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The convolution algebra $H^1(R)$

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Abstract. $H^1(R)$ is a Banach algebra which has better mapping properties under singular integrals than $L^1(R)$. We show that its approximate identity sequences are unbounded by constructing one unbounded approximate identity sequence $\{v_n\}$. We introduce a Banach algebra Q that properly lies between H^1 and L^1, and use it to show that

$$c(1+\ln n) \leq ||v_n||_{H^1} \leq Cn^{1/2}.$$

We identify the maximal ideal space of H^1 and give the appropriate version of Wiener's Tauberian theorem.

1. Introduction

The Hardy space $H^1(R)$ is known to have better mapping properties than $L^1(R)$; for example the Hilbert transform and other singular integral operators are unbounded on $L^1(R)$, but are bounded on $H^1(R)$. We investigate whether the properties of $H^1(R)$ as a convolution algebra are similarly better than those of $L^1(R)$, in particular for spectral synthesis. In general, $H^1(R)$ is apparently as badly behaved for synthesis as $L^1(R)$, but the cancellation properties of $H^1(R)$ ($\int f(x)\,dx = 0$, if $f \in H^1$) functions do sometimes cause differences of behavior. For example, multiplication by characters $\chi_\xi(x) = e^{-ix\xi}$ is bounded on L^1, but because of the cancellation

condition, can map H^1 into H^1 only if $\xi \in Z(\hat{f})$, despite the fact that multiplication by such a character has no effect on the molecular norm ([7]), and H^1 functions are sums of molecules. We discovered a number of such properties where there are major differences. The Banach algebra $L^1(R)$ has bounded approximate identity sequences, and we construct an unbounded approximate identity sequence for $H^1(R)$. Hence, by a result of Warner-Whitley ([17]), all approximate identity sequences of $H^1(R)$ are unbounded. We give a bound for the rate of increase that depends on the behavior of the approximate identity sequences in a convolution algebra Q lying between $H^1(R)$ and $L^1(R)$, but the bounds are not sharp. We identify the maximal ideal space $\Delta(H^1)$ (all continuous homomorphisms of H^1 onto the complex numbers) of $H^1(R)$ with $\hat{R} \setminus \{0\}$. Thus H^1 is a semisimple Banach algebra. We give appropriate versions of Wiener's ideal theorem (if I is a closed linear subspace of $H^1(R)$, then I is a closed ideal if and only if I is translation invariant) and Wiener's Tauberian theorem (if I is a closed ideal in $H^1(R)$, and

$$Z(I) = \cap_{f \in I}\{\xi \in \Delta(H^1) : \hat{f}(\xi) = 0\} = \emptyset,$$

then $I = H^1(R)$). Finally we show that some operators associated with linear fractional transformations, which preserve $L^1(R)$, also preserve $H^1(R)$.

We have since discovered an explanation for the above results (except for the bound on the growth of the norms) using essential ideals. We will discuss this in a paper to appear ([6]).

2. Preliminaries

The Hardy space $H^1(R)$ consists of the functions $f \in L^1(R)$ such that the Hilbert transform Hf is also in $L^1(R)$. The Hilbert transform is the principal value integral of the kernel $\frac{1}{x}$, that is

$$Hf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\, dy.$$

There are many equivalent norms on H^1 ([15]). If the norm on H^1 is defined by

$$||f||_{H^1} = ||f||_1 + ||Hf||_1,$$

then H^1 becomes a Banach space.

Theorem 2.1. *$H^1(R)$ is a commutative (regular) Banach algebra under convolution, with the H^1 norm.*

This follows immediately from the fact that H is a convolution operator, and $H(f*g) = Hf*g = f*Hg$. Consequently

$$\begin{aligned} ||f*g||_{H^1} &= ||f*g||_1 + ||H(f*g)||_1 \\ &\leq ||f||_1||g||_1 + ||Hf||_1||g||_1 \\ &= ||f||_{H^1}||g||_1 \leq ||f||_{H^1}||g||_{H^1}. \end{aligned}$$

Notice that the argument shows that $L^1 * H^1 \subseteq H^1$, since if $f \in L^1$, $g \in H^1$, $f*g \in H^1$, and $||f*g||_{H^1} \leq ||f||_1||g||_{H^1}$. Therefore H^1 is a module over L^1.

