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Generalized composition operators from Bloch type spaces to Q_K type spaces

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Abstract. This paper characterizes the boundedness and compactness of the generalized composition operator $(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi$ from Bloch type spaces to Q_K type spaces.

1. Introduction

Let φ be an analytic self-map of the unit disk D. For $g \in H(D)$, the class of all analytic functions on D, we define a linear operator as follows

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \qquad f \in H(D).$$

The operator C_φ^g is called the generalized composition operator which is introduced in [5] for the first time. When $g = \varphi'$, we see that this operator is essentially composition operator C_φ which is defined by $C_\varphi f = f \circ \varphi$. Therefore, C_φ^g is a generalization of the composition operator C_φ. One of the critical problems on composition operators is to relate function theoretic properties of φ to operator theoretic properties of the restriction of C_φ to various Banach spaces of analytic functions. The composition

operators on $B^\alpha, Q_p, F(p,q,s)$ and Q_k have been studied by some authors (see, for example, [3,6,7,10,12] and references therein). The purpose of this paper is to study the boundedness and compactness of the generalized composition operators from Bloch type spaces to Q_K type spaces by the $K-$Carleson measure, which can be viewed as a development of the study on spaces Q_K and $F(p,q,s)$. The corresponding problems for Q_K type spaces were studied in [2] and [4]. For $a \in D$, the Green's function with logarithmic singularity at a is denoted by $g(z,a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$. Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous function. For $0 < p < \infty, -2 < q < \infty$, we say $f \in Q_K(p,q)$ provided

$$\|f\|^p_{K,p,q} = \sup_{a\in D} \int_D |f'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < \infty,$$

and $f \in Q_{K,0}(p,q)$ provided

$$\lim_{|a|\to 1} \int_D |f'(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) = 0,$$

where dA means the normalized Lebesgue area measure on D such that $A(D) = 1$. $Q_{K,0}(p,q)$ is the subspace of $Q_K(p,q)$, and $Q_K(p,q)$ is a Banach space under the norm $\|f\|_{Q_K(p,q)} = |f(0)| + \|f\|_{K,p,q}$. Throughout the paper, we always assume that K satisfies the following conditions:

(a) K is nondecreasing;
(b) K is two times differentiable on $(0,1)$;
(c) $\int_0^{1/e} K(\log(1/r)) r dr < \infty$;
(d) $K(t) = K(1) > 0, t \geq 1$;
(e) $K(2t) \approx K(t), t \geq 0$.

Also, we assume that $\displaystyle\int_0^1 (1-r^2)^q K(\log \frac{1}{r}) r dr < \infty$. Otherwise, $Q_K(p,q)$ contains constant functions only (see [11]). In order to obtain the main results in this paper, we further assume that $\displaystyle\int_0^1 \varphi_k(s) \frac{ds}{s} < \infty$, where

$$\varphi_k(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \quad 0 < s \leq \infty.$$

If $p = 2, q = 0$, we have that $Q_K(p,q) = Q_K$. If $K(t) = t^s$, then $Q_K(p,q) = F(p,q,s)$. We know from [11] that $Q_K(p,q) \subset B^{\frac{q+2}{p}}$. For $0 < \alpha < \infty$, a function $f \in H(D)$ is said to belong to the Bloch type

space B^α with the norm

$$\|f\|_\alpha = \sup_{z\in D} |f'(z)|(1-|z|^2)^\alpha.$$

Let B_0^α denote the subspace of B^α consisting of those $f \in B^\alpha$ such that

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0.$$

B^α is also a Banach space under the norm $\|f\|_{B^\alpha} = |f(0)| + \|f\|_\alpha$.

