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Marcinkiewicz integrals with variable kernels on Hardy and weak Hardy spaces*

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Abstract. In this article, we consider the Marcinkiewicz integrals with variable kernels defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $\Omega(x,z) \in L^\infty(\mathbb{R}^n) \times L^q(\mathbb{S}^{n-1})$ for $q > 1$. We prove that the operator μ_Ω is bounded from Hardy space, $H^p(\mathbb{R}^n)$, to $L^p(\mathbb{R}^n)$ space; and is bounded from weak Hardy space, $H^{p,\infty}(\mathbb{R}^n)$, to weak $L^p(\mathbb{R}^n)$ space for $\max\{\frac{2n}{2n+1}, \frac{n}{n+\alpha}\} < p < 1$, if Ω satisfies the $L^{1,\alpha}$-Dini condition with any $0 < \alpha \le 1$.

1. Introduction

Let $\mathbb{R}^n (n \ge 2)$ be the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with induced Lebesgue measure $d\sigma$, and let $x' = \frac{x}{|x|}$ for any $x \ne 0$.

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In 1958, E. M. Stein [12] first introduced the following Marcinkiewicz integral μ_ω of higher dimension with convolution kernel,

$$\mu_\omega(f) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\omega(x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2},$$

where $\omega(x)$ is a homogeneous function of degree zero with $\omega \in L^1(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}} \omega(x')d\sigma(x') = 0$.

E. M. Stein proved that μ_ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le 2$ and is of weak type $(1,1)$ if $\omega \in \mathrm{Lip}_\alpha(\mathbb{S}^{n-1})$ with $0 < \alpha \le 1$. Subsequently, A. Benedek, A. Calderon and R. Panzone [3] showed that if $\omega \in C^1(\mathbb{S}^{n-1})$, then μ_ω is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Later on, the above results were improved by many authors under some weaker smoothness conditions on ω, see [1], [7, 8], [11], [14, 15, 16] for instance.

We remark that the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. If let $\phi(x) = \omega(x)|x|^{-n+1}\chi_B(x)$ and $\phi_t(x) = t^{-n}\phi(x/t)$, where B denotes the unit ball of $\mathbb{R}^n$ and χ_B denotes the characteristic function of B, then

$$\mu_\omega(f)(x) = \left(\int_0^\infty |\phi_t * f(x)|^2 \frac{dt}{t} \right)^{1/2} = g_\phi(f)(x).$$

In order to study non-smoothness partial differential equations with variable coefficients, mathematicians pay more attention to the singular integral with variable kernels, see [2], [4], [5] and [6] among others. Specially, in 1955 Calderón and Zygmund [4] considered the singular integral with variable kernel defined by

$$T_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)dy.$$

In this paper, we study the Marcinkiewicz integral with variable kernel defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2}.$$

We point out that μ_Ω can be interpreted as a Hilbert-valued function. In fact, denote the Hilbert space $\mathcal{H}$ by

$$\mathcal{H}=\left\{h(t):\|h\|_{\mathcal{H}}=\left(\int_0^\infty |h(t)|^2\frac{dt}{t}\right)^{1/2}<+\infty\right\},$$

and let $h_f(t,x)=t^{-1}F_{\Omega,t}(x)$, where

$$F_{\Omega,t}(x)=\int_{|x-y|\le t}\frac{\Omega(x,x-y)}{|x-y|^{n-1}}f(y)dy.$$

Then we obtain that $\mu_\Omega(f)(x)=\|h_f(\cdot,x)\|_{\mathcal{H}}$.

Before stating our theorems, we first introduce some definitions about the variable kernel $\Omega(x,z)$. A function $\Omega(x,z)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ is said to be in $L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$, $q\ge 1$, if $\Omega(x,z)$ satisfies the following three conditions:

(1) $\Omega(x,\lambda z)=\Omega(x,z)$, for any $x,z\in\mathbb{R}^n$ and any $\lambda>0$;

(2) $\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})}=\sup\limits_{r\ge 0,\, y\in\mathbb{R}^n}\left(\int_{\mathbb{S}^{n-1}}|\Omega(rz'+y,z')|^q d\sigma(z')\right)^{1/q}<\infty$;

(3) $\int_{\mathbb{S}^{n-1}}\Omega(x,z')d\sigma(z')=0$ for any $x\in\mathbb{R}^n$.

In [4], Calderón and Zygmund proved that if Ω satisfies (1), (3) and

$$\sup_{y\in\mathbb{R}^n}\left(\int_{\mathbb{S}^{n-1}}|\Omega(y,z')|^q d\sigma(z')\right)^{1/q}<\infty, \tag{2'}$$

then T_Ω is bounded on $L^2(\mathbb{R}^n)$ provided $q\ge 2(n-1)/n$. They also found that for no n can we replace the exponent $2(n-1)/n$ by a smaller one. Since the condition (2) implies (2'), so the $L^2(\mathbb{R}^n)$ boundedness of T_Ω holds if $\Omega\in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$ with $q\ge 2(n-1)/n$. Recently in [7], the authors proved that if $\Omega\in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$ with $q\ge 2(n-1)/n$, then μ_Ω is bounded on $L^2(\mathbb{R}^n)$; and they also showed the H^1-L^1 boundedness of μ_Ω under certain Dini condition.

