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Codomains for the Cauchy-Riemann and Laplace operators in $\mathbb{R}^2$

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Abstract. A codomain for a nonzero constant-coefficient linear partial differential operator $P(\partial)$ with fundamental solution E is a space of distributions T for which it is possible to define the convolution $E * T$ and thus solving the equation $P(\partial)S = T$. We identify codomains for the Cauchy-Riemann operator in $\mathbb{R}^2$ and Laplace operator in $\mathbb{R}^2$. The convolution is understood in the sense of the $\mathcal{S}'$-convolution.

1. Introduction

Let $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$. Consider a nonzero polynomial

$$P(x) = \sum_{|\alpha| \leq m} a_\alpha x^\alpha,$$

where $a_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. Let $\partial = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$. Then $P(\partial)$ is called a nonzero constant-coefficient linear partial differential operator and therefore by the Malgrange-Ehrenpreis theorem (see B. Malgrange [11], L. Ehrenpreis [6]) there exists a distribution E such that $P(\partial)E = \delta$, where δ is the Dirac distribution concentrated at the origin in $\mathbb{R}^n$. Such a distribution E is called a fundamental (or elementary) solution of the partial differential operator $P(\partial)$ (see, for example, M.A. Al-Gwaiz [1], p. 106). It can be shown that a fundamental solution E of the Laplace operator $\Delta = \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ in $\mathbb{R}^n$ is given by

$$E(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|\,, & \text{for } \ n = 2 \\[2ex] \dfrac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}(2-n)} |x|^{2-n}\,, & \text{for } \ n \neq 2 \end{cases}$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$, $\operatorname{Re} x > 0$ is the Gamma function (see, for example, M.A. Al-Gwaiz [1], p. 151). It can also be shown that a fundamental solution E of the Cauchy-Riemann operator $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$ in $\mathbb{R}^2$ is given by

$$E(x_1, x_2) = \frac{1}{\pi(x_1 + ix_2)}$$

(see M.A. Al-Gwaiz [1], pp. 142–143). These fundamental solutions are tempered distributions. In [8], L. Hörmander proved that a tempered fundamental solution of a nonzero constant-coefficient linear partial differential operator always exists.

Suppose we wish to solve

$$P(\partial)S = T, \tag{1}$$

where $P(\partial)$ is a nonzero constant-coefficient linear partial differential operator, T is a known distribution and S is an unknown distribution. Let E denote a fundamental solution of the operator $P(\partial)$. If T has compact support, then $S := E * T$ solves (1), where the convolution is interpreted in the classical sense as defined by L. Schwartz [13]. The convolution operator $T \to E * T$ becomes a two-sided inverse for the differential operator $P(\partial)$ in the space $\mathcal{E}'$ of distributions with compact support. That $S := E * T$ solves (1) is good news. Unfortunately, the condition that T must have compact support is very restrictive. There is

need for more general solvability conditions for (1). Given a specific nonzero constant-coefficient linear partial differential operator $P(\partial)$, we can weaken the condition that T has compact support. We achieve this by employing a certain type of convolution called $\mathcal{S}'$-convolution.

A codomain for the operator $P(\partial)$ is a space of distributions T for which it is possible to define the convolution $E * T$ and thus solving the equation $P(\partial)S = T$. It is important to point out that finding a codomain depends on having an explicit formula for a fundamental solution E of the operator $P(\partial)$ and on how we define the convolution of E with distributions T that may not have compact support.

There are different types of convolutions in the literature, each designed to achieve a specific purpose. It is expected that any genuine convolution operation should be commutative, should commute with the operation of differentiation and satisfy, in some sense, the Fourier exchange formula: $\mathcal{F}(S*T) = \mathcal{F}(S)\mathcal{F}(T)$, where S and T are tempered distributions. We will be working with a certain type of convolution called $\mathcal{S}'$-convolution defined on $\mathcal{S}'$, the space of tempered distributions. If S and T are tempered distributions, we have no reason to expect that $S * T$ exists in the classical sense or if it does exist, we cannot expect it to be a tempered distribution and as such we cannot expect the Fourier exchange formula to be valid. To ensure that the classical convolution is a tempered distribution, L. Schwartz [13] introduced the space $\mathcal{O}'_C$ of rapidly decreasing distributions. He proved that any tempered distribution is convolvable with any distribution in $\mathcal{O}'_C$. Moreover, he proved that the convolution satisfies the Fourier exchange formula. The definition of the $\mathcal{S}'$-convolution was developed by Y. Hirata and H. Ogata [7] with the purpose of extending the validity of the Fourier exchange formula $\mathcal{F}(S*T) = \mathcal{F}(S)\mathcal{F}(T)$, which was originally proved by L. Schwartz ([13], p. 268) for pairs of distributions in the Cartesian product $\mathcal{O}'_C \times \mathcal{S}'$. Y. Hirata and H. Ogata showed that the $\mathcal{S}'$-convolution satisfies the three conditions given above that are expected of a genuine convolution. Later R. Shiraishi [15] introduced an equivalent definition which is the definition we are going to use.

The goal of our work is to identify codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$ and Laplace operator in $\mathbb{R}^2$. Once a codomain for the operator $P(\partial)$ is obtained, it leads immediately to existence results for the equation $P(\partial)S = T$ when the distribution T belongs to the appropriate codomain. Moreover, a solution S is given explicitly by the formula $S = E * T$.