It is well known that the Hilbert transform maps L^p into L^p for $p > 1$, and is unbounded on L^1. Since $\widehat{Hf}(\xi) = -\imath\ sgn(\xi)\hat{f}(\xi)$, and the Fourier transform of an L^1 function is continuous, if $f \in L^1, \hat{f}(0) \neq 0$, $Hf \notin L^1$. Moreover $H^2 f = -f$, and therefore the Hilbert transform maps H^1 onto H^1.

We will repeatedly use two of Fefferman's results about H^1 ([2], [4], [5], [15]). The first is the characterization of the dual space of $H^1, (H^1)^* = BMO$. The second is his theorem (see also [2], p. 131) that if $f \in H^1(R)$,

$$\int \frac{|\hat{f}(\xi)|}{|\xi|}\, d\xi \leq C||f||_{H^1}.$$

Definition 2.2.

$$Q = \{f | f \in L^1, \int \frac{|\hat{f}(\xi)|}{|\xi|}\, d\xi < \infty.\}.$$

We use the obvious norm on Q,

$$\int \frac{|\hat{f}(\xi)|}{|\xi|}\, d\xi + ||f||_1.$$

It is easy to check that Q is a Banach space with this norm, and since the Fourier transform of an L^1 function is bounded, it follows that Q is an ideal in L^1. Fefferman's inequality ([4]) implies that $H^1 \subseteq Q \subseteq L^1_0$, where $L^1_0(R) = \{f | f \in L^1(R), \int_R f(x)\, dx = 0\}$. Since $H^1 \subseteq Q$, $Q^* \subseteq (H^1)^* = BMO$, and since BMO functions are L^∞ functions plus Hilbert transforms of L^∞ functions, they are tempered distributions and have Fourier transforms as distributions. An easy computation shows that

$$Q^* = \{F = G + \psi | G \in L^\infty, \xi\hat{\psi} \in L^\infty\}$$

It follows from the duality result and the known duality result for H^1 that the inclusion of H^1 in Q is proper, and that Q is properly included in L^1_0.

We will frequently need test functions in H^1. We will often construct them as molecules, based on the following result in García-Cuerva and Rubio de Francia ([7]):

Theorem 2.3. *If $g \in L^2(R), xg(x) \in L^2(R)$, and $\int_{-\infty}^{\infty} g(x)\,dx = 0$, then $g \in H^1(R)$, and $||g||_{H^1} \leq 2\sqrt{2}||g||_2^{1/2}||xg||_2^{1/2}$.*

Such functions are called (L^2) molecules centered at 0, (m is a molecule centered at x_0 if $m \in L^2, (x-x_0)m \in L^2$ and $\int_{-\infty}^{\infty} m(x)dx = 0$), and a general H^1 function is an l^1 sum of such molecules ([3]). The constant $c = (||g||_2||xg||_2)^{1/2}$ is called the molecular norm of g ([7]). In particular, any compactly supported L^2 function whose integral is zero is an H^1 molecule. The same proof shows that for any $p > 1$, and $q > 1$ we have:

Theorem 2.4. *If $g \in L^p(R), xg(x) \in L^q(R)$, $1 < p < \infty, 1 < q < \infty$, and $\int_{-\infty}^{\infty} g(x)\,dx = 0$, then $g \in H^1(R)$, and*

$$||g||_{H^1} \leq c_{p,q}||g||_p^{\frac{1/q}{1/p'+1/q}}||xg||_q^{\frac{1/p'}{1/p'+1/q}}.$$

If $f \in L^p$ is compactly supported and has integral zero, it will also be an H^1 molecule with molecular norm bounded by $a^{1/p'}||f||_p$, assuming the support of f is contained in a symmetric interval about 0 of length $2a$.

