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Bounded holomorphic projections for exponentially decreasing weights

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Abstract. We construct generalized Bergman projections on a large class of weighted L^∞–spaces. The examples include exponentially decreasing weights on the unit disc and complex plane.

1. Introduction

The classical Bergman projection $P : f \mapsto \int_{\mathbb{D}} f(\zeta)(1-z\bar{\zeta})^{-2} dA(\zeta)$, where dA is the normalized 2–dimensional Lebesgue measure on the open unit disc $\mathbb{D}$ of the complex plane, is a bounded operator on $L^p(\mathbb{D}) = L^p(\mathbb{D}, dA)$ for

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$1 < p < \infty$; it projects the space onto its closed subspace of analytic functions. In the case $p = 1$, any projection

$$(1) \qquad P_\alpha : f \mapsto (\alpha+1) \int_{\mathbb{D}} \frac{f(\zeta)(1-|\zeta|^2)^\alpha}{(1-z\bar{\zeta})^{2+\alpha}} dA(\zeta)\ , \quad \alpha > 0,$$

has the same role, but for $p = \infty$, no analogous bounded projection exists. See [8] for these classical facts. However, with weighted sup–norms the situation is different. It is easy to see that for every $\beta > 0$, the projection P_α is bounded for $\alpha + 1 > \beta > 0$ on the space $L_v^\infty(\mathbb{D})$ of measurable functions bounded with respect to the weighted sup–norm

$$(2) \qquad \|f\|_v := \operatorname{ess\,sup}_{z\in\mathbb{D}} v(z)|f(z)|, \quad \text{where } v(z) := (1-|z|)^\beta.$$

Other Bergman–type projections and also non–radial weights were recently considered in [2]. For more results, see [1], [5], [7], [8].

Not much is known about bounded projections in the case of weighted L^p–norms $(\int |f|^p v dA)^{1/p}$, if the weight function decreases rapidly as a function of the boundary distance, e.g. $v(z) := \exp(-1/1-|z|^2))$, $z \in \mathbb{D}$. In this case, the orthogonal projection of the space $L_v^2(\mathbb{D})$ is known to be unbounded in the spaces $L_v^p(\mathbb{D})$ for $p \neq 2$, and it seems that no bounded projection from $L_v^p(\mathbb{D})$ onto its subspace of analytic functions is known.

However, the first named author proved recently in [6] that for a large class of weights satisfying a condition (B), see below, the space Hv (the subspace of $L_v^\infty(\mathbb{D})$ consisting of analytic functions) is isomorphic as a Banach space to ℓ^∞. This implies, by [4], p.105, the existence of a bounded projection from L_v^∞ onto Hv. The exponential weight mentioned above satisfies the condition (B).

It is of course of interest to find out concrete formulas for such projections. This task is carried out in the present paper. We construct "canonical" bounded projections from L_v^∞ onto the subspace Hv of analytic functions for weights satisfying the condition (B). It is possible to use these projections to create a satisfactory theory of Toeplitz operators on these weighted spaces, see [3].

The projections will have series representations. To descibe them shortly in the case of $\mathbb{D}$, we shall associate to a given weight an increasing sequence $(s_n)_{n=1}^\infty$, $0 < s_n < 1$, and a sequence $(T_n)_{n=1}^\infty$ of finite rank operators, which are just multipliers on the sequence space of Fourier coefficients. (They essentially arise from two Cesaró summations.) If $f : \mathbb{D} \to \mathbb{C}$ is a *continuous* function, we form for each radius $r \in]0,1[$ the Fourier–coefficients $f_k(r)$ of

$f|_{r\partial\mathbb{D}}$ and write

$$f(z) = \sum_{k=-\infty}^{\infty} f_k(r) e_k(z), \tag{3}$$

which makes sense at least in L^2. Here $e_k(z) := r^{|k|} e^{ik\varphi}$ for $z = re^{i\varphi}$. The projection P_C is then formally given by

$$\begin{aligned} P_C f(z) &= \sum_{n=1}^{\infty} T_n \sum_{k\geq 0} f_k(s_n) e_k(z) \\ &= \sum_{k\geq 0} \left(\sum_{n=1}^{\infty} t_{nk} f_k(s_n) \right) z^k. \end{aligned} \tag{4}$$