In [14], L. Schwartz found a class of weighted distributions whose elements are $\mathcal{S}'$-convolvable with the distribution $pv\left(\frac{1}{x}\right)$. A weighted distribution space was also used by J. Horváth [9] and N. Ortner [12] to identify those tempered distributions that are $\mathcal{S}'$-convolvable with the n-dimensional

hypersingular kernels of the form $k\left(\frac{x}{|x|}\right)|x|^{-\lambda}$, where $\lambda \in \mathbb{C}$ and k is a bounded function on the unit sphere. J. Alvarez and L.E.S. Moyo [3] used weighted integrable distribution spaces to identify the maximal codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Laplace operator $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ associated with the Euclidean space $\mathbb{R}^n$, $n \neq 2$ and the product Laplace operator $\Delta_x \ \Delta_y := \left(\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}\right)\left(\sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2}\right)$ associated with the Cartesian product space $\mathbb{R}^m \times \mathbb{R}^n$, $m \neq 2$, $n \neq 2$.

Our work is organized in the following way. In Section 2, we collect definitions and results that will be used in the proving of our main results in Sections 3 and 4. In Section 3 and Section 4, we identify codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$ and Laplace operator in $\mathbb{R}^2$, respectively. Finally, in Section 5, we conjecture that it is impossible to identify maximal codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$ and Laplace operator in $\mathbb{R}^2$.

The notation that will be used is standard. The symbols C^∞, L^p, L^p_{loc}, $\mathcal{D}$, $\mathcal{S}$, $\mathcal{O}_M$, $\mathcal{D}'$, $\mathcal{S}'$, $\mathcal{E}'$, $\mathcal{O}'_C$, etc, indicate the usual spaces of functions or distributions defined on $\mathbb{R}^n$ with values in $\mathbb{C}$. The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$, whereas $||\cdot||_p$ denotes the L^p-norm. The conjugate exponent of p will be denoted by p'. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and a point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we will use the following notations: $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j}$, $\partial_{x_j} = \frac{\partial}{\partial x_j}$, $\partial = (\partial_{x_1}, \ldots, \partial_{x_n})$, $\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $\overline{\partial} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ is the Cauchy-Riemann operator in $\mathbb{R}^2$, $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the Laplace operator in $\mathbb{R}^n$, $\alpha! = \alpha_1! \ldots \alpha_n!$, and $\binom{\alpha}{\beta} = \prod_{j=1}^{n} \frac{\alpha_j!}{\beta_j!(\alpha_j - \beta_j)!}$, where $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}_0^n$, with $\beta_j \leq \alpha_j$, for $j = 1, \ldots, n$. Given a function g, the function $x \to g(-x)$ will be denoted by $\overset{\vee}{g}$. Given a distribution T, the distribution $\phi \to \langle T, \overset{\vee}{\phi} \rangle$, where ϕ is an appropriate test function ϕ, will be denoted by $\overset{\vee}{T}$. The Fourier transform of a function g will be denoted by $\mathcal{F}(g)$ or $\widehat{g}$. The letter C will indicate a positive constant which may be different at different instances. Subindices appended to C will indicate quantities on which the constant C depends. We will introduce other notation at the appropriate time.

2. Preliminary definitions and results

In this section, we collect definitions and results that will be used in the proving of our main results in Section 3 and Section 4. We begin with the following lemmas.

Lemma 1 ([16], p. 38)**.** *Let* $w(x) = \sqrt{1+|x|^2}$, $x \in \mathbb{R}^n$. *Let* $m \in \mathbb{R}$. *For each* $\alpha \in \mathbb{N}_0^n$, *we have*

$$|\partial^\alpha w^m(x)| \leq C_{m,\alpha} w^{m-|\alpha|}(x).$$

Proof. See M.W. Wong [16], pp. 38–39. □

Lemma 2 (Peetre's inequality, [4], p. 163)**.** *Let* $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}$. *Then*

$$\left(\frac{1+|x|^2}{1+|y|^2}\right)^s \leq 2^{|s|}(1+|x-y|^2)^{|s|}.$$

Proof. See J. Barros-Neto [4], p. 163. □

Next, we define the space $\mathcal{D}'_{L^p}$, $1 \leq p \leq \infty$, as introduced by L. Schwartz in [13] and the weighted space $w^\lambda \mathcal{D}'_{L^1}$. Before we define these spaces, we need the following definitions.

Definition 3. $\mathcal{D}_{L^p} := \{g \in C^\infty : \forall \alpha \in \mathbb{N}_0^n, \partial^\alpha g \in L^p(\mathbb{R}^n), 1 \leq p < \infty\}$,
$\mathcal{D}_{L^\infty} = \mathcal{B} := \{g \in C^\infty$: *for each* $\alpha \in \mathbb{N}_0^n, \partial^\alpha g$ *is bounded*$\}$,
$\overset{\bullet}{\mathcal{B}} := \{g \in C^\infty$: *for each* $\alpha \in \mathbb{N}_0^n, \partial^\alpha g(x)$ *tends to zero as* $|x| \to \infty\}$.