We will use the ideals associated with synthesis ([12]),

$$I(E) = \{\hat{f} | f \in L^1(R),\ \hat{f}(\xi) = 0, \xi \in E\},$$

$$k(E) = \{f | f \in L^1(R),\ \hat{f}(\xi) = 0, \xi \in E\},$$

$$j(E) = \{f | f \in L^1(R),\ \text{supp}\ \hat{f} \text{ is compact and } \text{supp}\ \hat{f} \cap E = \emptyset\}.$$

H^1 is very similar to $L^1(R)$ (more precisely, to $L^1_0(R)$). As examples of this resemblance, we mention these properties of $H^1(R)$:

(a) Every S-set E for $H^1(R)$ is an S-set for $L^1(R)$. Since $E \subset \Delta(H^1) = \hat{R} \setminus \{0\}, 0 \notin E$. Every compact S-set for $L^1(R)$ can be translated so that 0 is not in the translate, and thus the translate is an S-set for H^1, and the same idea can be applied to compact non-S-sets.

(b) Every Helson set E for H^1 is a Helson set for L^1 (or L^1_0)),

(c) Every C-set for H^1 is a C-set for L^1.

(d) Strong Ditkin sets in the maximal ideal space of $H^1(R)$ are strong Ditkin sets for $L^1(R)$ (see Theorem 3.6).

For general references on synthesis, see ([8], [12],[13]).

There is a difference that complicates matters. In the Banach algebra L^1, every approximate identity sequence is bounded, whereas we show that all approximate identity sequences for H^1 are unbounded.

3. Approximate identities

The definition of an approximate identity frequently includes boundedness as a condition. However, there are also spaces such as $L^1(G)\cap L^2(G)$ which have unbounded approximate identities. The natural norm in $L^1(G)\cap L^2(G)$ is

$$||f||_{L^1(G)\cap L^2(G)} = ||f||_1 + ||f||_2.$$

When $G = R^n$, this space has mixed homogeneity with respect to dilation: if $f_\lambda(x) = \lambda^{-n} f(\lambda x)$, then $||f_\lambda|| = ||f||_1 + \lambda^{-n/2}||f||_2$. Unlike $L^1(R)\cap L^2(R)$, the space $H^1(R)$ shares the homogeneity of L^1, $||f_\lambda||_{H^1} = ||f||_{H^1}$. Nevertheless, its approximate identity sequence is unbounded. We shall modify the de la Vallée-Poussin kernel so that the modified function is in H^1.

Theorem 3.1. *H^1 has an unbounded approximate identity.*

Let u be the de la Vallée-Poussin kernel so that $\hat{u} = 1$, in $[-1,1]$, and $\hat{u} = 0, |x| > 2$. Moreover, $\hat{u}$ is piecewise linear, with $||u||_1 \le 3/2$ ([1]). If $\lambda > 0$, set $u_\lambda(x) := \lambda u(\lambda x)$, so that $||u_\lambda||_1 = ||u||_1$, and set $v_n = u_n - u_{1/n}$. Now $||v_n||_1 \le 3$.

Lemma 3.1.1. $v_n \in H^1$.

Since $\widehat{v_n}$ has compact support and $0 \notin$ supp $\widehat{v_n}$, $v_n \in H^1$ follows from a general fact about functions whose Fourier transform, $\hat{f}$, have compact support not containing zero. The proof of the general fact is an exercise in Torchinsky ([16]); we will prove below that v_n is a molecule and compute an estimate of its molecular norm.

Lemma 3.1.2. $||u_{1/n} * f||_1 \to 0$, *if* $f \in L^1, \hat{f}(0) = 0$.