For $0<\alpha\le 1$, a function $\Omega\in L^\infty(\mathbb{R}^n)\times L^1(\mathbb{S}^{n-1})$ is called to satisfy the $L^{1,\alpha}$*-Dini condition* if

$$\int_0^1\frac{\varpi(\delta)}{\delta^{1+\alpha}}d\delta<\infty, \tag{1.1}$$

where

$$\varpi(\delta)=\sup_{r>0,\, y\in\mathbb{R}^n\, |O|<\delta}\int_{\mathbb{S}^{n-1}}|\Omega(rz'+y,Oz')-\Omega(rz'+y,z')|d\sigma(z'), \tag{1.2}$$

and O is a rotation in $\mathbb{R}^n$ with $|O|=\|\,O-I\,\|$, where I is the identity operator. For the special case $\alpha=0$, it reduces to the L^1-Dini condition

Our first aim is to show that the Marcinkiewicz integral μ_Ω with variable kernel is bounded on Hardy spaces $H^p(\mathbb{R}^n)$ with some $p<1$.

Theorem 1.1. *Let $\Omega(x,z)\in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$ with $q>2(n-1)/n$, and let $\Omega(x,z')$ satisfy the $L^{1,\alpha}$-Dini condition with $0<\alpha\le 1$. Then, if $\max\{\frac{2n}{2n+1},\frac{n}{n+\alpha}\}<p<1$, there exists an absolute constant C independent of f such that*

$$\|\mu_\Omega(f)\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{H^p(\mathbb{R}^n)}. \tag{1.3}$$

Corollary 1.2. *Let $\omega(z)\in L^q(\mathbb{S}^{n-1})$ with $q>2(n-1)/n$, and let $\omega(z')$ satisfy $L^{1,\alpha}$-Dini condition with $0<\alpha\le 1$. Then, μ_ω is bounded from Hardy space $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $\max\{\frac{2n}{2n+1},\frac{n}{n+\alpha}\}<p<1$.*

Another aim of the paper is to derive that μ_Ω is bounded from weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ to weak $L^p(\mathbb{R}^n)$ space, $L^{p,\infty}(\mathbb{R}^n)$, for some $p<1$. Let us first recall the definition of weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$.

Definition 1.3. Let $\varphi\in C_0^\infty$ with $\int\varphi(x)dx\neq 0$. Denote by $f_+^*(x)=\sup\limits_{t>0}|(\varphi_t*f)(x)|$, where $\varphi_t(x)=t^{-n}\varphi(\frac{x}{t})$. A distribution f is said to belong to the weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ if $f_+^*(x)$ belongs to the weak $L^p(\mathbb{R}^n)$ space, $L^{p,\infty}(\mathbb{R}^n)$, i.e, there is a constant $C>0$ such that

$$|\{x\in R^n: f_+^*(x)>\beta\}|\le\frac{C^p}{\beta^p},\qquad\forall\ \beta>0.$$

The smallest constant C satisfying the above inequality is called the $H^{p,\infty}$ norm of f, and is denoted by $\|f\|_{H^{p,\infty}}$.

Theorem 1.4. *Let $\Omega(x,z)\in L^\infty(\mathbb{R}^n)\times L^q(\mathbb{S}^{n-1})$ with $q>2(n-1)/n$, and let $\Omega(x,z')$ satisfy the $L^{1,\alpha}$-Dini condition with $0<\alpha\le 1$. Then, if $\max\{\frac{2n}{2n+1},\frac{n}{n+\alpha}\}<p<1$, there exists a constant C independent of f and β such that*

$$|\{x:|\mu_\Omega f(x)|>\beta\}|\le C\frac{\|f\|^p_{H^{p,\infty}}}{\beta^p}. \tag{1.4}$$

Corollary 1.5. *Let $\omega(z)\in L^q(\mathbb{S}^{n-1})$ for $q>2(n-1)/n$, and let $\omega(z')$ satisfy $L^{1,\alpha}$-Dini condition with $0<\alpha\le 1$. Then μ_ω is bounded from weak Hardy space $H^{p,\infty}(\mathbb{R}^n)$ to weak space $L^{p,\infty}(\mathbb{R}^n)$ for $\max\{\frac{2n}{2n+1},\frac{n}{n+\alpha}\}<p<1$.*

Remark 1.6. It's easy to see that $\mathrm{Lip}_\alpha\subset L^{1,\alpha}$-Dini for any $0<\alpha\le 1$. Thus, the conclusions of Corollary 1.2 and 1.5 may be regarded as an improvement and extension of Stein's results about the Marcinkiewicz integrals with convolution kernels in [12] and [8].

Remark 1.7. It is worthy noting that the $H^1 - L^1$ boundedness of μ_Ω may be regarded as the limit case of Theorem 1.1 by choosing $p = 1$ and letting $\alpha \to 0$. Hence, Theorem 1.3 and Corollary 1.7 in [7] are the special cases of above Theorem 1.1.

Throughout the paper, C always denotes a positive constant not necessarily the same at each occurrence. We use $a \sim b$ to mean the equivalence of a and b; that is, there exists a positive constant C independent of a, b such that $C^{-1}a \le b \le Ca$.