The space $\overset{\bullet}{\mathcal{B}}$ is a closed subspace of $\mathcal{B}$. We equip $\mathcal{D}_{L^p}$ with the topology generated by the countable family of norms

$$||\phi||_{m,p} := \sup_{|\alpha| \leq m} ||\partial^\alpha \phi||_{L^p}, \quad m \in \mathbb{N},\ 1 \leq p \leq \infty.$$

A sequence $\{\phi_k\}$ in $\mathcal{D}_{L^p}$ converges to 0 in $\mathcal{D}_{L^p}$ if $||\phi_k||_{m,p} \to 0$ as $k \to \infty$. The space $\mathcal{B}_c$ is the space $\mathcal{B}$ endowed with β-convergence. Following W. Kierat and U. Sztaba [10], pp. 75–76, a sequence $\{\phi_k\}$, $\phi_k \in \mathcal{B}$ is said to be β-convergent to zero in $\mathcal{B}$ if it satisfies the following two conditions:

(1) For each $\alpha \in \mathbb{N}_0^n$, there exists a positive real number C_α such that $||\partial^\alpha \phi_k||_{L^\infty} \leq C_\alpha$.
(2) For each $\alpha \in \mathbb{N}_0^n$, the sequence $\{\partial^\alpha \phi_k\}$ converges uniformly to zero on each compact set $K \subset \mathbb{R}^n$.

Now, let us define the space $\mathcal{D}'_{L^p}$. For $1 < p \leq \infty$, put $p' = \frac{p}{p-1}$. Then $\mathcal{D}'_{L^p}$ is the topological dual of $\mathcal{D}_{L^{p'}}$. The space $\mathcal{D}'_{L^1}$ is the topological

dual of both $\overset{\bullet}{\mathcal{B}}$ and $\mathcal{B}_c$. The space $\mathcal{D}$ is dense in $\overset{\bullet}{\mathcal{B}}$ and in $\mathcal{B}_c$ (see, for example, W. Kierat and U. Sztaba [10], pp. 75-76) and in $\mathcal{D}_{L^p}$, for $p \in [1, \infty)$. We state a characterization theorem, due to L. Schwartz [13], for the distribution space $\mathcal{D}'_{L^p}$.

Theorem 4 ([4], p. 173)**.** *Given $T \in \mathcal{D}'$ and $p \in [1, \infty]$ fixed, the following statements are equivalent:*

(1) $T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha$, *where* $f_\alpha \in L^p$, $m \in \mathbb{N}_0$.

(2) $T \in \mathcal{D}'_{L^p}$.

Proof. See J. Barros-Neto [4], pp. 173–175. □

As a consequence of Theorem 4, the space $\mathcal{D}'_{L^p}$ is contained in the space $\mathcal{S}'$.

Our next theorem says that the space $\mathcal{D}'_{L^p}$ is closed with respect to multiplication by functions in $\mathcal{B}$. The theorem will play a pivotal role in the proofs of the main results in Sections 3 and 4.

Theorem 5 ([13], p. 203)**.** *For $p \in [1, \infty]$ fixed, the space $\mathcal{D}'_{L^p}$ is closed with respect to multiplication by functions in $\mathcal{B}$.*

Proof. Let $T \in \mathcal{D}'_{L^p}$ and $g \in \mathcal{B}$. Since T belongs to $\mathcal{D}'_{L^p}$, by Theorem 4, we have

$$T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha,$$

where each f_α belongs to L^p, $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^n$. Therefore

$$gT = \sum_{|\alpha| \leq m} g \partial^\alpha f_\alpha. \tag{2}$$

We wish to show that gT belongs to $\mathcal{D}'_{L^p}$. In other words, by Theorem 4, we wish to show that $gT = \sum_{|\beta| \leq q} \partial^\beta h_\beta$, where each h_β belongs to L^p, $q \in \mathbb{N}_0$, and $\beta \in \mathbb{N}_0^n$. Now, for all ϕ in $\mathcal{D}$, we have

$$\langle g \partial^\alpha f_\alpha, \phi \rangle = \langle \partial^\alpha f_\alpha, g\phi \rangle = (-1)^{|\alpha|} \langle f_\alpha, \partial^\alpha (g\phi) \rangle .$$

By Leibniz's formula, we have

$$\begin{aligned} (-1)^{|\alpha|} \langle f_\alpha, \partial^\alpha (g\phi) \rangle &= (-1)^{|\alpha|} \left\langle f_\alpha, \sum_{0 \leq \alpha' \leq \alpha} C_{\alpha,\alpha'} \left(\partial^{\alpha'} g \right) \partial^{\alpha - \alpha'} \phi \right\rangle \\ &= \sum_{0 \leq \alpha' \leq \alpha} C_{\alpha,\alpha'} \langle f_\alpha, (\partial^{\alpha'} g) \partial^{\alpha - \alpha'} \phi \rangle \end{aligned}$$

$$= \sum_{0\le\alpha'\le\alpha} C_{\alpha,\alpha'}\langle f_\alpha \partial^{\alpha'} g, \partial^{\alpha-\alpha'}\phi\rangle$$
$$= \sum_{0\le\alpha'\le\alpha} C_{\alpha,\alpha'}\langle \partial^{\alpha-\alpha'}(f_\alpha \partial^{\alpha'} g), \phi\rangle.$$

Hence, in the sense of distributions, we have

$$g\partial^\alpha f_\alpha = \sum_{0\le\alpha'\le\alpha} C_{\alpha,\alpha'}\partial^{\alpha-\alpha'}(f_\alpha \partial^{\alpha'} g).$$

It remains to show that $f_\alpha \partial^{\alpha'} g$ belongs to L^p. We consider two cases.