Proof. The proof actually shows that $||u_\alpha * f||_1 \to 0, \alpha \to 0+$. For any $\epsilon > 0$, choose $K > 0$ such that

$$\int_{|y|>K} |f(y)|\, dy < \frac{\epsilon}{6}.$$

Since $\hat{f}(0) = 0$,

$$\begin{aligned} ||u_\alpha * f||_1 &= \int |u_\alpha * f(x) - u_\alpha(x)\hat{f}(0)|\, dx \\ &\le \int\int |u_\alpha(x-y) - u_\alpha(x)||f(y)|\, dy\, dx, \end{aligned}$$

after an application of Fubini's theorem. Since $u \in L^1(R)$, there exists a $\delta > 0$ such that if $|z| < \delta$, then $\int |u(w+z) - u(w)|\, dw < \frac{\epsilon}{2||f||_1}$.

Choose $\alpha > 0$, sufficiently small that $|\alpha K| < \delta$, and observe that if $|y| \le K, |\alpha y| < \delta$,

$$\int |u(\alpha x - \alpha y) - u(\alpha x)|\, dx < \frac{\epsilon}{2||f||_1}.$$

Thus, we have

$$\begin{aligned} ||u_\alpha * f||_1 &\le \int_{|y|\le K} + \int_{|y|>K} \alpha|u(\alpha x - \alpha y) - u(\alpha x)||f(y)|\, dx\, dy, \\ &\le \int_{|y|\le K} \alpha\frac{\epsilon}{2||f||_1}|f(y)|\, dy + \int_{|y|>K} ||2u_\alpha||_1|f(y)|\, dy, \\ &\le \frac{\alpha\epsilon}{2} + \frac{3\epsilon}{6} < \epsilon. \end{aligned}$$

This holds for every $\epsilon > 0$, concluding the proof. □

Lemma 3.1.3. *The sequence $\{v_n\}$ is an approximate identity for H^1.*

Proof. Note that $v_n * f - f \to 0$ in L^1 norm, $v_n \in H^1$, and $H(v_n * f - f) = v_n * Hf - Hf \to 0$ in L^1, since $Hf \in L^1$, because we are assuming $f \in H^1$. Using the above observations,

$$\begin{aligned} ||v_n * f - f||_{H^1} &= ||v_n * f - f||_1 + ||H(v_n * f - f)||_1 \\ &= ||v_n * f - f||_1 + ||v_n * Hf - Hf||_1 \to 0. \end{aligned}$$

by the above remarks. Thus $\{v_n\}$ is an approximate identity for H^1. □

Lemma 3.1.4. *H^1 is a proper subset of L^1_0.*

Lemma 3.1.5. *The sequence $\{v_n\}$ is not a bounded approximate identity.*

Suppose $||v_n||_{H^1} \le M$, and let $f \in L^1$ with $\hat{f}(0) = 0$. Since we know that for such $f \in L^1$, $v_n * f \to f$ in L^1,

$$||v_n * f - f||_1 \to 0.$$

There is a subsequence $\{v_{n_k}\}$ such that $v_{n_k} * f \to f$ a.e. We apply the Jones-Journé theorem ([10]) to $\{v_{n_k} * f\}$. It is bounded in H^1 by $M||f||_1$, and tends a.e. to f. Thus $f \in H^1$, and $||f||_{H^1} \leq M||f||_1$. In particular, it follows that $Hf \in L^1$, and hence $f \in H^1$. Since f is arbitrarily chosen, $H^1 = L^1_0$, a contradiction. This concludes the proof of Theorem 3.1. □

Since there is an unbounded (in H^1) approximate identity sequence in H^1, every approximate identity sequence is unbounded ([17]). Segal algebras other than L^1 have unbounded approximate identities. However, the fact that $H^1 \subset \overline{(H^1)}^{L^1} = L^1_0$ shows that H^1 is not a Segal algebra.