2. Proof of theorem 1.1

In order to show the $H^p - L^p$ boundedness of μ_Ω, we will use the atomic decomposition theory of the real Hardy space $H^p(\mathbb{R}^n)$ for $\frac{n}{n+1} < p \le 1$, see for instance [13]. A function $a(x)$ is said to be $(p, 2, 0)$ atom if it satisfies the following three conditions:

(i) $\mathrm{supp}(a) \subset B(x_0, r)$, where $B(x_0, r) = \{y \in \mathbb{R}^n : |y - x_0| \le r\}$ is a ball in $\mathbb{R}^n$;

(ii) $\|a\|_{L^2} \le |B(x_0, r)|^{1/2-1/p}$;

(iii) $\int_{\mathbb{R}^n} a(x)dx = 0$.

It is well known that every $f \in H^p(\mathbb{R}^n)$, $\frac{n}{n+1} < p \le 1$, has an atomic decomposition $f = \sum_k \lambda_k a_k$, which converges in H^p norm and in the sense of distributions; where $(\sum_k |\lambda_k|^p)^{1/p} \cong \|f\|_{H^p}$, and all $a_k(x)$ are $(p, 2, 0)$ atoms.

Start with f in a nice dense class of function, say $f \in H^p(\mathbb{R}^n) \cap C_0^\infty(\mathbb{R}^n)$. If we denote the kernel by $K(x, z) = \frac{\Omega(x,z)}{|z|^{n-1}}$ and set

$$\mu_{\Omega,\varepsilon}(f)(x) = \left(\int_0^\infty |F_{\Omega,t,\varepsilon}(x)|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where

$$F_{\Omega,t,\varepsilon}(x) = \int_{\varepsilon<|x-y|\le t} K(x, x-y) f(y) dy.$$

Now following [13] (Page 115), we write the distribution kernel $K = K_0 + K_\infty$, where K_0 has compact support, and thus the distribution $F_{\Omega,t,\varepsilon}$ is well defined for every fixed ε and t, and

$$F_{\Omega,t,\varepsilon}(x) = \sum_k \lambda_k \int_{\varepsilon<|x-y|\le t} K(x, x-y) a_k(y) dy.$$

We claim that, for almost every $x \in \mathbb{R}^n$, $\mu_\Omega(f)(x) = \lim_{\varepsilon\to 0} \mu_{\Omega,\varepsilon}(f)(x)$. To see the claim, we use the cancellation condition of Ω and the fact $\Omega(x,z) \in L^\infty(\mathbb{R}^n) \times L^r(\mathbb{S}^{n-1})$ for any $1 \le r < 2(n-1)/n$ to obtain

$$\begin{aligned} F_{\Omega,t,\varepsilon}(x) &= \int_{\varepsilon<|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}}(f(y)-f(x))dy \\ &\le C\int_{|x-y|\le t} \frac{|\Omega(x,x-y)|}{|x-y|^{n-2}}dy \le Ct^2. \end{aligned}$$

On the other hand, Hölder inequality gives

$$\begin{aligned} F_{\Omega,t,\varepsilon}(x) &\le C\left(\int_{|x-y|\le t} \frac{|\Omega(x,x-y)|^r}{|x-y|^{(n-1)r}}dy\right)^{1/r} \\ &\le Ct^{-(n-1)+(n/r)}, \end{aligned}$$

for any $1 < r < n/(n-1)$. Therefore

$$t^{-3}|F_{\Omega,t,\varepsilon}(x)|^2 \le C(t\chi_{(0,1)}(t) + t^{-1-2n+(2n/r)}\chi_{(1,\infty)}(t))$$

uniformly on ε. So by the Lebesgue dominated convergence theorem we get that $\mu_\Omega(f)(x) = \lim_{\varepsilon\to 0} \mu_{\Omega,\varepsilon}(f)(x)$.

Thus, by similar approximation arguments as in [9] (Theorem 7.3) and in [13] (Page 115), we can obtain

$$\mu_\Omega(f) \le \sum_k |\lambda_k|\mu_\Omega(a_k). \tag{2.1}$$

Therefore, to derive the inequality (1.3) for any $f \in H^p(\mathbb{R}^n)$ and prove Theorem 1.1, it suffices to show that for any $(p,2,0)$ atom $a(x)$, there exists a constant $C > 0$ independent of $a(x)$ such that

$$\|\mu_\Omega(a)\|_{L^p}^p \le C. \tag{2.2}$$

Without loss of generality, we let the support of the atom $a(x)$ is $B = B(0,r)$, and denote $B^* = B(0,8r)$. Using the L^2 boundedness of μ_Ω, we have

$$\begin{aligned} \int_{B^*} |\mu_\Omega(a)(x)|^p dx &\le |B^*|^{1-p/2} \parallel \mu_\Omega(a) \parallel_{L^2}^p \\ &\le C|B^*|^{1-p/2} \parallel a \parallel_{L^2}^p \quad \le C. \end{aligned} \tag{2.3}$$

Note that $(u+v)^s \le u^s + v^s$ for any $u, v \ge 0$ and $0 \le s \le 1$. It is left to give the estimate for the integral

$$\begin{aligned}
I &= \int_{(B^*)^c} |\mu_\Omega(a)(x)|^p dx \\
&= \int_{(B^*)^c} \left(\int_0^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y)dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx \\
&\le \int_{(B^*)^c} \left(\int_0^{|x|+2r} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y)dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx \\
&\quad + \int_{(B^*)^c} \left(\int_{|x|+2r}^\infty \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y)dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx \\
&:= I_1 + I_2.
\end{aligned}$$

We note that, for $y \in B$ and $x \in (B^*)^c$, $|x-y| \sim |x| \sim |x| + 2r$. Thus by the mean value theorem we have

$$\left| \frac{1}{|x-y|^2} - \frac{1}{(|x|+2r)^2} \right| \le C \frac{r}{|x-y|^3}.$$