Case 1. $1 \le p < \infty$.
Since g belongs to $\mathcal{B}$, given any γ in $\mathbb{N}_0^n$, there exists a positive constant C_γ such that $|\partial^\gamma g(x)| \le C_\gamma$. Now,

$$\int_{\mathbb{R}^n} |f_\alpha \partial^{\alpha'} g|^p dx = \int_{\mathbb{R}^n} |\partial^{\alpha'} g|^p |f_\alpha|^p dx$$
$$\le C_{\alpha'}^p \int_{\mathbb{R}^n} |f_\alpha|^p dx < \infty.$$

This shows that $f_\alpha \partial^{\alpha'} g$ belongs to L^p.

Case 2. $p = \infty$.
Since g belongs to $\mathcal{B}$, given any γ in $\mathbb{N}_0^n$, there exists a positive constant C_γ such that

$$|\partial^\gamma g(x)| \le C_\gamma. \tag{3}$$

Also, since each f_α belongs to L^∞, there exists a positive constant C_α such that

$$|f_\alpha(x)| \le C_\alpha, \tag{4}$$

for almost all x in $\mathbb{R}^n$. Now, by using (3) and (4), we have

$$|(f_\alpha \partial^{\alpha'} g)(x)| = |\partial^{\alpha'} g(x)|\ |f_\alpha(x)| \le C_{\alpha'} C_\alpha,$$

for almost all x in $\mathbb{R}^n$. This shows that $f_\alpha \partial^{\alpha'} g$ belongs to L^∞. Up to this point, we have shown that

$$g\partial^\alpha f_\alpha = \sum_{0\le\alpha'\le\alpha} C_{\alpha,\alpha'}\partial^{\alpha-\alpha'}(f_\alpha \partial^{\alpha'} g),$$

where $f_\alpha \partial^{\alpha'}(g)$ belongs to L^p. Hence, by Theorem 4, we conclude that $g\partial^\alpha f_\alpha$ belongs to $\mathcal{D}'_{L^p}$. Finally, by (2), we conclude that gT belongs to $\mathcal{D}'_{L^p}$. □

Next, we define the space $w^\lambda \mathcal{D}'_{L^1}$.

Definition 6 ([2], p. 153)**.** *For $n \geq 1$, set $w(x) = \sqrt{1+|x|^2}$, $x \in \mathbb{R}^n$. Then for fixed $\lambda \in \mathbb{R}$, we define*

$$w^\lambda \mathcal{D}'_{L^1} := \{T \in \mathcal{S}' : w^{-\lambda} T \in \mathcal{D}'_{L^1}\} \tag{5}$$

with the topology on $w^\lambda \mathcal{D}'_{L^1}$ induced by the map

$$\begin{array}{rcl} w^\lambda \mathcal{D}'_{L^1} & \to & \mathcal{D}'_{L^1} \\ T & \mapsto & w^{-\lambda} T. \end{array}$$

By examining the definition of $w^\lambda \mathcal{D}'_{L^1}$ in (5), we have $w^{-\lambda} T \in \mathcal{D}'_{L^1}$ so that by invoking Theorem 4, we have $w^{-\lambda} T = \sum_{|\alpha| \leq m} \partial^\alpha f_\alpha$, where f_α belongs to L^1. This gives $T = \sum_{|\alpha| \leq m} w^\lambda \partial^\alpha f_\alpha$. Thus, we may also define $w^\lambda \mathcal{D}'_{L^1}$ as follows:

$$w^\lambda \mathcal{D}'_{L^1} = \left\{ T \in \mathcal{S}' : T = \sum_{|\alpha| \leq m} w^\lambda \partial^\alpha f_\alpha,\ f_\alpha \in L^1 \right\}.$$

We may also view $w^\lambda \mathcal{D}'_{L^1}$ as the topological dual of $w^{-\lambda}\mathcal{B}_c$ and $w^{-\lambda}\dot{\mathcal{B}}$. It is clear from Definition 6 that the space $w^\lambda \mathcal{D}'_{L^1}$ consists of tempered distributions. The space $w^\lambda \mathcal{D}'_{L^1}$ is referred to as the space of weighted integrable distributions. The space $w^\lambda \mathcal{D}'_{L^1}$ will be used in Section 3 to identify a codomain in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$. The same space $w^\lambda \mathcal{D}'_{L^1}$, with a modified weight function, will be used in Section 4 to identify a codomain in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Laplace operator in $\mathbb{R}^2$. For more information on the space $w^\lambda \mathcal{D}'_{L^1}$, consult J. Alvarez and L.E.S. Moyo [3].

Finally, it is time to define $\mathcal{S}'$-convolution, a type of convolution that we will be working with. Here is the original definition of $\mathcal{S}'$-convolution, due to Y. Hirata and H. Ogata [7].

Definition 7 ([7], p. 148)**.** Two distributions $S, T \in \mathcal{S}'$ are $\mathcal{S}'$-convolvable if $(S * \phi)(\check{T} * \psi) \in L^1$, for all $\phi, \psi \in \mathcal{S}$.

In [15], R. Shiraishi proved that Definition 7 is equivalent to the following definition which is the definition that we are going to use.

Definition 8 ([15], p. 26; [5], p. 193)**.** Two distributions $S, T \in \mathcal{S}'$ are $\mathcal{S}'$-convolvable if $(\phi * \check{S})T \in \mathcal{D}'_{L^1}$, for all $\phi \in \mathcal{S}$. In this case, the $\mathcal{S}'$-convolution $S * T$ is defined by

$$\langle S * T, \phi \rangle = \langle (\phi * \check{S})T, 1 \rangle,$$

for $\phi \in \mathcal{S}$.