As we remarked above, H^1 is an L^1 module, and many results requiring an approximate identity can be handled by considering H^1 as such a module, taking an approximate identity from L^1. Newman's Theorem ([11], [14]) is one result in which knowledge of an approximate identity in H^1 is useful. Newman's proof ([11]) was for the torus, and was extended to compact abelian groups in Rudin ([14]) for the complex version of H^1.

Theorem 3.2. (D. J. Newman) *There is no continuous projection $P : L^1 \mapsto H^1$ such that $Pf = f, f \in H^1$.*

Proof. Suppose there were such a P. Take an $f \in L^1_0$. We showed in our proof of the approximate identity that $f * v_n \to f$ in L^1. Thus $P(f * v_n) \to Pf$ in H^1. But since $v_n \in H^1, f * v_n \in H^1$, and hence $P(f * v_n) = f * v_n$. But we proved that for $f \in L^1_0$, $f * v_n \to f$ in L^1. Using L^1 convergence, $Pf = f$ and hence, $f \in H^1$. This implies that $L^1_0 = H^1$, a contradiction of Lemma 3.1.4. □

We can give a bound for the rate of growth of the approximate identity by a simple computation. Recall that $u_n(x) = 2K_{2n}(x) - K_n(x)$, where K_n is the usual Fejér kernel,

$$K_n(x) = \frac{1}{2\pi}\frac{1}{n}\frac{\sin^2(nx/2)}{(x/2)^2}.$$

Theorem 3.3. $||v_n||_{H^1} \leq Cn^{1/2}$.

Proof. We can write

$$\begin{aligned} & v_n(x) \\ = \quad & \frac{1}{2\pi}\frac{1}{n}\left(\frac{\sin^2(2nx/2)}{(x/2)^2} - \frac{\sin^2(nx/2)}{(x/2)^2}\right) - \frac{1}{2\pi}n\left(\frac{\sin^2(x/4n)}{(x/2)^2} - \frac{\sin^2(x/2n)}{(x/2)^2}\right) \\ = \quad & v^1_n(x) - v^2_n(x). \end{aligned}$$

The terms in parentheses are handled similarly. Use the double angle formula, square, and subtract to obtain

$$v_n^1 = \frac{1}{2\pi}\frac{1}{n}\phi_n(x)\frac{\sin^2(nx/2)}{(x/2)^2},$$

where $\phi_n(x) = 4\cos^2(nx/2) - 1$ is a function that is bounded in absolute value independently of n, $1 \le \phi(x) \le 3$ for all x real. The L^2 norm of v_n^1 is bounded by

$$\frac{1}{n}\left(\int_R \frac{\sin^4(nx/2)}{(x/2)^4}\,dx\right)^{1/2},$$

and a change of variables shows that this term is bounded by $Cn^{1/2}$. A similar procedure shows that the L^2 norm of v_n^2 is bounded by $Cn^{-1/2}$.

The other term we need to control in the molecular norm is the L^2 norm of xv_n^1. In the expression for v_n^1, the L^2 norm of xv_n^1 is bounded by

$$\frac{1}{n}\left(\int_R \frac{\sin^4(nx/2)}{(x/2)^2}\,dx\right)^{1/2},$$

and a change of variables shows that this term is bounded by $Cn^{-1/2}$. A similar procedure shows that the L^2 norm of xv_n^2 is bounded by $C(1/n)^{-1/2} = Cn^{1/2}$. Combining these terms, we see that

$$||v_n||_2||xv_n||_2 \le Cn^{1/2}n^{1/2} = Cn,$$

and since the integral of v_n is zero, v_n is a molecule with molecular norm less than or equal to $||v_n||_2^{1/2}||xv_n||_2^{1/2}$. Since the H^1 norm is bounded by the molecular norm, we conclude that

$$||v_n||_{H^1} \le Cn^{1/2}.$$

The lower bound requires the use of the ideal Q of L^1.