The convergence of this series will be proven below. For each fixed n, there are only finitely many nonzero numbers t_{nk}; they are the coefficients of the multiplier operators T_n see (6). Also, given a fixed degree k, there exist at most 2 nonzero t_{nk}. Actually $0 \leq t_{nk} \leq 1$, and, for an increasing sequence $(m_n)_{n=1}^{\infty}$, for a fixed n, the value of t_{nk} grows linearly from 0 at $k \cong m_{n-1}$ to 1 at $k \cong m_n$, and then decreases back to 0 at $k \cong m_{n+1}$. This interplay of the degree m_n on one hand, and the radius s_n on the other hand, is of essential importance for the proof of the boundedness of the projection P_C.

If f is just an L^∞–function, the point evaluations of the coefficients f_k have to be replaced by integral means.

It is probably difficult to find integral kernels in terms of elementary functions for P_C. One of the reasons is that the definition of the numbers s_n and m_n is not completely trivial.

2. Preliminaries, notations

We consider function spaces defined on Ω which is either $\mathbb{D}$ or $\mathbb{C}$. By a weight we mean a continuous, radial function $v : \Omega \to]0, \infty[$, which is also strictly decreasing as $r := |z|$ increases. We assume that $\lim_{r\to 1} v(r) = 0$, if $\Omega = \mathbb{D}$, and $\lim_{r\to\infty} r^m v(r) = 0$ for any $m \geq 0$, if $\Omega = \mathbb{C}$. We define the spaces

$$\begin{aligned} L_v^\infty &:= \left\{ f : \Omega \to \mathbb{C} \text{ measurable} \mid \|f\|_v := \operatorname{ess\,sup}_{z\in\Omega} v(z)|f(z)| < \infty \right\} \\ hv &:= h_v^\infty := \{ f : \Omega \to \mathbb{C} \text{ harmonic } \mid \|f\|_v < \infty \} \\ Hv &:= H_v^\infty := \{ f \in hv \mid f \text{ holomorphic} \} \end{aligned}$$

We also denote, for functions f on $\mathbb{D}$ or $\mathbb{C}$,

$$M_\infty(f,r) := \sup_{|z|=r} |f(z)| \tag{5}$$

We shall need some notations and results from [6]. In [6], the first named author showed the boundedness of the Riesz projection $R : \sum_{k=-\infty}^{\infty} f_k e_k \mapsto \sum_{k\geq 0} f_k z^k$ from hv onto Hv provided the following condition (B) holds for the weight v:

Definition 1. The weight v satisfies the condition (B), if

$$\forall\, b_1 > 1\ \exists\, b_2 > 1\ \exists\, c > 0\ \forall\, m,n > 0$$
$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \ \text{ and } \ m,n,|m-n| \geq c \ \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \leq b_2$$

Note that m and n need not be integers. Here the number $r_n > 0$ denotes the global maximum point of the function $r \mapsto r^n v(r)$. It is easy to see that $r_n < r_m$ for $n < m$ and that r_n tends to the radius of the domain, as $n \to \infty$.

From now on we fix a weight v having the property (B).