R. Shiraishi [15] proved that if S and T are $\mathcal{S}'$-convolvable, where S and T belong to $\mathcal{S}'$, then the map

$$\begin{array}{rcl} \mathcal{S} & \to & \mathbb{C} \\ \phi & \to & \langle (\phi * \check{S})T, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} \end{array}$$

defines a tempered distribution, denoted by $S * T$. It turns out that S and T are $\mathcal{S}'$-convolvable if and only if T and S are $\mathcal{S}'$-convolvable. That is,

$$\langle (\phi * \check{T})S, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c} = \langle (\phi * \check{S})T, 1 \rangle_{\mathcal{D}'_{L^1}, \mathcal{B}_c},$$

for all ϕ in $\mathcal{S}$. We also have $\partial^\alpha (S * T) = \partial^\alpha S * T = S * \partial^\alpha T$, for all $\alpha \in \mathbb{N}_0^n$, and $\mathcal{F}(S * T) = \mathcal{F}(S)\mathcal{F}(T)$, as one would expect from a genuine convolution operation (see Y. Hirata and H. Ogata [7], p. 151).

We observe that in Definition 8, the multiplicative product $(\phi * \check{S})T$ is well-defined because the regularization $\phi * \check{S}$ is a C^∞-function with the property that for every $\alpha \in \mathbb{N}_0^n$, $\partial^\alpha (\phi * \check{S})$ is bounded by a polynomial (see L. Schwartz [13], p. 248). In other words, $\phi * \check{S}$ is an element of the space $\mathcal{O}_M$ of functions which, together with all their derivatives, are slowly increasing at infinity.

For additional equivalent definitions of the $\mathcal{S}'$-convolution, see P. Dierolf and J. Voigt [5], p. 193 or R. Shiraishi [15], p. 26.

3. Codomain for the Cauchy-Riemann operator in $\mathbb{R}^2$

In this section, we identify a codomain in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$. We recall that $E(x_1, x_2) = \frac{1}{\pi(x_1 + i x_2)}$ is a fundamental solution of the Cauchy-Riemann operator in $\mathbb{R}^2$.

Theorem 9. *Let* $T \in \mathcal{S}'\left(\mathbb{R}^2\right)$. *Let* $h(x_1, x_2) = \frac{1}{x_1+ix_2}$. *If* $T \in w\mathcal{D}'_{L^1}\left(\mathbb{R}^2\right)$, *then* T *is* $\mathcal{S}'$*-convolvable with* h.

Proof. Assume that T belongs to $w\mathcal{D}'_{L^1}\left(\mathbb{R}^2\right)$. We wish to prove that T is $\mathcal{S}'$-convolvable with h. That is, we wish to show that

$$(\phi * \overset{\vee}{h})T \in \mathcal{D}'_{L^1}(\mathbb{R}^2), \tag{6}$$

for every ϕ in $\mathcal{S}(\mathbb{R}^2)$. But then for any ϕ in $\mathcal{S}(\mathbb{R}^2)$, we have $(\phi * \overset{\vee}{h})T = w^{-1}Tw(\phi * \overset{\vee}{h})$. By the hypothesis, we know that $w^{-1}T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. So by Theorem 5, to prove (6), it suffices to prove that $w(\phi * \overset{\vee}{h})$ is an element of $\mathcal{B}(\mathbb{R}^2)$. First, we prove that $w(\phi * \overset{\vee}{h})$ is bounded in $\mathbb{R}^2$. Now

$$\begin{aligned} |w(\phi * \overset{\vee}{h})|(x) &= \left| \int_{\mathbb{R}^2} \frac{(1+|x|^2)^{\frac{1}{2}}}{y_1+iy_2} \phi(x-y)dy \right| \\ &\leq \int_{\mathbb{R}^2} \frac{(1+|x|^2)^{\frac{1}{2}}}{|y_1+iy_2|} |\phi(x-y)|\, dy. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \frac{(1+|x|^2)^{\frac{1}{2}}}{|y_1+iy_2|} |\phi(x-y)|\, dy \\ &\leq \int_{\mathbb{R}^2} 2^{\frac{1}{2}}(1+|x-y|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy = I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{|y|<1} 2^{\frac{1}{2}}(1+|x-y|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy,$$

and

$$I_2 = \int_{|y|\geq 1} 2^{\frac{1}{2}}(1+|x-y|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy.$$

We will prove that $w(\phi * \overset{\vee}{h})$ is bounded in $\mathbb{R}^2$ by proving that both I_1 and I_2 are bounded in $\mathbb{R}^2$. We first prove that I_1 is bounded in $\mathbb{R}^2$.

$$\begin{aligned} I_1 &= \int_{|y|<1} 2^{\frac{1}{2}}(1+|x-y|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy \\ &\leq C \int_{|y|<1} (1+|x-y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy \\ &= C \int_{|y|<1} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \frac{1}{|y_1+iy_2|} dy, \end{aligned}$$

so that

$$I_1 \leq C \int_{|y|<1} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \frac{1}{|y_1+iy_2|} dy. \tag{7}$$