For a subarc $I \subset \partial D$, the boundary of D, let

$$S(I) = \{r\xi \in D : 1-|I| < r < 1, \xi \in I\},$$

where $|I|$ denotes the arc length of $I \subset \partial D$. If $|I| \geq 1$ then we set $S(I) = D$. A positive measure μ on D is said to be a $K-$Carleson measure if

$$\sup_{I\subset\partial D} \int\limits_{S(I)} K(\frac{1-|z|}{|I|})d\mu(z) < \infty.$$

If

$$\lim_{|I|\to 0} \int\limits_{S(I)} K(\frac{1-|z|}{|I|})d\mu(z) = 0,$$

then we say μ is a vanishing $K-$Carleson measure. Clearly, if $K(t) = t^p, 0 < p < \infty$, then μ is a $K-$Carleson measure if and only if $(1-|z|^2)^p d\mu(z)$ is a $p-$Carleson measure. Note that $p = 1$ give the classical Carleson measure.

In this paper, two quantities A_1 and A_2 are said to be equivalent if there exist two finite positive constants C_1 and C_2 such that $C_2A_2 \leq A_1 \leq C_1A_2$, written as $A_1 \approx A_2$. Throughout this paper, C always denote positive constants and may be different at different occurrences.

2. Preliminaries and lemmas

Lemma 2.1. [7] *Let $\alpha > 0$, there are two functions $f_1, f_2 \in B^\alpha$ such that*

$$|f_1'(z)| + |f_2'(z)| \geq \frac{1}{(1-|z|^2)^\alpha}$$

for all $z \in D$.

Lemma 2.2. *A positive measure μ on D is a $K-$Carleson measure if and only if*

$$\sup_{a\in D} \int_D K(1-|\varphi_a(z)|^2)d\mu(z) < \infty,$$

and a positive measure μ on D is a vanishing $K-$ Carleson measure if and only if

$$\lim_{|a|\to 1}\int_D K(1-|\varphi_a(z)|^2)d\mu(z)=0.$$

Proof. It can be obtained from the definitions of $K-$Carleson measure, compact $K-$Carleson measure and Theorem 2.1 in [9]. □

Lemma 2.3. *Let $0<p<\infty, -2<q<\infty$. Then*

$$\|f\|^p_{K,p,q}\approx \sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z),$$

and $f\in Q_{K,0}(p,q)$ if and only if

$$\lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z)=0.$$

Proof. It can be obtained from the proof of Theorem 2 in [11]. □

Lemma 2.4. *Let $0<p<\infty, -2<q<\infty$. Then a closed set K in $Q_{K,0}(p,q)$ is compact if and only if it is bounded and satisfies*

$$\lim_{|a|\to 1}\sup_{f\in K}\int_D |f'(z)|^p(1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z)=0.$$

Proof. It can be proved similar to Lemma 1 in [8]. □

Lemma 2.5. **[5]** *Let $0<\alpha<\infty, g\in H(D)$ and φ be an analytic self-map of D. Suppose that X is a Banach space. Then $C^g_\varphi : B^\alpha_0\to X$ is compact if and only if $C^g_\varphi : B^\alpha_0\to X$ is weakly compact.*

Lemma 2.6. *Let $\alpha>0, 0<p<\infty, -2<q<\infty, g\in H(D)$ and φ be an analytic self-map of D. Then the operator $C^g_\varphi : B^\alpha\to Q_K(p,q)$ is compact if and only if for any bounded sequence $\{f_n\}$ in B^α which converges to zero uniformly on compact subsets of D as $n\to\infty$, we have $\|C^g_\varphi f_n\|_{Q_K(p,q)}\to 0$ as $n\to\infty$.*

Proof. It can be proved similar to Proposition 3.11 in [1]. □

Lemma 2.7. *Let $\alpha>0, 0<p<\infty, -2<q<\infty, g\in H(D)$ and φ be an analytic self-map of D. Suppose that $C^g_\varphi : B^\alpha\to Q_K(p,q)$ is compact, then for every $a\in D$,*

$$\lim_{r\to 1}\int_{|\varphi|>r} |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)=0.$$