We soon define (in Theorem 1 below) an increasing sequence $(m_n)_{n=1}^\infty$ of real numbers. Given such a sequence, we shall define the operators T_n acting on harmonic functions $f(z) := \sum_{k\in\mathbb{Z}} f_k e_k(z)$ by

$$\begin{aligned} T_n f \quad &:= \sum_{m_{n-1}<|k|\leq m_n} \frac{|k| - [m_{n-1}]}{[m_n] - [m_{n-1}]} f_k e_k \\ &\quad + \sum_{m_n<|k|\leq m_{n+1}} \frac{[m_{n+1}] - |k|}{[m_{n+1}] - [m_n]} f_k e_k \\ &=: \sum_{k\in\mathbb{Z}} t_{nk} f_k e_k \end{aligned} \tag{6}$$

Here $[a]$ denotes the largest integer $\leq a$. So the operator T_n is a multiplier on the space of Fourier–coefficients. Given a sequence $(m_n)_{n=1}^\infty$, we also denote

$$s_n := r_{m_n}. \tag{7}$$

We collect in the following theorem everything that is needed from [6]. Recall that v is assumed to satisfy (B).

Theorem 1. *There are numbers* $0 < m_1 < m_2 < \dots$ *and constants* $d_1, d_2 > 0$ *such that for any* $f \in hv$ *we have*

$$(8)\quad d_1 \sup_n M_\infty(T_n f, s_n) v(s_n) \le \|f\|_v \le d_2 \sup_n M_\infty(T_n f, s_n) v(s_n)$$

and

$$(9)\quad d_1 M_\infty(T_n f, s_n) v(s_n) \le \|T_n f\|_v \le d_2 \Big(\sup_k \|T_k\|\Big) M_\infty(T_n f, s_n) v(s_n)$$

for all n.

The operators T_n *are uniformly bounded with respect to* $M_\infty(\cdot, 1)$.

Finally, the Riesz projection $R : hv \to Hv$ *is bounded with respect to* $\|\cdot\|_v$.

Remark 1. (i) Our operators T_n are equal to $V_{m_{n+1},m_n} - V_{m_n,m_{n-1}}$ of [6], see (3.1) in the reference. They have the properties $f = \sum_n T_n f$ for every trigonometric polynomial f, and

$$(10)\qquad T_n T_m = 0$$

for $|n - m| \ge 2$.

(ii) For any series $\sum_{k=0}^\infty f_k e_k$, $f_k \in \mathbb{C}$, for any n, only finitely many summands of $\sum_{k=0}^\infty T_n f_k e_k$ are nonzero. (Hence there are absolutely no convergence problems with the latter series.)

(iii) Theorem 1 implies that $\sum_{k=0}^\infty f_k e_k$ is the Taylor series (converging at least uniformly on compacta of $\mathbb{D}$) of an analytic function $f \in Hv$, if the coefficients f_k are such that

$$(11)\quad \sup_n M_\infty\Big(\sum_{k=0}^\infty T_n f_k e_k, s_n\Big) v(s_n) = \sup_n M_\infty\Big(\sum_{k=0}^\infty t_{nk} f_k e_k, s_n\Big) v(s_n) < \infty.$$

Proof of Theorem 1. We obtain the numbers m_n from Lemma 5.1 of [6]. In particular, one of the quotients

$$\left(\frac{s_n}{s_{n+1}}\right)^{m_n} \frac{v(s_n)}{v(s_{n+1})} \quad \text{or} \quad \left(\frac{s_{n+1}}{s_n}\right)^{m_{n+1}} \frac{v(s_{n+1})}{v(s_n)}$$

is equal to the constant $b > 0$ of the reference. Condition (B) yields a constant $d > b$ (independent of n) such that both quotients are numbers in the interval $[b, d]$. Now, Proposition 5.2. of [6] yields constants $c_1, c_2 > 0$

such that

$$\begin{aligned} & c_1 \sup_n \sup_{s_{n-1}\le r\le s_{n+1}} M_\infty(T_n f, r)v(r) \\ (12) \qquad & \le \|f\|_v \le c_2 \sup_n \sup_{s_{n-1}\le r\le s_{n+1}} M_\infty(T_n f, r)v(r). \end{aligned}$$

This already implies the first inequality of (8). Using [6], Proposition 4.1, we see that either $m_{n+1} - m_{n-1} \le c$ for all n (c as in condition (B)) or there are η, $\kappa > 0$, independent of n, such that

$$\eta \le \frac{m_{n+1} - m_n}{m_n - m_{n-1}} \le \kappa.$$

Lemma 3.3.(c) of [6] then tells us that, in any case, the operators T_n are uniformly bounded with respect to $M_\infty(\cdot, 1)$. (If $m_{n+1} - m_{n-1} \le c$, then rank $T_n \le c$.)