Applying this inequality and the Minkowski's inequality, we obtain that

$$\begin{aligned}
I_1 &= \int_{(B^*)^c} \left(\int_0^{|x|+2r} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} a(y)dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx \\
&\le C \int_{(B^*)^c} \left(\int_B \left(\int_{|x-y|}^{|x|+2r} \frac{|\Omega(x,x-y)|^2}{|x-y|^{2(n-1)}} |a(y)|^2 \frac{1}{t^3} dt \right)^{1/2} dy \right)^p dx \\
&= C \int_{(B^*)^c} \left(\int_B \frac{|\Omega(x,x-y)|}{|x-y|^{n-1}} |a(y)| \left| \frac{1}{|x-y|^2} - \frac{1}{(|x|+2r)^2} \right|^{1/2} dy \right)^p dx \\
&\le C r^{\frac{p}{2}} \int_{(B^*)^c} \left(\int_B \frac{|\Omega(x,x-y)|}{|x-y|^{n+\frac{1}{2}}} |a(y)| dy \right)^p dx.
\end{aligned}$$

Since $1 > p > \frac{n}{n+\frac{1}{2}}$, so we can choose ε satisfying $0 < \varepsilon < n + \frac{1}{2} - \frac{n}{p}$. Using Hölder inequality for integrals, we have

$$I_1 \le C r^{\frac{p}{2}} \int_{(B^*)^c} \left(\int_B \frac{|\Omega(x,x-y)|}{|x-y|^{n+\varepsilon}} \cdot |x-y|^{\varepsilon - \frac{1}{2}} |a(y)| dy \right)^p dx$$

$$
\begin{aligned}
&= Cr^{\frac{p}{2}} \int_{(B^*)^c} \left(\int_B \frac{|\Omega(x, x-y)|}{|x-y|^{n+\varepsilon}} \cdot |a(y)| dy \right)^p \cdot |x|^{(\varepsilon - \frac{1}{2})p} dx \\
&\leq Cr^{\frac{p}{2}} \left(\int_{(B^*)^c} \int_B \frac{|\Omega(x, x-y)|}{|x-y|^{n+\varepsilon}} \cdot |a(y)| dy dx \right)^p \left(\int_{(B^*)^c} |x|^{\frac{(\varepsilon - \frac{1}{2})p}{1-p}} dx \right)^{1-p} \\
&\leq Cr^{\frac{p}{2}} \left(\int_B \int_{(B^*)^c} \frac{|\Omega(x, x-y)|}{|x-y|^{n+\varepsilon}} dx \cdot |a(y)| dy \right)^p \left(\int_{8r}^{\infty} t^{\frac{(\varepsilon - \frac{1}{2})p}{1-p}} \cdot t^{n-1} dt \right)^{1-p} \\
&\leq Cr^{\frac{p}{2}} \|\Omega\|^p_{L^\infty \times L^1} \left(\int_B |a(y)| dy \int_{8r}^{+\infty} t^{-n-\varepsilon} t^{n-1} dt \right)^p \cdot r^{(\varepsilon - \frac{1}{2})p + n(1-p)} \\
&\leq Cr^{n(1-p)} \left(\int_B |a(y)| dy \right)^p \leq Cr^{n(1-p)} |B|^{\frac{p}{2}} \|a\|^p_{L^2} \leq C.
\end{aligned}
$$

where we have used that $|x-y| \sim |x|$.

As to the estimate of I_2. Noting that if $t \geq |x| + 2r$, then $B \subset \{y : |x-y| < t\}$. So by the cancellation condition (iii) of a, we have

$$
\int_{|x-y|<t} a(y) dy = 0.
$$

From this and Minkowski's inequality for integrals, we obtain

$$
\begin{aligned}
I_2 = &\int_{(B^*)^c} \left(\int_{|x|+2r}^{\infty} \left| \int_{|x-y|\leq t} \left(\frac{\Omega(x, x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x-y|^{n-1}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \frac{\Omega(x,x)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}} \right) \cdot a(y) dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx \\
&\leq I_{21} + I_{22},
\end{aligned}
$$

where

$$
I_{21} = \int_{(B^*)^c} \left(\int_{|x|+2r}^{\infty} \left| \int_{|x-y|\leq t} \frac{|\Omega(x, x-y) - \Omega(x,x)|}{|x-y|^{n-1}} |a(y)| dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx,
$$

and

$$
I_{22} = \int_{(B^*)^c} \left(\int_{|x|+2r}^{\infty} \left| \int_{|x-y|\leq t} \left| \frac{\Omega(x,x)}{|x-y|^{n-1}} - \frac{\Omega(x,x)}{|x|^{n-1}} \right| |a(y)| dy \right|^2 \frac{1}{t^3} dt \right)^{p/2} dx.
$$

Applying Minkowski inequality for integrals, Hölder inequality for integrals and Fubini theorem successively, we can obtain

$$
\begin{aligned}
I_{21} &\leq \int_{(B^*)^c} \left| \int_B \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^n}|a(y)|dy \right|^p dx \\
&\leq \sum_{j=3}^{+\infty} \int_{2^j r\leq |x|<2^{j+1}r} \left| \int_B \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^n}|a(y)|dy \right|^p dx \\
&\leq \sum_{j=3}^{+\infty} \left(\int_{2^j r\leq |x|<2^{j+1}r} \int_B \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^n}|a(y)|dydx \right)^p (2^j r)^{n(1-p)} \\
&= \sum_{j=3}^{+\infty} \left(\int_B \left(\int_{2^j r\leq |x|<2^{j+1}r} \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^n}dx \right) |a(y)|dy \right)^p (2^j r)^{n(1-p)}.
\end{aligned}
$$