Proof. Let $f_n(z) = n^{\alpha-1}z^n$, we can check that $\lim_{n\to\infty}\sup_{z\in D}|f_n'(z)|(1-|z|^2)^\alpha = (2\alpha)^\alpha e^{-\alpha}$, and thus $\{f_n\}$ is norm bounded in B^α and converges to zero uniformly on compact subsets of D. In view of Lemma 2.6 it follows that $\|C_\varphi^g f_n\|_{Q_K(p,q)} \to 0$, as $n\to\infty$. Therefore, for given $\varepsilon > 0$ and each $a \in D$, there is an $N > 1$ such that if $n \geq N$, then

$$\int_D |f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

Given $r \in (0,1)$, we have

$$n^{\alpha p}r^{(n-1)p}\int_{|\varphi|>r}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

Choosing r so that $n^{\alpha p}r^{(n-1)p} = 1$, we complete the proof. □

Lemma 2.8. *Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and φ be an analytic self-map of D. Suppose $C_\varphi^g : B^\alpha \to Q_K(p,q)$ is compact, then for every $a \in D$,*

$$\lim_{r\to 1}\sup_{\|f\|_{B^\alpha}\leq 1}\int_{|\varphi|>r}|f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

Proof. Let B_α denote the unit ball of B_α and $f_t(z) = f(tz), t \in (0,1)$. Suppose $f \in B_\alpha$, then $f_t \to f$ uniformly on compact subsets of D as $t \to 1$ and $\{f_t\}$ is bounded on B^α, thus for any $\varepsilon > 0$ and $a \in D$ there is a $t \in (0,1)$ such that

$$\int_D |f_t'(\varphi(z)) - f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \frac{\varepsilon}{2^p}.$$

On the other hand, by Lemma 2.7, for above ε there is a $\delta = \delta(f,\varepsilon), \delta \in (0,1)$ such that as $r \in [\delta, 1)$

$$\int_{|\varphi|>r}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \frac{\varepsilon}{2^p\sup_t\|f_t'\|_\infty^p}.$$

Therefore, by the triangle inequality, we obtain that

$$\begin{aligned}
&\int_{|\varphi|>r}|f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
=\ &\int_{|\varphi|>r}|f'(\varphi(z)) - f_t'(\varphi(z)) + f_t'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
<\ &\varepsilon.
\end{aligned}$$

Next we will show that above $\delta = \delta(f, \varepsilon)$ in fact is independent of f. Since $C_\varphi(B_\alpha)$ is relatively compact in $Q_K(p,q)$. It means that there are $f_1, f_2, \cdots, f_m \in B_{B_\alpha}$ such that for any $\varepsilon > 0$ and each $f \in B_{B_\alpha}$ there is a $k, k = 1, 2, \cdots, m$ such that

$$\int_D |f'(\varphi(z)) - f_k'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < \varepsilon.$$

If $\max\limits_{1\le k\le m} \delta_k(f_k, \varepsilon) = \delta < r < 1$, from above proof we have for all $k = 1, 2, \cdots, m$,

$$\int_{|\varphi|>r} |f_k'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < \varepsilon.$$

Hence for any $f \in B_{B_\alpha}$, by the triangle inequality, we obtain that

$$\int_{|\varphi|>r} |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a)) dA(z) < 2^p \varepsilon.$$

Thus the proof is completed. □

3. Main results

Theorem 3.1. *Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and φ be an analytic self-map of D. Then following statements are equivalent:*

(1) $C_\varphi^g : B^\alpha \to Q_K(p,q)$ *is bounded;*

(2) $C_\varphi^g : B_0^\alpha \to Q_K(p,q)$ *is bounded;*

(3) $\sup\limits_{a\in D} \int_D \dfrac{|g(z)|^p (1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a)) dA(z) < \infty$;

(4) $\frac{|g(z)|^p (1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} dA(z)$ *is a $K-$Carleson measure.*

Proof. (1) ⇒ (2) It is obvious.