Using Lemma 3.1. of [6] we see that here

$$\sup_{s_{n-1}\le r\le s_{n+1}} M_\infty(T_n f, r)v(r) \le 2bdM_\infty(T_n f, s_n)v(s_n).$$

Hence, the second inequality of (8) follows.

The first inequality in (9) is a triviality. Concerning the second, we use (10) and (12) to obtain for a fixed n

$$\begin{aligned} \|T_n f\|_v &\le 2bd^2c_2(M_\infty(T_{n-1}T_n f, s_{n-1})v(s_{n-1}), \\ &\qquad M_\infty(T_n^2 f, s_n)v(s_n), M_\infty(T_{n+1}T_n f, s_{n+1})v(s_{n+1})) \\ &\le 2bd^2c_2(\sup_k \|T_k\|) \sup(M_\infty(T_n f, s_{n-1})v(s_{n-1}), \\ &\qquad M_\infty(T_n f, s_n)v(s_n), M_\infty(T_n f, s_{n+1})v(s_{n+1})) \\ &\le 4b^2d^4c_2(\sup_k \|T_k\|)M_\infty(T_n f, s_n)v(s_n) \end{aligned}$$

In the last inequality we used Lemma 3.1 of [6] again.

The boundedness of the Riesz projection is proven in Proposition 6.8 of [6]. □

3. Main result with examples

Assume that a weight v on $\Omega = \mathbb{D}$ or $\mathbb{C}$, satisfying (B), is given; let the sequences $(m_n)_{n=1}^\infty$ and $(s_n)_{n=1}^\infty$ and also the multiplier sequences $(t_{nk})_{k=1}^\infty$ be as above.

Given a continuous function f on Ω, let $f_k(r)$ be the kth Fourier coefficient of $f|_{r\partial\mathbb{D}}$, i.e.

$$f_k(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})r^{-|k|}e^{-ik\varphi}d\varphi. \tag{13}$$

Hence, for $z \in \mathbb{D}$ and $r = |z|$, we have

$$f(z) = \sum_k f_k(r)e_k(z) \tag{14}$$

at least in the L^2–sense on every $r\partial\mathbb{D}$. We define for all n the operators

$$W_n f = W_n \sum_k f_k e_k := \sum_k t_{nk} f_k(s_n) e_k \tag{15}$$

Notice that the sum on the right hand side has only finitely many terms. We define the projection

$$P_C f := \sum_{n=1}^{\infty} RW_n f = \sum_{k\geq 0}\sum_{n=1}^{\infty} t_{nk} f_k(s_n) e_k, \tag{16}$$

provided the series in (16) converges at least uniformly on compact subsets of $\mathbb{D}$. (It is then the Taylor series of an analytic function.) This is true, if we can show that $\|P_C f\|_v < \infty$, see Theorem 2 below.

To define a corresponding projection P_M on the larger space of bounded functions we choose for every n the interval $I_n := [s_n, s_n + \epsilon_n]$ with $I_n \subset]0,1[$, if $\Omega = \mathbb{D}$, and $I_n \subset]0,\infty[$, if $\Omega = \mathbb{C}$. Here ϵ_n is fixed so small that $v(s_n) \geq v(r) \geq v(s_n)/2$ for all $r \in I_n$ (and that $s_n + \epsilon_n < s_{n+1}$). We define

$$Z_n f := Z_n \sum_k f_k e_k := \sum_k \frac{t_{nk}}{\epsilon_n}\left(\int_{I_n} f_k(s)ds\right)e_k, \tag{17}$$

and, as in (16),

$$P_M f := \sum_{n=1}^{\infty} RZ_n f = \sum_{k\geq 0}\sum_{n=1}^{\infty} \frac{t_{nk}}{\epsilon_n}\left(\int_{I_n} f_k(s)ds\right)e_k, \tag{18}$$

provided the series converges at least uniformly on compact subsets of $\mathbb{D}$.