We have an explicit formula for $\widehat{v_n}$ which allows for the easy computation of $||v_n||_Q = 4\ln n$. Cutting out a notch of size $\frac{1}{n}$ around zero results in the $\ln n$ behavior. This also proves that $Q \ne L_0^1$, since the norm of v_n in L_0^1 is bounded by 3. $\square$

Theorem 3.4. $2 + 4\ln n \le ||v_n||_{H^1} \le Cn^{1/2}$.

The characterization of the dual space of $H^1(R)$ allows us to determine the maximal ideal space of $H^1(R)$.

Theorem 3.5. *The maximal ideal space $\Delta(H^1)$ of $H^1(R)$ can be identified with $\hat{R} \setminus \{0\}$. This implies that H^1 is semi-simple.*

Proof. If $\xi \in \hat{R} \setminus \{0\}, \phi(f) = \int f(x)e^{-\imath\xi x}\,dx$. Note that ϕ is multiplicative, and $|\phi(f)| \leq 1||f||_1 \leq ||f||_{H^1}$. Thus $\phi \in \Delta(H^1)$.

On the other hand, if 0 is in the maximal ideal space, there is a homomorphism ϕ such that $\phi(f) = \hat{f}(0)$. It follows that $\phi \equiv 0$ on H^1, since $\hat{f}(0) = 0$ for every $f \in H^1$, and so $\phi \notin \Delta(H^1)$.

To prove the converse, let $\phi \in \Delta(H^1)$. This means that ϕ is a continuous linear functional on H^1, and hence it may be written as $\phi(f) = \int f(x)\overline{\psi(x)}\,dx$, where ψ is measurable since $\psi \in BMO$. Moreover, if $f, g \in H^1$, we have

$$\phi(f)\phi(g) = \phi(f * g) = \int (f * g)(x)\overline{\psi(x)}\,dx.$$

Since ψ is measurable, a computation shows that

$$\psi(x + y) = \psi(x)\psi(y), x, y \in R.$$

Thus ψ is an exponential, $\psi(x) = e^{(a+\imath b)x}$. The fact that $\psi \in BMO$ implies that $|\psi| \in BMO$, and hence that $\int_R |\psi(x)|/(1 + |x|)^{-2}\,dx < \infty$. Considering this at ∞, we conclude that $a \leq 0$, and at $-\infty$, that $a \geq 0$. Hence $\psi \in L^\infty$ and $||\psi||_\infty = 1$. The remark above shows that $\xi \in \hat{R} \setminus \{0\}$, because otherwise $\phi \equiv 0$.

For each $\xi \in \Delta(H^1)$ and f in L^1 with $\hat{f}(0) = 0$, there is a compact neighborhood $V(\xi) \in \Delta$ and $g \in H^1$ such that $\hat{g} = \hat{f}$ on $V(\xi)$.

$H^1(R)$ is Tauberian (Condition D at infinity), satisfies Condition D at 0, and satisfies Condition D at each $\xi \in \Delta(H^1)$. [This asserts that every $\{\xi\}$ is a C-set, even if ξ is the point at infinity.] □

Theorem 3.6. *$H^1(R)$ satisfies condition D at each $\xi_0 \in \Delta(H^1)$.*

Proof. Let $\epsilon > 0$ and suppose that $f \in H^1$, and $\hat{f}(\xi_0) = 0$. Then $\widehat{Hf}(\xi_0) = 0$, and so there is a $u \in j(\{\xi_0\})$ such that

$$||u * f - f||_1 < \epsilon/4, ||u * Hf - Hf||_1 < \epsilon/4,$$

and hence,

$$||u * f - f||_{H^1} < \epsilon/2.$$

Since $u * f \in H^1$, there is an n_0 such that for every $n \geq n_0$,

$$||v_n * u * f - u * f||_{H^1} < \frac{\epsilon}{||u||_1}.$$

It follows that

$$||v_n * u * f - f||_{H^1} \leq ||v_n * u * f - u * f||_{H^1} + ||u * f - f||_{H^1},$$

which is less than or equal to

$$||u||_1||v_n * f - f||_{H^1} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