(2) ⇒ (3) For $f \in B^\alpha$ if we set $f_s(t) = f(st)$ for $0 < s < 1$, then $f_s \in B_0^\alpha$ and $\|f_s\|_{B^\alpha} \le \|f\|_{B^\alpha}$. Thus, by the condition (2) for all $f \in B^\alpha$, we have

$$\|C_\varphi^g(f_s)\|_{Q_K(p,q)} \le \|C_\varphi^g\| \|f_s\|_{B^\alpha} \le C\|f\|_{B^\alpha}.$$

From Lemma 2.1, there exist f_1 and $f_2 \in B^\alpha$ such that

$$\begin{aligned}
&\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|s\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)\\
&\quad\le\ 2^{p-1}\sup_{a\in D}\int_D |f_1'(s\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&\qquad +2^{p-1}\sup_{a\in D}\int_D |f_2'(s\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(g(z,a))dA(z)\\
&\quad\le\ C\|C_\varphi^g(f_{1s})\|_{Q_K(p,q)}+C\|C_\varphi^g(f_{2s})\|_{Q_K(p,q)}\\
&\quad\le\ C\|C_\varphi^g(f_1)\|_{B^\alpha}+C\|C_\varphi^g(f_2)\|_{B^\alpha}<\infty.
\end{aligned}$$

.

Applying Fatou's lemma to the above inequality, we get

$$\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)<\infty.$$

<u>(3) ⇒ (4)</u> From properties of K and the condition (3), we obtain

$$\begin{aligned}
&\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(1-|\varphi_a(z)|^2)dA(z)\\
&\qquad\le\ \sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(2g(z,a))dA(z)\\
&\qquad\approx\ \sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)<\infty.
\end{aligned}$$

Thus, by Lemma 2.2, $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}dA(z)$ is a K-Carleson measure.

<u>(4) ⇒ (1)</u> For any $f\in B^\alpha$, we have

$$\begin{aligned}
&\sup_{a\in D}\int_D |f'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^qK(1-|\varphi_a(z)|^2)dA(z)\\
&\qquad\le\ \|f\|_{B^\alpha}^p\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(1-|\varphi_a(z)|^2)dA(z).
\end{aligned}$$

In addition, that $(C_\varphi^g f)(0)=0$. From Lemma 2.2 and Lemma 2.3, we have that $C_\varphi^g: B^\alpha\to Q_K(p,q)$ is bounded. □

Theorem 3.2. *Let $\alpha>0, 0<p<\infty, -2<q<\infty, g\in H(D)$ and φ be an analytic self-map of D. Then following statements are equivalent:*

(1) $C_\varphi^g: B^\alpha\to Q_{K,0}(p,q)$ *is bounded;*

(2) $C_\varphi^g : B^\alpha \to Q_{K,0}(p,q)$ *is compact;*

(3) $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ *is weakly compact;*

(4) $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ *is compact;*

(5) $\lim\limits_{|a|\to 1} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a))dA(z) = 0;$

(6) $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} dA(z)$ *is a vanishing* $K-$ *Carleson measure.*

Proof. (1) $\Leftrightarrow$ (3) Since $(C_\varphi^g)^{**}((B_0^\alpha)^{**}) = C_\varphi^g(B^\alpha) \subset Q_{K,0}(p,q)$, it follows from Gantmacher's theorem.

(3) $\Leftrightarrow$ (4) It follows from Lemma 2.5.