To estimate the inner integral above, we note $|y|<r$ and $|x|>2r$, which implies,

$$
\begin{aligned}
\left| \frac{x-y}{|x-y|} - \frac{x}{|x|} \right| &= \left| \frac{(|x|-|x-y|)x-|x|y}{|x||x-y|} \right| \\
&\leq \frac{||x|-|x-y||+|y|}{|x-y|} \leq 4\frac{|y|}{|x|}.
\end{aligned} \tag{2.4}
$$

And thus

$$
\begin{aligned}
&\int_{\mathbb{S}^{n-1}} |\Omega(x,x-y)-\Omega(x,x)|d\sigma(x') \\
&= \int_{\mathbb{S}^{n-1}} \left| \Omega\left(x, \frac{x-y}{|x-y|}\right) - \Omega\left(x, \frac{x}{|x|}\right) \right| d\sigma(x') \leq \varpi\left(4\frac{|y|}{|x|}\right).
\end{aligned}
$$

This and a direct computation give

$$
\begin{aligned}
&\int_{2^j r\leq |x|<2^{j+1}r} \frac{|\Omega(x,x-y)-\Omega(x,x)|}{|x-y|^n}dx \\
&\leq C\int_{2^j r}^{2^{j+1}r} t^{-1}\varpi\left(\frac{4|y|}{t}\right)dt \\
&= C\int_{\frac{|y|}{2^{j-1}r}}^{\frac{|y|}{2^{j-2}r}} \frac{\varpi(\delta)}{\delta}d\delta.
\end{aligned} \tag{2.5}
$$

Now using the condition $\int_0^1 \frac{\varpi(\delta)}{\delta^{1+\alpha}}d\delta<\infty$, and $1>p>\frac{n}{n+\alpha}$, we can get

$$
\begin{aligned}
I_{21} &\le C\sum_{j=3}^{+\infty}\left(\int_B |a(y)|\int_{\frac{|y|}{2^{j-1}r}}^{\frac{|y|}{2^{j-2}r}}\frac{\varpi(\delta)}{\delta^{1+\alpha}}\delta^{\alpha}d\delta dy\right)^p (2^j r)^{n(1-p)}\\
&\le C\sum_{j=3}^{+\infty}(2^j r)^{n(1-p)}(2^j)^{-p\alpha}\left(\int_B |a(y)|dy\right)^p\\
&\le Cr^{n(1-p)}\left(\int_B |a(y)|dy\right)^p \le C.
\end{aligned}
$$

At last, we give the estimate of I_{22}. Obviously, mean value theorem gives

$$
\left|\frac{1}{|x-y|^{n-1}}-\frac{1}{|x|^{n-1}}\right| \le C\frac{|y|}{|x|^n}.
$$

Applying this inequality, Minkowski's inequality for integrals, and the fact $\frac{n}{n+1}<p<1$, we deduce that

$$
\begin{aligned}
I_{22} &\le \int_{|x|>8r}\left(\int_{|x|+2r}^{\infty}\left(\int_{|x-y|\le t}\frac{|\Omega(x,x)||y|}{|x|^n}|a(y)|dy\right)^2\frac{dt}{t^3}\right)^{p/2}dx\\
&\le C\int_{|x|>8r}\left(\int_B \frac{|\Omega(x,x)||y|}{|x|^{n+1}}|a(y)|dy\right)^p dx\\
&\le Cr^p\left(\int_{|x|>8r}|\Omega(x,x)|^p|x|^{(-n-1)p}dx\right)\left(\int_B |a(y)|dy\right)^p\\
&\le Cr^p\, r^{(-n-1)p+n}\|\Omega\|^p_{L^\infty\times L^p}\left(\int_B |a(y)|dy\right)^p\\
&\le C.
\end{aligned}
$$

Combining (2.3), and the estimates of I_1, I_{21} and I_{22}, we get (2.2) and then complete the proof of Theorem 1.1.

3. Proof of theorem 1.4

In order to prove theorem 1.4, we need the following decomposition theorem for distributions in $H^{p,\infty}(\mathbb{R}^n)$.

Lemma 3.1. [10] *Given a distribution $f\in H^{p,\infty}(\mathbb{R}^n)$, there exits a bounded function sequence $\{f_k\}_{k=-\infty}^{+\infty}$ which has the following properties:*

(a) $f=\sum_k f_k$ *in the sense of distributions;*

(b) *Each* f_k *may be further decomposed in* L^p *as* $f_k = \sum_i h_i^k$, *where* $\{h_i^k\}$ *satisfies the following three conditions:*

(i) $\mathrm{supp}(h_i^k) \subset B_i^k := B(x_i^k, r_i^k)$, *where* $B(x,r)$ *denotes the ball in* $\mathbb{R}^n$ *with the center at* x *and radius* r. *Moreover,* $\sum_i |B_i^k| \le C_1 2^{-kp}$, *where* $C_1 \sim \| f \|_{H^{p,\infty}}^p$ *and* $\sum_i \chi_{B_i^k}(x) \le C$;

(ii) $\| h_i^k \|_\infty \le C2^k$;

(iii) $\int h_i^k(x)dx = 0$.