Theorem 2. *Let v satisfy (B). Then P_M is a bounded projection $L_v^\infty \to Hv$, and P_C is a bounded projection from $L_v^\infty \cap C(\Omega)$ onto Hv.*

Here $C(\Omega)$ is the space of all continuous functions $f : \Omega \to \mathbb{C}$.

Remark 2. Consider the case $\Omega = \mathbb{D}$. Then the condition (B) is necessary and sufficient for Theorem 2. Indeed, if (B) does not hold, then Hv is isomorphic to H^∞, by [6]. In this case Hv cannot be complemented in L^∞_v and hence, in particular, P_M cannot be bounded.

Proof 1°. We start by showing that P_M and P_C are bounded. Concerning P_M, we want to prove first

$$M_\infty(Z_n f, s_n)v(s_n) \le C\|T_n f\|_v \tag{19}$$

for $f \in L^\infty_v$. For $f(z) = \sum_k f_k(r)e_k(z)$ we have

$$M_\infty(Z_n f, s_n)v(s_n) = \operatorname{ess}\sup_{\varphi\in[0,2\pi]} \left|\sum_{k\in\mathbb{Z}} \frac{t_{nk}}{\epsilon_n}\left(\int_{I_n} f_k(s)ds\right)s_n^{|k|}e^{ik\varphi}\right| v(s_n)$$

$$\begin{aligned} &\le \operatorname{ess}\sup_{\varphi\in[0,2\pi]} \left|\frac{1}{\epsilon_n}\int_{I_n}\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s_n^{|k|}e^{ik\varphi}ds\right| v(s_n) \\ &\le \operatorname{ess}\sup_{\varphi\in[0,2\pi]}\sup_{s\in I_n} \left|\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s_n^{|k|}e^{ik\varphi}\right| v(s_n) \\ &= \sup_{s\in I_n}\sup_{\varphi\in[0,2\pi]} \left|\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s_n^{|k|}e^{ik\varphi}\right| v(s_n). \end{aligned} \tag{20}$$

We obtain

$$\sup_{\varphi\in[0,2\pi]} \left|\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s_n^{|k|}e^{ik\varphi}\right| \le \sup_{\varphi\in[0,2\pi]} \left|\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s^{|k|}e^{ik\varphi}\right|$$

for every $s \in I_n$. (To see this, we have $r_k \le s$ for these s. We apply the maximum modulus principle to the harmonic function $h(re^{i\varphi}) = \sum_{k\in\mathbb{Z}} t_{nk}f_k(s)r^{|k|}e^{ik\varphi}$ to get $\sup_\varphi |h(s_k e^{i\varphi})| \le \sup_\varphi |h(se^{i\varphi})|$.)

By the choice of the interval I_n, $v(s_n) \le 2v(s)$ for $s \in I_n$. Using these we obtain (19): we bound (20) by

$$2\sup_{s\in I_n}\sup_{\varphi\in[0,2\pi]} \left|\sum_{k\in\mathbb{Z}} t_{nk}f_k(s)s^{|k|}e^{ik\varphi}\right| v(s) \le 2\|T_n f\|_v. \tag{21}$$