(4) $\Rightarrow$ (5) Assume that $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ is compact, then $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ is weakly compact. From above proofs we have that $C_\varphi^g : B^\alpha \to Q_{K,0}(p,q)$ is bounded. Hence, as in the proof of Theorem 3.1, there exist $f_1, f_2 \in B^\alpha$ such that

$$\begin{aligned}
&\lim_{|a|\to 1} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a)))dA(z) \\
&\quad \leq \; 2^{p-1} \lim_{|a|\to 1} \int_D |f_1'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\
&\qquad +2^{p-1} \lim_{|a|\to 1} \int_D |f_2'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\
&\quad = \; 0.
\end{aligned}$$

(5) $\Rightarrow$ (6) From properties of K and the condition (5), we obtain

$$\begin{aligned}
&\lim_{|a|\to 1} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(1-|\varphi_a(z)|^2)dA(z) \\
&\quad \leq \; \lim_{|a|\to 1} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(2g(z,a))dA(z) \\
&\quad \approx \; \lim_{|a|\to 1} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a))dA(z) \\
&\quad = \; 0.
\end{aligned}$$

Thus, by Lemma 2.2, $\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} dA(z)$ is a compact K-Carleson measure.

(6) $\Rightarrow$ (2) It follows from Lemma 2.4 that C_φ^g is compact if and only if

$$\lim_{|a|\to 1} \sup_{\|f\|_{B^\alpha}\leq 1} \int_D |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

For $\|f\|_{B^\alpha} \le 1$, by Lemma 2.3, we have

$$\begin{aligned}
&\int_D |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\
&\approx \int_D |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(1-|\varphi_a(z)|^2)dA(z) \\
&= \int_D |f'(\varphi(z))|^p (1-|\varphi(z)|^2)^{p\alpha} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(1-|\varphi_a(z)|^2)dA(z).
\end{aligned}$$

Since $\sup\limits_{\|f\|_{B^\alpha}\le 1} |f'(w)|(1-|w|^2)^\alpha = 1$ for each $w \in D$, from Lemma 2.2 we get the desired result.

$\underline{(2) \Rightarrow (1)}$ It is obvious. □

Theorem 3.3. *Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and φ be an analytic self-map of D. Then following statements are equivalent:*

(1) $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ *is bounded;*

(2)

$$\lim_{|a|\to 1} \int_D |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0$$

and

$$\sup_{a\in D} \int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a))dA(z) < \infty.$$

Proof. $\underline{(2) \Rightarrow (1)}$ Suppose that condition (2) holds and let $f \in B_0^\alpha$. Then for any $\varepsilon > 0$ there is a $r \in (0,1)$ such that as $r < |w| < 1$, $|f'(w)|^p(1-|w|^2)^{p\alpha} < \varepsilon$. Thus

$$\begin{aligned}
&\sup_{a\in D} \int_{|\varphi|>r} |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\
&\le \varepsilon \sup_{a\in D} \int_{|\varphi|>r} \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}} K(g(z,a))dA(z) \\
&\le C\varepsilon.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\lim_{|a|\to 1} \int_{|\varphi|\le r} |f'(\varphi(z))|^p |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) \\
&\le \frac{\|f\|_{B^\alpha}^p}{(1-r^2)^{p\alpha}} \lim_{|a|\to 1} \int_{|\varphi|\le r} |g(z)|^p (1-|z|^2)^q K(g(z,a))dA(z) = 0.
\end{aligned}$$

It follows that $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ is bounded.

$\underline{(1) \Rightarrow (2)}$ Suppose that $C_\varphi^g : B_0^\alpha \to Q_{K,0}(p,q)$ is bounded, then $C_\varphi^g : B_0^\alpha \to Q_K(p,q)$ is bounded. By Theorem 3.1, we get that

$$\sup_{a\in D}\int_D \frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z) < \infty.$$

Let $f(z) = z \in B_0^\alpha$, the boundedness of C_φ implies

$$\lim_{|a|\to 1}\int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) = 0.$$

□

Theorem 3.4. *Let $\alpha > 0, 0 < p < \infty, -2 < q < \infty, g \in H(D)$ and φ be an analytic self-map of D. Then following statements are equivalent:*

(1) $C_\varphi^g : B^\alpha \to Q_K(p,q)$ *is compact;*

(2) $C_\varphi^g : B_0^\alpha \to Q_K(p,q)$ *is compact;*

(3) $\sup_{a\in D}\int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \infty$ *and*

$$\lim_{r\to 1}\sup_{a\in D}\int_{|\varphi|>r}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z) = 0.$$

Proof. $\underline{(1) \Rightarrow (2)}$ It is obvious.