We now give the proof of Theorem 1.4. For any $f \in H^{p,\infty}(\mathbb{R}^n)$ and $\beta > 0$, we choose k_0 satisfying $2^{k_0} \le \beta < 2^{k_0+1}$. Applying Lemma 3.1, we can write

$$f = \sum_{k=-\infty}^{k_0} f_k + \sum_{k=k_0+1}^{+\infty} f_k = F_1 + F_2, \qquad f_k = \sum_i h_i^k,$$

where h_i^k satisfies the properties (i), (ii) and (iii) of Lemma 3.1.

Let $A_k = \mathrm{supp}(f_k)$, then $A_k = \bigcup_i B_i^k$ and $|A_k| \le \sum_i |B_i^k| \le C2^{-kp}\|f\|_{H^{p,\infty}}^p$, also $\|f_k\|_\infty \le C2^k$, so we can get

$$\begin{aligned}\|F_1\|_2 &\le \sum_{k=-\infty}^{k_0} \|f_k\|_2 \le C \sum_{k=-\infty}^{k_0} 2^k |A_k|^{\frac{1}{2}} \\ &\le \sum_{k=-\infty}^{k_0} 2^{k(1-\frac{p}{2})} \|f\|_{H^{p,\infty}}^{\frac{p}{2}} \le C\|f\|_{H^{p,\infty}}^{\frac{p}{2}} \beta^{1-\frac{p}{2}}.\end{aligned}$$

By the L^2-boundedness of μ_Ω, it follows

$$|\{x : \mu_\Omega(F_1)(x) > \beta\}| \le \frac{\|\mu_\Omega(F_1)\|_2^2}{\beta^2} \le C\frac{\|F_1\|_2^2}{\beta^2} \le C\frac{\|f\|_{H^{p,\infty}}^p}{\beta^p}. \tag{3.1}$$

On the other hand, we denote $\overline{B}_i^k = B(x_i^k, 2R_i^k)$ and $R_i^k = (\frac{4}{3})^{\frac{(k-k_0)}{n}} r_i^k$, and let $B_{k_0} = \bigcup_{k=k_0+1}^{+\infty} \bigcup_i \overline{B}_i^k$, we can get

$$\begin{aligned}|B_{k_0}| &\le C \sum_{k=k_0+1} \sum_i |\overline{B}_i^k| \le C \sum_{k=k_0+1} \sum_i 2^n (\frac{4}{3})^{k-k_0} |B_i^k| \\ &\le C \sum_{k=k_0+1}^{+\infty} \sum_i (\frac{4}{3})^{k-k_0} |B_i^k| \le C \sum_{k=k_0+1}^{+\infty} (\frac{4}{3})^{k-k_0} 2^{-kp} \|f\|_{H^{p,\infty}}^p \le C\frac{\|f\|_{H^{p,\infty}}^p}{\beta^p},\end{aligned} \tag{3.2}$$

where the last inequality holds owing to $p > \frac{n}{n+\frac{1}{2}}$. Therefore, in order to prove Theorem 1.4, it suffices to show

$$|\{x \in (B_{k_0})^c : \mu_\Omega(F_2)(x) > \beta\}| \leq C\frac{\|f\|^p_{H^{p,\infty}}}{\beta^p}. \tag{3.3}$$

Firstly, a similar argument as the one used in (2.1) and Minkowski's inequality for series give

$$\begin{aligned}\int_{(B_{k_0})^c} |\mu_\Omega(F_2)(x)|^p dx &\leq \int_{(B_{k_0})^c} \sum_{k=k_0+1}^{+\infty} \sum_i |\mu_\Omega(h_i^k)(x)|^p dx \\ &\leq C \sum_{k=k_0+1}^{+\infty} \sum_i (J_1 + J_2),\end{aligned} \tag{3.4}$$

where

$$J_1 = \int_{(B_{k_0})^c} \left| \int_0^{|x-x_i^k|+2r_i^k} \left| \int_{|x-y|\leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right|^{\frac{p}{2}} dx,$$

and

$$J_2 = \int_{(B_{k_0})^c} \left| \int_{|x-x_i^k|+2r_i^k}^{+\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x, x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right|^{\frac{p}{2}} dx.$$

It is obvious that when $y \in B_i^k, x \in B_{k_0}^c$, we will have $|x-x_i^k| \sim |x-y| \sim |x-x_i^k| + 2r_i^k$ and

$$\left| \frac{1}{(|x-x_i^k| + 2r_i^k)^2} - \frac{1}{|x-y|^2} \right| \leq C\frac{r_i^k}{|x-y|^3}.$$