Let $f \in L^\infty_v$ be arbitrary. Notice that $Z_n f$ is always of the form $T_n g$ for some $g \in hv$: we can write the right hand side of (17) as

$$T_n \sum_{|k|\le N_n} \frac{1}{\epsilon_n}\int_{I_n} f_k(s)ds e_k$$

for an $N_n \in \mathbb{N}$ large enough. We thus can use (10) to obtain

$$\begin{aligned}
\|P_M f\|_v &\leq \|R\| \cdot \|\sum_n Z_n f\|_v \\
&\leq d_2 \|R\| \sup_n M_\infty(T_n Z_{n-1} f + T_n Z_n f + T_n Z_{n+1} f, s_n) v(s_n) \\
&\leq d_2 \|R\| (\sup_k \|T_k\|) \sup_n \Big(M_\infty(Z_{n-1} f, s_n) v(s_n) + \\
&\qquad M_\infty(Z_n f, s_n) v(s_n) + M_\infty(Z_{n+1} f, s_n) v(s_n) \Big).
\end{aligned}$$

This is bounded by a v–dependent constant times $\sup_n \|Z_n f\|_v$. Since $Z_n f = T_n g$ for some $g \in hv$, the second inequality of Theorem 1, (9), applies and yields, together with (19), the bound

$$C \sup_n M_\infty(Z_n f, s_n) v(s_n) \quad \leq \quad C' \sup_n \|T_n f\|_v \leq C' (\sup_n \|T_n\|) \|f\|_v.$$

By Remark 1, (iii), the right hand side of (18) is the Taylor series of an analytic function and hence $P_M f$ is well–defined.

The proof for P_C is easier, since the analogue of (19) is quite trivial:

$$M_\infty(W_n f, s_n) v(s_n) = \sup_{|z|=s_n} \Big| \sum_{k \in \mathbb{Z}} t_{nk} f_k(z) e_k(z) \Big| v(z) \leq \|T_n f\|_v. \tag{22}$$

The rest of the proof goes as that for P_M.

2°. We show that P_M and P_C are projections. To this end fix $f \in L_v^\infty$ and fix n. Put $h = R Z_n f$. Then we have $h = \sum_{k=[m_{n-1}]}^{[m_{n+1}]} h_k e_k$ for certain constants h_k and hence $Z_n h = T_n h$ for any n. We obtain

$$\begin{aligned}
P_M h &= \sum_{k=1}^{\infty} R T_k h \\
&= T_{n-1} h + T_n h + T_{n+1} h \\
&= (V_{m_{n+2}, m_{n+1}} - V_{m_{n-1}, m_{n-2}}) h = h
\end{aligned}$$

This proves that P_M is a projection. The proof for P_C is the same, replacing Z_n by W_n. □

Example 1. If $\Omega := \mathbb{D}$ and $v(z) = \exp(-1/(1-r))$, then

$$r_m := 1 - \frac{2}{1 + \sqrt{4m+1}} \cong 1 - \frac{1}{\sqrt{m}}, \tag{23}$$

and the sequence $(m_n)_{n=1}^{\infty}$ is defined by $m_{n+1} := m_n + \mathcal{O}(m_n^{3/4})$ for all n. We refer to Example 2.3 of [6].

Example 2. If $\Omega := \mathbb{C}$ and $v(z) = \exp(-e^r)$, then the number r_m is the solution of the equation $xe^x = m$, i.e. $x + \log x = \log m$, hence, r_m equals $\log m+$ very small corrections. The sequence $(m_n)_{n=1}^{\infty}$ is obtained from $m_{n+1} := m_n + \mathcal{O}(\sqrt{m_n}\log m_n)$. For details, see [6], Example 2.1.

Example 3. If $\Omega := \mathbb{C}$ and $v(z) = \exp(-(\log r)^{\rho})$, $|z| \geq 1$, $\rho \geq 2$ fixed, and $v(z) = 1$ for $|z| < 1$ then

$$r_m := \exp((n/\rho)^{1/(\rho-1)}) \tag{24}$$

and the sequence $(m_n)_{n=1}^{\infty}$ is defined by $m_{n+1} := m_n + \beta m_n^{(\rho-2)/(\rho-1)}$ for a positive constant β. See [6], Example 2.2.

