\mathrm{id}_2\,\big)\,.
$$

On the one hand Lemma 2 with $\delta = s_0 - s_1 - d(\frac{1}{p_0} - \frac{1}{p_1})$ and $\alpha = d-1$ asserts that

(37)

$$
a_k\big(\,\mathrm{id}_1\,\big) \;\le\; a_k\Big(\,\mathrm{id}\,:\, \ell_{q_0}(2^{j\delta}\ell^{2^{j+2}}_{p_0}(w_{d-1})) \to \ell_{q_1}(\ell^{2^{j+2}}_{p_1}(w_{d-1}))\Big) \;\le\; Ck^{-\varkappa},
$$

where $\varkappa$ is given by (8). On the other hand if we take α such that $\big(\alpha+1\big)\big(\frac{1}{p_0} - \frac{1}{p_1}\big) \;=\; s_0 - s_1$ then $\alpha > d-1$ and

$$
\begin{aligned}
(38)\qquad a_k\big(\,\mathrm{id}_2\,\big) \;&\le\; a_k\Big(\ell_{q_0}(2^{j\delta}\widetilde{\ell}^{2^{j+2}}_{p_0}(w_\alpha) \hookrightarrow \ell_{q_1}(\widetilde{\ell}^{2^{j+2}}_{p_1}(w_\alpha))\Big)\\
&\le\; a_k\Big(\ell_{q_0}(2^{j\delta}\ell_{p_0}(w_{\alpha/p}) \hookrightarrow \ell_{q_1}(\ell_{p_1})\Big)\\
&\le\; Ck^{-\varkappa}\ ,
\end{aligned}
$$

where the last inequality was proved in [23] with $\varkappa$ given by (8). Now the lemma follows from (35)-(38). □

3.1.3. Approximation numbers for function spaces: global analysis. We will need the estimates of approximation numbers of Sobolev embeddings of weighted function Besov spaces on $\mathbb{R}$ with exponential weights. Let v_{p_0}, v_{p_1}, v_p be the exponential weights defined by (5). For $\alpha = \frac{\vartheta}{p}$, let v_α be the weight for the sequence space (13)-(14). It was proved in [18] that the Besov space $B^{s_0}_{p_0,q_0}(\mathbb{R}, v_p)$ is isomorphic to the sequence

space $\ell_{q_0}\big(2^{j\delta_0}\ell_{p_0}(v_\alpha)\big)$, $0 < s_0 < \infty$ and $\delta_0 = s_0 + \frac{1}{2} - \frac{1}{p_0}$, cf. [18, Theorem 1]. Thus for the approximation numbers we get

$$\begin{aligned} a_k\big(B^{s_0}_{p_0,q_0}(\mathbb{R}, v_{p_0}) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}, v_{p_1})\big) \\ \sim a_k\big(B^{s_0}_{p_0,q_0}(\mathbb{R}, v_p) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R})\big) \\ \sim a_k\big(\ell_{q_0}\big(2^{j\delta_0}\ell_{p_0}(v_\alpha)\big) \hookrightarrow \ell_{q_1}\big(2^{j\delta_1}\ell_{p_1}\big)\big) \\ \sim a_k\big(\ell_{q_0}\big(2^{j\delta}\ell_{p_0}(v_\alpha)\big) \hookrightarrow \ell_{q_1}\big(\ell_{p_1}\big)\big) \end{aligned} \tag{39}$$

$\delta_1 = s_1 + \frac{1}{2} - \frac{1}{p_1}$ and $\delta = s_0 - s_1 + \frac{1}{p_1} - \frac{1}{p_0}$. Now (39), Lemma 3 and the lift property for the Besov spaces imply the following proposition

Proposition 2. *Suppose* $1 \le p_0 < p_1 \le \infty$, $1 \le q_0, q_1 \le \infty$, $\alpha > 0$ *and* $\delta = s_0 - s_1 - (\frac{1}{p_0} - \frac{1}{p_1}) > 0$, *then*

$$a_k\Big(B^{s_0}_{p_0,q_0}(\mathbb{R}, v_{p_0}) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}, v_{p_1})\Big) \sim k^{-\varkappa},$$

where $\varkappa$ *is the same as in Lemma* 3.

Now Theorem 3 follows from (9), (11), (12), Theorem 2, Proposition 1 and Proposition 2.

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