Since ϕ belongs to $\mathcal{S}\left(\mathbb{R}^2\right)$, we conclude that the function $(1+|x|^2)^{\frac{1}{2}}\phi$ is bounded in $\mathbb{R}^2$. Hence, from (7), we have

$$I_1 \leq C \int_{|y|<1} \frac{1}{|y_1+iy_2|} dy. \tag{8}$$

By switching to polar coordinates, we obtain

$$\begin{aligned} \int_{|y|<1} \frac{1}{|y_1+iy_2|} dy &= \int_0^{2\pi} \int_0^1 \frac{1}{|r\cos\theta + ir\sin\theta|} r dr d\theta \\ &= \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{r^2\left(\cos^2\theta+\sin^2\theta\right)}} dr d\theta = 2\pi. \end{aligned}$$

Therefore, from (8), we have $I_1 \leq C$. This completes the proof that I_1 is bounded in $\mathbb{R}^2$. Next, we prove that I_2 is bounded in $\mathbb{R}^2$.

$$\begin{aligned} I_2 &= \int_{|y|\geq 1} 2^{\frac{1}{2}} (1+|x-y|^2)^{\frac{1}{2}} (1+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy \\ &\leq \int_{|y|\geq 1} 2^{\frac{1}{2}} (1+|x-y|^2)^{\frac{1}{2}} (|y|^2+|y|^2)^{\frac{1}{2}} \frac{|\phi(x-y)|}{|y_1+iy_2|} dy \\ &\leq C \int_{|y|\geq 1} (1+|x-y|^2)^{\frac{1}{2}} |y| \frac{|\phi(x-y)|}{|y_1+iy_2|} dy \\ &= C \int_{|y|\geq 1} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \, dy \\ &\leq C \int_{\mathbb{R}^2} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \, dy, \end{aligned}$$

so that

$$I_2 \leq C \int_{\mathbb{R}^2} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \, dy. \tag{9}$$

Since ϕ belongs to $\mathcal{S}(\mathbb{R}^2)$, we conclude that the function $(1+|x|^2)^{\frac{1}{2}}\phi$ is integrable in $\mathbb{R}^2$. Therefore $\int_{\mathbb{R}^2} (1+|x-y|^2)^{\frac{1}{2}} |\phi(x-y)| \, dy$ is finite. Consequently, from (9), we have $I_2 \leq C$. This shows that I_2 is bounded in $\mathbb{R}^2$. Up to this point, we have proved that I_1 and I_2 are bounded in $\mathbb{R}^2$. Therefore $p(x) := (w(\phi * \check{h}))(x)$ is bounded in $\mathbb{R}^2$. We claim that each

derivative of the $C^\infty(\mathbb{R}^2)$-function p is also bounded in $\mathbb{R}^2$. In fact, for each α in $\mathbb{N}_0^2$, by Leibniz's formula, we have

$$\begin{aligned}\partial^\alpha p &= \sum_{0\le\alpha'\le\alpha} C_{\alpha',\alpha}(\partial^{\alpha'}w)(\partial^{\alpha-\alpha'}\phi * \check{h}) \\ &= \sum_{0\le\alpha'\le\alpha} C_{\alpha',\alpha}w^{-1}(\partial^{\alpha'}w)w(\partial^{\alpha-\alpha'}\phi * \check{h}).\end{aligned}$$

We show that $w^{-1}\partial^{\alpha'}(w)$ and $w(\partial^{\alpha-\alpha'}\phi * \check{h})$ are bounded in $\mathbb{R}^2$. We begin by showing that $w(\partial^{\alpha-\alpha'}\phi * \check{h})$ is bounded in $\mathbb{R}^2$. We know that if ϕ belongs to $\mathcal{S}(\mathbb{R}^2)$, then $\partial^\gamma\phi$ belongs to $\mathcal{S}(\mathbb{R}^2)$, for all γ in $\mathbb{N}_0^2$. Therefore, $\partial^{\alpha-\alpha'}\phi$ belongs to $\mathcal{S}(\mathbb{R}^2)$. But then we have already shown above that $w(\phi * \check{h})$ is bounded in $\mathbb{R}^2$ for every ϕ in $\mathcal{S}(\mathbb{R}^2)$. Hence, $w(\partial^{\alpha-\alpha'}\phi * \check{h})$ is bounded in $\mathbb{R}^2$. Next, we show that $w^{-1}\partial^{\alpha'}w$ is bounded in $\mathbb{R}^2$. By Lemma 1 we have

$$|\partial^{\alpha'}w(x)| \le C_{\alpha'}w^{1-|\alpha'|}(x).$$

Therefore

$$|w^{-1}\partial^{\alpha'}w(x)| \le C_{\alpha'}w^{-|\alpha'|}(x),$$

from which we conclude that $w^{-1}\partial^{\alpha'}w$ is bounded in $\mathbb{R}^2$. Up to this point, we have shown that both $w(\partial^{\alpha-\alpha'}\phi * \check{h})$ and $w^{-1}(\partial^{\alpha'}w)$ are bounded in $\mathbb{R}^2$. Therefore $\partial^\alpha p$ is bounded in $\mathbb{R}^2$ for each α in $\mathbb{N}_0^2$. Thus, $w(\phi * \check{h})$ belongs to $\mathcal{B}(\mathbb{R}^2)$. Now, let us put things together. By the hypothesis, we know that $w^{-1}T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. Also, we have just shown that $w(\phi * \check{h})$ belongs to $\mathcal{B}(\mathbb{R}^2)$. Therefore, by Theorem 5, we conclude that $w(\phi * \check{h})w^{-1}T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. But $w(\phi * \check{h})w^{-1}T = (\phi * \check{h})T$. Hence, $(\phi * \check{h})T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$, for every ϕ in $\mathcal{S}(\mathbb{R}^2)$. Therefore, T is $\mathcal{S}'$-convolvable with $h(x_1, x_2) = \frac{1}{x_1+ix_2}$. □

4. Codomain for the Laplace operator in $\mathbb{R}^2$

In this section we identify a codomain in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Laplace operator in $\mathbb{R}^2$. We recall that $E(x) = \frac{1}{2\pi}\ln|x|$ is a fundamental solution of the Laplace operator in $\mathbb{R}^2$.