A similar argument proves the assertion made in the introduction about strong Ditkin sets. We use the definition of strong Ditkin sets given in Reiter and Stegeman ([12], p. 294). E is a strong Ditkin set if there is a sequence $\{f_n\}$ in $j(E)$ such that for every $f \in k(E)$, $\lim_{n\to\infty} f * f_n = f$.

Theorem 3.7. *Let E be closed in $\Delta(H^1)$. Then E is a strong Ditkin set for $H^1(R)$ if and only if E is a strong Ditkin set for $L^1(R)$.*

Proof. Let $E \subset \Delta(H^1)$ be a strong Ditkin set for $L^1(R)$. Then there are functions $\{u_n\} \subset j(E)$ such that for every $f \in k(E)$, $||u_n * f - f||_1 \to 0$.

Since $Hf \in k(E)$, we also have $||u_n * Hf - Hf||_1 \to 0$, and hence $||u_n * f - f||_{H^1} \to 0$. However, the functions in the sequence $\{u_n\}$, which only belong to L^1, need to be smoothed. Notice that for every $g \in H^1$,

$$||v_k * g||_{H^1} \leq ||v_k||_1||g||_{H^1} \leq 6||g||_{H^1}.$$

Hence, for every $\epsilon > 0$, there is an n_0 such that if

$$n \geq n_0, \quad ||u_n * f - f||_{H^1} < \epsilon/12,$$

so that

$$||v_k * u_n * f - f||_{H^1} \leq ||v_k * (u_n * f - f)||_{H^1} + ||v_k * f - f||_{H^1}.$$

The sequence $\{v_k\}$ is an approximate identity (sequence), so there is a k_0 such that if $k \geq k_0$, $||v_k * f - f||_{H^1} < \epsilon/2$. For any such k,

$$||v_k * (u_n * f - f)||_{H^1} < 6\frac{\epsilon}{12},$$

and combining this inequality with the previous inequality, we see that if $k \geq k_0, n \geq n_0$, then

$$||v_k * u_n * f - f||_{H^1} < \epsilon,$$

which completes the proof. □

The fact that $H^1(R) = L^1(R) * H^1(R)$ follows from Theorem 32.22 of Hewitt and Ross ([8]). Hewitt and Ross gives the result for a Banach algebra A and a left module L over A. We take $A = L^1, L = H^1$, and note that L^1 is a Banach algebra with a bounded approximate identity which we may

take as $\{v_n\}$ above (in particular, $||v_n||_1 \leq 3$), and since

$$||v_n * f - f||_{H^1} \to 0,\ f \in H^1,$$

the result of Hewitt and Ross shows that $A*H^1$ is a closed linear subspace in H^1. However, since $v_n * f \in A*H^1$, $A*H^1$ is also dense in H^1, which gives the above result. The same result implies, further, that if $f \in H^1, \delta > 0$ are given, there is a $g \in L^1$, $f \in H^1$ such that $f = g * h, ||g||_1 \leq 3, h \in I(f)$, and $||f - h||_{H^1} \leq \delta$.

Using Besov spaces, we could show that $H^1 * H^1$ is a proper subset of H^1, but this would introduce an additional layer of complexity. However, we remark that it would give an alternate proof that H^1 is not equal to L^1_0 because $L^1_0 * L^1_0 = L^1_0$.

4. Mapping properties

The function $F = f + iHf$, for $f \in H^1$, is the boundary value of an ordinary analytic H^1 function in the upper half plane (i.e. the Poisson integral of F). Analytic functions are invariant under linear fractional transformations. Consequently, it is reasonable to ask if their boundary values retain this property. We show that they do. We also show that the Fourier transforms of H^1 functions agree locally with the Fourier transforms of L^1 functions (the points 0 and ∞ excluded).