Because of $\frac{n}{n+\frac{1}{2}} < p < 1$, one can choose proper ε such that $0 < \varepsilon < n + \frac{1}{2} - \frac{n}{p}$. Combining the above inequality and Minkowski's inequality for integrals, the size condition of h_i^k and Hölder inequality for integrals, we deduce that

$$\begin{aligned}J_1 &\leq \int_{(B_{k_0})^c} \left| \int_{B_i^k} \frac{|\Omega(x, x-y)|}{|x-y|^{n-1}} |h_i^k(y)| \frac{(r_i^k)^{\frac{1}{2}}}{|x-y|^{\frac{3}{2}}} dy \right|^p dx \\ &\leq C2^{kp}(r_i^k)^{\frac{p}{2}} \int_{(B_{k_0})^c} \left| \int_{B_i^k} \frac{|\Omega(x, x-y)|}{|x-y|^{n+\varepsilon}} |x-x_i^k|^{\varepsilon-\frac{1}{2}} dy \right|^p dx\end{aligned}$$

$$
\begin{aligned}
&= C2^{kp}(r_i^k)^{\frac{p}{2}} \int_{(B_{k_0})^c} \left| \int_{B_i^k} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\varepsilon}} dy \right|^p |x-x_i^k|^{(\varepsilon-\frac{1}{2})p} dx \\
&\le C2^{kp}(r_i^k)^{\frac{p}{2}} \left(\int_{(B_{k_0})^c} \int_{B_i^k} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\varepsilon}} dy dx \right)^p \left(\int_{(B_{k_0})^c} |x-x_i^k|^{\frac{p(\varepsilon-\frac{1}{2})}{1-p}} dx \right)^{1-p} \\
&= C2^{kp}(r_i^k)^{\frac{p}{2}} \left(\int_{B_i^k} \left| \int_{(B_{k_0})^c} \frac{|\Omega(x,x-y)|}{|x-y|^{n+\varepsilon}} dx \right| dy \right)^p \left(\int_{2R_i^k}^{+\infty} r^{\frac{(\varepsilon-\frac{1}{2})p}{1-p}+n-1} dr \right)^{1-p} \\
&\le C2^{kp}(r_i^k)^{\frac{p}{2}} |B_i^k|^p \parallel \Omega \parallel_{L^\infty \times L^1}^p \left(\int_{2R_i^k}^{+\infty} r^{-\varepsilon-1} dr \right)^p (2R_i^k)^{(\varepsilon-\frac{1}{2})p+n(1-p)} \\
&\le C2^{kp}(r_i^k)^{\frac{p}{2}} |B_i^k|^p \left(2(\frac{4}{3})^{\frac{k-k_0}{n}} r_i^k \right)^{n(1-p)-\frac{p}{2}} \le C2^{kp}|B_i^k| \left(\frac{3}{4}\right)^{\frac{(k-k_0)(np+\frac{p}{2}-n)}{n}}.
\end{aligned}
$$

It follows

$$
\sum_{k=k_0+1}^{+\infty} \sum_i J_1 \le C \sum_{k=k_0+1}^{+\infty} \left(\frac{3}{4}\right)^{\frac{(k-k_0)(np+\frac{p}{2}-n)}{n}} \|f\|_{H^{p,\infty}}^p \le C\|f\|_{H^{p,\infty}}^p \tag{3.5}
$$

Next we estimate I_2. One notes that $B_i^k \subset \{y; |x-y| < t\}$ since $x \in B_{k_0}^c$ and $t > |x-x_i^k| + 2r_i^k$. By the cancellation of h_i^k and Minkowski's inequality for integrals, we can get

$$
\begin{aligned}
J_2 &= \int_{(B_{k_0})^c} \left| \int_{|x-x_i^k|+2r_i^k}^{+\infty} \left| \int_{|x-y|\le t} \frac{\Omega(x,x-y)}{|x-y|^{n-1}} h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right|^{\frac{p}{2}} dx \\
&= \int_{(B_{k_0})^c} \left| \int_{|x-x_i^k|+2r_i^k}^{+\infty} \left| \int_{|x-y|\le t} \left(\frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x-y)}{|x-x_i^k|^{n-1}} \right.\right.\right. \\
&\qquad \left.\left.\left. + \frac{\Omega(x,x-y)}{|x-x_i^k|^{n-1}} - \frac{\Omega(x,x-x_i^k)}{|x-x_i^k|^{n-1}} \right) h_i^k(y) dy \right|^2 \frac{dt}{t^3} \right|^{\frac{p}{2}} dx \\
&\le C2^{kp} \int_{(B_{k_0})^c} \left| \frac{1}{|x-x_i^k|} \int_{B_i^k} \left| \frac{\Omega(x,x-y)}{|x-x_i^k|^{n-1}} - \frac{\Omega(x,x-x_i^k)}{|x-x_i^k|^{n-1}} \right| dy \right|^p dx \\
&\quad + C2^{kp} \int_{(B_{k_0})^c} \left| \frac{1}{|x-x_i^k|} \int_{B_i^k} \left| \frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x-y)}{|x-x_i^k|^{n-1}} \right| dy \right|^p dx \\
&= C2^{kp}(J_{2,1} + J_{2,2}).
\end{aligned}
$$

Decompose $J_{2,1}$ as following

$$\begin{aligned}
J_{2,1} &= \sum_{j=1}^{+\infty} \int_{2^j R_i^k \le |x-x_i^k| \le 2^{j+1} R_i^k} \left| \int_{B_i^k} \left| \frac{\Omega(x, x-y)}{|x-x_i^k|^n} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^n} \right| dy \right|^p dx \\
&\le \sum_{j=1}^{+\infty} \left(2^j R_i^k\right)^{n(1-p)} \left(\int_{2^j R_i^k \le |x-x_i^k| \le 2^{j+1} R_i^k} \left| \int_{B_i^k} \left| \frac{\Omega(x, x-y)}{|x-x_i^k|^n} \right.\right.\right. \\
&\qquad \left.\left.\left. - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^n} \right| dy \right| dx \right)^p .
\end{aligned}$$