Theorem 10. *Let $h(x) = \ln|x|$, where $x \in \mathbb{R}^2$. Set $w_a(x) = \ln(a+|x|^2)$, where $a > 1$ and $x \in \mathbb{R}^2$. Let $T \in \mathcal{S}'(\mathbb{R}^2)$. If $T \in \frac{1}{w_a}\mathcal{D}'_{L^1}(\mathbb{R}^2)$, then T is $\mathcal{S}'$-convolvable with h.*

Proof. Assume that T belongs to $\frac{1}{w_a}\mathcal{D}'_{L^1}(\mathbb{R}^2)$. We wish to prove that T is $\mathcal{S}'$-convolvable with h. That is, we wish to show that

$$(\phi * \overset{\vee}{h})T \in \mathcal{D}'_{L^1}(\mathbb{R}^2), \tag{10}$$

for every ϕ in $\mathcal{S}(\mathbb{R}^2)$. But then for any ϕ in $\mathcal{S}(\mathbb{R}^2)$, we have

$$(\phi * \overset{\vee}{h})T = w_a T \frac{1}{w_a}(\phi * \overset{\vee}{h}). \tag{11}$$

Let us examine the right hand side of (11). By the hypothesis, $w_a T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. So by Theorem 5, to prove (10), it suffices to prove that $g(x) := \frac{1}{w_a}(\phi * \overset{\vee}{h})(x)$ is an element of $\mathcal{B}(\mathbb{R}^2)$. First, we show that the $C^\infty(\mathbb{R}^2)$-function g is bounded in $\mathbb{R}^2$.

$$\begin{aligned}
|g(x)| &= \left|\frac{1}{w_a(x)}(\phi * \overset{\vee}{h})(x)\right| \\
&= \left|\frac{1}{w_a(x)}\int_{\mathbb{R}^2} \phi(x-y)\overset{\vee}{h}(y)dy\right| \\
&= \frac{1}{w_a(x)}\left|\int_{\mathbb{R}^2} \phi(x-y)\ln|y|\,dy\right| \\
&\leq \frac{1}{w_a(x)}\int_{\mathbb{R}^2} |\phi(x-y)\ln|y||\,dy \\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 = \frac{1}{w_a(x)}\int_{|y|<1} |\phi(x-y)\ln|y||\,dy,$$

and

$$I_2 = \frac{1}{w_a(x)}\int_{|y|\geq 1} |\phi(x-y)\ln|y||\,dy.$$

We prove that g is bounded in $\mathbb{R}^2$ by proving that both I_1 and I_2 are bounded in $\mathbb{R}^2$. We begin by proving that I_1 is bounded in $\mathbb{R}^2$. Since ϕ belongs to $\mathcal{S}(\mathbb{R}^2)$, we conclude that ϕ is bounded in $\mathbb{R}^2$. So, from

$$I_1 = \frac{1}{w_a(x)}\int_{|y|<1} |\phi(x-y)\ln|y||\,dy,$$

we have

$$I_1 \leq \frac{C}{w_a(x)} \int_{|y|<1} |\ln|y||\, dy.$$

By switching to polar coordinates, we obtain

$$\begin{aligned} I_1 &\leq \frac{C}{w_a(x)} \int_{|y|<1} |\ln|y||\, dy \\ &= \frac{C}{w_a(x)} \int_0^{2\pi} \int_0^1 r\,|\ln r|\, dr d\theta \\ &= \frac{C}{w_a(x)} \lim_{\epsilon\to 0^+} \int_\epsilon^1 r\,|\ln r|\, dr. \end{aligned}$$

But

$$\begin{aligned} \lim_{\epsilon\to 0^+} \int_\epsilon^1 r\,|\ln r|\, dr &= -\lim_{\epsilon\to 0^+} \int_\epsilon^1 r \ln r dr \\ &= -\lim_{\epsilon\to 0^+} \left[\frac{r^2}{2} \ln r \,\Big|_\epsilon^1 - \frac{1}{2} \int_\epsilon^1 r^2 \frac{1}{r} dr \right] \\ &= -\lim_{\epsilon\to 0^+} \left[-\frac{1}{2}\epsilon^2 \ln\epsilon - \left[\frac{1}{4} r^2\right] \Big|_\epsilon^1 \right] \\ &= -\lim_{\epsilon\to 0^+} \left[-\frac{1}{2}\epsilon^2 \ln\epsilon - \frac{1}{4} + \frac{1}{4}\epsilon^2 \right] \\ &= \frac{1}{4}. \end{aligned}$$