Theorem 4.1. *If $f \in H^1$, $Tf(x) = \frac{1}{x^2} f(\frac{1}{x}) \in H^1(R)$.*

Proof. The proof relies on the fact that if $f \in L^1$, then $Tf \in L^1$ by a simple change of variables. Moreover, if the integral of f on R is zero, the integral of Tf is also zero. We further note that $xTf(x) = \frac{1}{x} f(\frac{1}{x}) = Uf(x)$, where U is a unitary map on L^2 that commutes with the Hilbert transform up to sign ([9]). For functions with integral zero, like Tf, $H(xTf) = xHTf$. However, $H(xTf) = HUf = -UHf(x)$, and we see that $HTf = \frac{1}{x} H(xTf) = \frac{1}{x} HUf = -\frac{1}{x} UHf = THf(x)$. Thus if $f \in H^1$, $f \in L^1, Hf \in L^1, Tf \in L^1, H(Tf) = -THf \in L^1$; we conclude that $Tf \in H^1$, and that

$$||Tf||_{H^1} = ||Tf||_1 + ||HTf||_1 = ||Tf||_1 + ||THf||_1 = ||f||_1 + ||Hf||_1 = ||f||_{H^1}.$$

□

The fact that any linear fractional transformation can be written as a composition of linear transformations and the above transformation leads to the following corollary.

Corollary 4.2. *If g is a linear fractional transformation, $g(x) = \frac{ax+b}{cx+d}$, where a, b, c, d are real numbers, $T_g f(x) = |g'(x)| f(g(x))$, then $T_g : H^1 \mapsto H^1$, and $||T_g f||_{H^1} \leq ||f||_{H^1}$.*

The local behavior at points of R determines H^1 in the same way it determines L^1.

Theorem 4.3. *Let $f \in L^1_0$. If $\hat{f}$ belongs locally to $\widehat{H^1}$ at 0 and ∞, then $f \in H^1$.*

Lemma 4.3.1. [13] *If $f \in L^1_0$, and $0 < |\xi| < \infty$, then $\hat{f}$ belongs locally to $\widehat{H^1}$ at ξ.*

Lemma 4.3.2. *If $\hat{f}$ has compact support and $\hat{f}$ belongs locally to $\widehat{H^1}$ at 0, then $f \in H^1$.*

Proof. Since $\hat{f} \in \widehat{H^1}$ locally at 0, there is a $w \in H^1$ such that supp $(\hat{w})$ is compact, and $\hat{f} = \hat{w}$ in a compact neighborhood of 0. Thus supp $(\widehat{f - w})$ is compact and does not contain 0, so by Lemma 3.1.1, $f - w \in H^1$, and therefore $f \in H^1$. □

Proof of Theorem 4.3. Since $\hat{f}$ belongs to $\widehat{H^1}$ at infinity, there exists a $k \in H^1$ such that $\hat{f} = \hat{k}$ off some compact set, so supp $(\widehat{f - k})$ is compact; thus $(\widehat{f - k})$ has compact support and $(\widehat{f - k})$ belongs locally to $\widehat{H^1}$ at 0. Therefore $(f - k) = h \in H^1$ by Lemma 4.3.2, and $f = k + h \in H^1$.

This is now an alternate proof of Lemma 4.3.2 (really the same proof). □

Corollary 4.4. *Let $f \in L^1(R), \hat{f}(0) = 0$ and suppose that supp $\hat{f}$ is compact. If for each $\xi \in$ supp $\hat{f}$, there exists a compact neighborhood $V(\xi)$ and an $h \in H^1(R)$ such that $\hat{f} = \hat{h}$ on V, then $f \in H^1(R)$.*

Proof. Clearly f satisfies the hypotheses of Theorem 4.3. □

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