We finish our paper by the remark that in the case $\Omega = \mathbb{D}$ the condition (B) can be described in a somewhat easier way. We have

Proposition 1. *Let $\Omega = \mathbb{D}$. Then the following are equivalent:*

(i) *Condition (B)*

(ii) $\forall\ b_1 > 1\ \exists\ b_2 > 1\ \forall\ m, n \geq 1$

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \leq b_1 \Rightarrow \left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \leq b_2$$

Proof. We only have to show that condition (B) implies (ii). To this end let $b_1 > 1$ and $m, n \geq 1$ be such that

$$\left(\frac{r_m}{r_n}\right)^m \frac{v(r_m)}{v(r_n)} \leq b_1.$$

We show that there exists b_2, independent of m and n, such that $(r_n/r_m)^{n-m} \leq b_2$. This implies (ii). Indeed, then we obtain

$$\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} = \left(\frac{r_n}{r_m}\right)^{n-m} \frac{r_n^m v(r_n)}{r_m^m v(r_m)} \leq \left(\frac{r_n}{r_m}\right)^{n-m} \leq b_2.$$

Let $c = c(b_1)$ be the constant of condition (B). Recall, that here we have $r_k < 1$ for all k. We consider several cases.

If $m \geq c$, $n \geq c$ and $|m - n| \geq c$ then (B) implies the existence of a constant b_3, independent of m and n, with

$$\left(\frac{r_n}{r_m}\right)^n \frac{v(r_n)}{v(r_m)} \leq b_3$$

and hence $(r_n/r_m)^{n-m} \le b_1 b_3$.

If $|m-n| < c$ then $(r_n/r_m)^{n-m} \le r_1^{-c}$ where the right-hand side is independent of m and n.

Fix N such that $r_{c+N} > r_c$. If $m \le c+N$ and $n \le c+N$ then similarly $(r_n/r_m)^{n-m} \le r_1^{-2c-2N}$.

Now consider the case $n < c < c+N \le m$. We have

$$\begin{aligned}\left(\frac{r_{c+N}}{r_c}\right)^{m-c-N} &\le \left(\frac{r_{c+N}}{r_n}\right)^{m-c-N} \frac{r_{c+N}^{c+N} v(r_{c+N})}{r_n^{c+N} v(r_n)} \\ &\le \left(\frac{r_{c+N}}{r_n}\right)^{m-c-N} \left(\frac{r_{c+N}}{r_n}\right)^{c+N-m} \left(\frac{r_m}{r_n}\right)^{m} \frac{v(r_m)}{v(r_n)} \\ &\le b_1.\end{aligned}$$

Since $\lim_{k\to\infty} (r_{c+N}/r_c)^{k-c-N} = \infty$ we find k_0, independent of m and n, such that $\left(\frac{r_{c+N}}{r_c}\right)^{k-c-N} > b_1$ for all $k \ge k_0$. Hence $m \le k_0$ and we obtain

$$\left(\frac{r_n}{r_m}\right)^{n-m} = \left(\frac{r_m}{r_n}\right)^{m-n} \le \left(\frac{1}{r_1}\right)^{k_0}.$$

Finally, assume $m < c < c+N \le n$. Since $\lim_{l\to\infty} v(r_l) = 0$ we find l_0 such that $r_l^j v(r_l) \le v(r_l) < r_c^c v(r_c)/b_1$ for all $j > 0$ and all $l \ge l_0$. Since

$$\frac{r_c^c v(r_c)}{r_n^m v(r_m)} \le r_c^{c-m} \frac{r_m^m v(r_m)}{r_n^m v(r_n)} \le b_1$$

we have $r_c^c v(r_c)/b_1 \le r_n^m v(r_n)$. Hence $n \le l_0$ and we obtain $(r_n/r_m)^{n-m} \le r_1^{-l_0}$.

Now put $b_2 = \max(b_1 b_3, r_1^{-\max(2c+2N, l_0, k_0)})$. Then we have in any case $(r_n/r_m)^{n-m} \le b_2$. □

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