Therefore,

$$I_1 \leq \frac{C}{w_a(x)} \leq \frac{C}{\ln a} \leq C.$$

This shows I_1 is bounded in $\mathbb{R}^2$. Next, we prove that I_2 is bounded in $\mathbb{R}^2$. But then

$$\begin{aligned} I_2 &= \frac{1}{w_a(x)} \int_{|y|\geq 1} |\phi(x-y) \ln|y||\, dy \\ &\leq \frac{1}{w_a(x)} \int_{|y|\geq 1} |y\phi(x-y)|\, dy \\ &\leq \frac{1}{w_a(x)} \int_{\mathbb{R}^2} |y\phi(x-y)|\, dy, \end{aligned}$$

so that

$$I_2 \leq \frac{1}{w_a(x)} \int_{\mathbb{R}^2} |y\phi(x-y)|\, dy. \tag{12}$$

Since ϕ belongs to $\mathcal{S}(\mathbb{R}^2)$, we conclude that the function $y\phi$ is integrable in $\mathbb{R}^2$. Hence,

$$\int_{\mathbb{R}^2} |y\phi(x-y)|\, dy$$

is finite. Therefore, from (12), we have

$$I_2 \leq \frac{C}{w_a(x)} \leq \frac{C}{\ln a} \leq C.$$

Up to this point, we have shown that both I_1 and I_2 are bounded in $\mathbb{R}^2$. Therefore

$$|g(x)| \leq C, \tag{13}$$

for all $x \in \mathbb{R}^2$. Next, we claim that each derivative of g is also bounded in $\mathbb{R}^2$. For each α in $\mathbb{N}_0^2$, by Leibniz's formula, we have

$$\begin{aligned} \partial^\alpha g &= \sum_{0\leq\alpha'\leq\alpha} C_{\alpha',\alpha}\partial^{\alpha'}\left(\frac{1}{w_a}\right)(\partial^{\alpha-\alpha'}\phi * \overset{\vee}{h}) \\ &= \sum_{0\leq\alpha'\leq\alpha} C_{\alpha',\alpha} w_a(x)\partial^{\alpha'}\left(\frac{1}{w_a}\right)\left[\frac{1}{w_a}(\partial^{\alpha-\alpha'}\phi * \overset{\vee}{h})\right]. \end{aligned}$$

We show that $w_a\partial^{\alpha'}(\frac{1}{w_a})$ and $\frac{1}{w_a}(\partial^{\alpha-\alpha'}\phi * \overset{\vee}{h})$ are bounded in $\mathbb{R}^2$. We begin by showing that $\frac{1}{w_a}(\partial^{\alpha-\alpha'}\phi * \overset{\vee}{h})$ is bounded in $\mathbb{R}^2$. From (13), we know that $g(x) := \frac{1}{w_a}(\phi * \overset{\vee}{h})(x)$ is bounded in $\mathbb{R}^2$ for any ϕ in $\mathcal{S}(\mathbb{R}^2)$. But then, we know that if ϕ belongs to $\mathcal{S}(\mathbb{R}^2)$ then $\partial^\gamma\phi$ also belongs to $\mathcal{S}(\mathbb{R}^2)$ for any $\gamma \in \mathbb{N}_0^2$. Therefore $\partial^{\alpha-\alpha'}\phi$ belongs to $\mathcal{S}(\mathbb{R}^2)$. Hence, $\frac{1}{w_a}(\partial^{\alpha-\alpha'}\phi * \overset{\vee}{h})$ is bounded in $\mathbb{R}^2$. Also, in the spirit of Lemma 1, it can be shown that $w_a\partial^{\alpha'}(\frac{1}{w_a})$ is bounded in $\mathbb{R}^2$ for any α' in $\mathbb{N}_0^2$. Therefore $\partial^\alpha g$ is bounded in $\mathbb{R}^2$ for each α in $\mathbb{N}_0^2$. This proves our claim. We therefore conclude that $g(x) := \frac{1}{w_a}(\phi * \overset{\vee}{h})(x)$ belongs to $\mathcal{B}(\mathbb{R}^2)$. Let us put things together. By the hypothesis, we know that w_aT belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. Also, we have just shown that $g(x) := \frac{1}{w_a}(\phi * \overset{\vee}{h})(x)$ belongs to $\mathcal{B}(\mathbb{R}^2)$. Therefore, by Theorem 5, we conclude that w_agT belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$. But $w_a(x)gT = (\phi * \overset{\vee}{h})T$. Hence, $(\phi * \overset{\vee}{h})T$ belongs to $\mathcal{D}'_{L^1}(\mathbb{R}^2)$, for every $\phi \in \mathcal{S}(\mathbb{R}^2)$. Thus, T is $\mathcal{S}'$-convolvable with $h(x) = \ln|x|$. $\square$

5. Conjecture

J. Alvarez and L.E.S. Moyo [3] used weighted integrable distribution spaces to identify the maximal codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Laplace operator $\Delta := \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ associated with the Euclidean space $\mathbb{R}^n$, $n \neq 2$ and the product Laplace operator $\Delta_x \ \Delta_y := \left(\sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}\right)\left(\sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2}\right)$ associated with the Cartesian product space $\mathbb{R}^m \times \mathbb{R}^n$, $m \neq 2$, $n \neq 2$. We conjecture that it is impossible to identify maximal codomains in the context of the $\mathcal{S}'$-convolution as defined by R. Shiraishi [15] for the Cauchy-Riemann operator in $\mathbb{R}^2$ and Laplace operator in $\mathbb{R}^2$.

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