Observe that $|y-x_i^k| < r_i^k$ and $|x-x_i^k| > R_i^k \ge 2r_i^k$. Using a similar argument as in inequalities (2.4) and (2.5), we can deduce that

$$\left| \frac{x-y}{|x-y|} - \frac{x-x_i^k}{|x-x_i^k|} \right| \le 4 \frac{|y-x_i^k|}{|x-x_i^k|},$$

and then

$$\begin{aligned}
&\int_{2^j R_i^k \le |x-x_i^k| \le 2^{j+1} R_i^k} \left| \frac{\Omega(x, x-y)}{|x-x_i^k|^n} - \frac{\Omega(x, x-x_i^k)}{|x-x_i^k|^n} \right| dx \\
&= \int_{2^j R_i^k}^{2^{j+1} R_i^k} \int_{\mathbb{S}^{n-1}} \left| \Omega(rx'+x_i^k, \frac{rx'+x_i^k-y}{|rx'+x_i^k-y|}) - \Omega(rx'+x_i^k, rx') \right| d\sigma(x') r^{-1} dr \\
&\le C \int_{2^j R_i^k}^{2^{j+1} R_i^k} \varpi(\frac{4|y-x_i^k|}{r}) r^{-1} dr \\
&= C \int_{\frac{|y-x_i^k|}{2^{j-1} R_i^k}}^{\frac{|y-x_i^k|}{2^{j-2} R_i^k}} \left(\frac{|y-x_i^k|}{\delta} \right)^{-1} \varpi(\delta) \frac{|y-x_i^k|}{\delta^2} d\delta \\
&= C \int_{\frac{|y-x_i^k|}{2^{j-1} R_i^k}}^{\frac{|y-x_i^k|}{2^{j-2} R_i^k}} \frac{\varpi(\delta)}{\delta^{1+\alpha}} \delta^\alpha d\delta \le C \left(\frac{r_i^k}{2^j R_i^k} \right)^\alpha .
\end{aligned}$$

This follows

$$\begin{aligned}
J_{2,1} &\le C \sum_{j=1}^{+\infty} (2^j R_i^k)^{n(1-p)} \left(\frac{r_i^k}{2^j R_i^k} \right)^{\alpha p} |B_i^k|^p \\
&\le C |B_i^k| \left(\frac{3}{4} \right)^{\frac{k-k_0}{n}(np-n+\alpha p)} \cdot \sum_{j=1}^{+\infty} 2^{j(n-np-\alpha p)} \le C |B_i^k| \left(\frac{3}{4} \right)^{\frac{k-k_0}{n}(np-n+\alpha p)} .
\end{aligned}$$

To deal with $J_{2,2}$, we use the mean value theorem, Hölder inequality for integrals, and the condition $\frac{n}{n+\frac{1}{2}} < p < 1$ to give

$$\begin{aligned}
J_{2,2} &= \int_{(B_{k_0})^c} \left(\frac{1}{|x-x_i^k|} \int_{B_i^k} \left| \frac{\Omega(x,x-y)}{|x-y|^{n-1}} - \frac{\Omega(x,x-y)}{|x-x_i^k|^{n-1}} \right| dy \right)^p dx \\
&\le C \int_{(B_{k_0})^c} \left(\frac{1}{|x-x_i^k|} \int_{B_i^k} \frac{|\Omega(x,x-y)|}{|x-x_i^k|^n} r_i^k dy \right)^p dx \\
&= C(r_i^k)^p \int_{(B_{k_0})^c} |x-x_i^k|^{-\frac{p}{2}} \left(\int_{B_i^k} \frac{|\Omega(x,x-y)|}{|x-x_i^k|^{n+\frac{1}{2}}} dy \right)^p dx \\
&\le (r_i^k)^p \left(\int_{(B_{k_0})^c} \left| \int_{B_i^k} \frac{|\Omega(x,x-y)|}{|x-x_i^k|^{n+\frac{1}{2}}} dy \right| dx \right)^p \left(\int_{(B_{k_0})^c} |x-x_i^k|^{-\frac{p}{2(1-p)}} dx \right)^{1-p} \\
&\le C(r_i^k)^p |B_i^k|^p \left(2 \left(\frac{4}{3}\right)^{\frac{k-k_0}{n}} r_i^k \right)^{-\frac{p}{2}-\frac{p}{2}+n(1-p)} = C|B_i^k| \left(\frac{3}{4}\right)^{\frac{(k-k_0)(np+p-n)}{n}}.
\end{aligned}$$

Combining the estimates of $J_{2,1}$ and $J_{2,1}$, we obtain that

$$\begin{aligned}
&\sum_{k=k_0+1}^{+\infty} \sum_i J_2 \le C \sum_{k=k_0+1}^{+\infty} \sum_i 2^{kp} (J_{2,1} + J_{2,2}) \\
&\le C \sum_{k=k_0+1}^{+\infty} \left(\left(\frac{3}{4}\right)^{\frac{(k-k_0)(np+\alpha p-n)}{n}} + \left(\frac{3}{4}\right)^{\frac{(k-k_0)(np+p-n)}{n}} \right) \| f \|_{H^{p,\infty}}^p \qquad (3.6) \\
&\le C \| f \|_{H^{p,\infty}}^p .
\end{aligned}$$

Obviously, the inequalities (3.5), (3.6) and (3.4) yield the inequality (3.3) and the result follows. □

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