$\underline{(2) \Rightarrow (3)}$ Let $\|f\|_{B^\alpha} \le 1$ and $f_t(z) = f(tz)$, then $\|f_t\|_{B^\alpha} \le 1$. Fixed $t \in (0,1)$, set $B_{B^\alpha}^t = \{f_t, f \in B_{B^\alpha}\}$. Then $B_{B^\alpha}^t \subset B_{B_0^\alpha}$. The compactness of C_φ^g implies that $C_\varphi^g(B_{B^\alpha}^t)$ is a relative compact subset of $Q_K(p,q)$. By Lemma 2.8, we see that for any $\varepsilon > 0$ and $a \in D$ there is a $\delta \in (0,1)$ such that as $r \in [\delta, 1)$,

$$\sup_{\|f\|_{B^\alpha}\le 1}\int_{|\varphi|>r}|f_t'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z) < \varepsilon.$$

As in the proof of Theorem 3.1, there exist $f_1, f_2 \in B^\alpha$ such that

$$\begin{aligned}
&\int_{|\varphi|>r}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|t\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)\\
&\quad\le\ C\int_{|\varphi|>r}|f_1'(t\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
&\qquad +C\int_{|\varphi|>r}|f_2'(t\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\
&\quad <\ 2C\varepsilon.
\end{aligned}$$

By Fatou's lemma, we have

$$\lim_{r\to 1}\sup_{a\in D}\int_{|\varphi|>r}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)=0.$$

On the other hand, by choosing $f=z\in B_0^\alpha$ we obtain

$$\sup_{a\in D}\int_D |g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)<\infty.$$

$\underline{(3)\Rightarrow(1)}$ Assume that $\{f_n\}\subset B^\alpha, \|f_n\|_{B^\alpha}\le 1$ and $f_n\to 0$ uniformly on compact subsets of D. By condition (3), for any $\varepsilon>0$ there is a $\delta\in(0,1)$ such that as $r\in[\delta,1)$,

$$\begin{aligned}&\sup_{a\in D}\int_{|\varphi|>r}|f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\ &\qquad\le\ \|f_n\|_{B^\alpha}^p\sup_{a\in D}\int_{|\varphi|>r}\frac{|g(z)|^p(1-|z|^2)^q}{(1-|\varphi(z)|^2)^{p\alpha}}K(g(z,a))dA(z)<\varepsilon.\end{aligned}$$

On the other hand, since $f_n'(\varphi(z))\to 0$ uniformly on $\{z:|\varphi(z)|\le r\}$, for the above $\varepsilon>0$ there is an integer $N>1$ such that as $n\ge N$,

$$\begin{aligned}&\sup_{a\in D}\int_{|\varphi|\le r}|f_n'(\varphi(z))|^p|g(z)|^p(1-|z|^2)^q K(g(z,a)))dA(z)\\ &\quad\le\ \sup_{|\varphi|\le r}|f_n'(\varphi(z))|^p\sup_{a\in D}\int_{|\varphi|\le r}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\ &\quad\le\ \varepsilon\sup_{a\in D}\int_{|\varphi|\le r}|g(z)|^p(1-|z|^2)^q K(g(z,a))dA(z)\\ &\quad\le\ C\varepsilon.\end{aligned}$$

Since $(C_\varphi^g f)(0)=0$, then we obtain $\|C_\varphi^g f_n\|_{Q_K(p,q)}\to 0$, as $n\to\infty$. Therefore $C_\varphi^g: B^\alpha\to Q_K(p,q)$ is compact. □

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