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L^∞-Estimates of the Bergman projection in the Lie ball of $\mathbb{C}^n$

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Abstract. In this paper, we consider estimates with loss for the Bergman projections of bounded symmetric domains of $\mathbb{C}^n$ in their Harish-Chandra realizations. This paper is twofold: on one side we develop transfer methods between these bounded domains and their Cayley transform; on the other side we give a new range of q such that the Bergman projection is bounded from $L^\infty(\mathcal{B}_n)$ to $L^q(\mathcal{B}_n)$ where $\mathcal{B}_n$ is the Lie ball of $\mathbb{C}^n$.

1. Introduction

Let $n \geq 3$. We consider the Lie ball

$$\mathcal{B}_n = \left\{ z \in \mathbb{C}^n : \left| \sum_{j=1}^n z_j^2 \right|^2 < 1, 1 - 2|z|^2 + \left| \sum_{j=1}^n z_j^2 \right|^2 > 0 \right\}$$

of $\mathbb{C}^n$, which is a bounded symmetric domain in $\mathbb{C}^n$ of rank 2 isomorphic to the tube domain $T_{\Lambda_n} = \mathbb{R}^n + i\Lambda_n$ over the Lorentz cone

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1 > 0,\ y_1^2 - y_2^2 - \cdots - y_n^2 > 0\}.$$

Note that the Lie ball $\mathcal{B}_n$ is a bounded domain of $\mathbb{C}^n$ while the tube domain $T_{\Lambda_n} = \mathbb{R}^n + i\Lambda_n$ is unbounded.

Let $q \geq 1$ and dv be the Lebesgue measure of $\mathbb{C}^n$; let D be a domain of $\mathbb{C}^n$. The *Bergman space* $A^q(D, dv)$ is the closed subspace of $L^q(D, dv)$ consisting of holomorphic functions on D. The space $A^2(D, dv)$ is therefore a Hilbert space. The orthogonal projection of the Hilbert space $L^2(D, dv)$ onto its closed subspace $A^2(D, dv)$ is called the *Bergman projection* on $A^2(D, dv)$.

In what follows, we shall denote $\mathcal{P}$ (resp. P) the Bergman projection on $A^2(\mathcal{B}_n, dv)$ (resp. $A^2(T_{\Lambda_n}, dv)$. Recall that the Bergman projection $\mathcal{P}$ (resp. P) is an integral operator associated with the kernel $\mathbf{B}(\cdot,\cdot)$ (resp. $B(\cdot,\cdot)$) called the *Bergman kernel* on $\mathcal{B}_n$ (resp. T_{Λ_n}) i.e the reproducing kernel of $A^2(\mathcal{B}_n, dv)$ (resp. $A^2(T_{\Lambda_n}, dv)$).

Since $\mathcal{B}_n$ is a bounded domain, one can as well ask for (L^p, L^q) estimates with $1 < q \leq p$. The case $q < p = \infty$ is of special interest because the Bergman projection of $L^\infty(\mathcal{B}_n)$ can be described as the *Bloch space of holomorphic functions*. (Cf. [11]). In the paper [1], David Békollé and Aline Bonami proved that the Bergman projection $\mathcal{P}$ is bounded on $L^q(\mathcal{B}_n, dv)$ whenever $\frac{2n-2}{n} < q < \frac{2n-2}{n-2}$. Since $\mathcal{B}_n$ is bounded, we conclude that the Bergman projection is bounded from $L^\infty(\mathcal{B}_n)$ to $A^q(\mathcal{B}_n, dv)$ whenever $\frac{2n-2}{n} < q < \frac{2n-2}{n-2}$. However, D. Békollé and A. Bonami made an interesting observation about the behaviour of the Bergman projection $\mathcal{P}$; they realized that the Bergman projection $\mathcal{P}$ behaves differently when it is considered as an operator defined from $L^\infty(\mathcal{B}_n)$ to $A^q(\mathcal{B}_n, dv)$. Precisely, they consider the positive Bergman operator $\mathcal{P}^+$ i.e the Bergman operator with positive Bergman kernel. Note that the boundedness of the positive Bergman operator implies the boundedness of the Bergman operator and the converse is not always true. They established that $\mathcal{P}^+$ is bounded from $L^\infty(\mathcal{B}_n)$ to $L^q(\mathcal{B}_n, dv)$ if and only if some integral on the intersection of the unit ball of $\mathbb{R}^n$ and the Lorentz cone is finite. They used a transfer principle and the formula of change of variable of Bergman kernels to carry the problem from the Lie ball to its unbounded realization. A good estimation of this integral led to the necessary and sufficient condition $q < \frac{2n}{n-2}$ for $\mathcal{P}^+$, which is sufficient for $\mathcal{P}$. Moreover still in [1], authors proved this negative result: the Bergman projection $\mathcal{P}$ is unbounded from $L^\infty(\mathcal{B}_n, dv)$ on $A^q(\mathcal{B}_n, dv)$ for $q > \frac{4n}{n-2}$. Then what is the behaviour of the Bergman projection $\mathcal{P}$ when

$q \in \left[\frac{2n}{n-2}, \frac{4n}{n-2}\right]$? In this paper, we give a partial answer to this question. We prove the following:

Theorem 1.1. *The Bergman Projection $\mathcal{P}$ is bounded from $L^\infty(\mathcal{B}_n, dv)$ to $A^q(D, dv)$ whenever $q < \frac{2n}{n-2} + \frac{n-2}{n-1}$. Furthermore, the Bloch space of holomorphic functions defined in $\mathcal{B}_n$ is continuously included in $A^q(\mathcal{B}_n, dv)$ whenever $q < \frac{2n}{n-2} + \frac{n-2}{n-1}$.*

In order to prove the result above, we follow exactly what have been done in [1], i.e we carry the problem from the bounded symmetric domain $\mathcal{B}_n$ to its unbounded realization $T_{\Lambda_n} = \mathbb{R}^n + i\Lambda_n$ via a transfer principle based on two tools:

(i) the biholomorphic transformation Φ given in [6] which exchanges the two domains;

(ii) the following well-known change of variable formula for the Bergman kernel

$$B_{\mathcal{B}}(z', w') = B_{T_{\Lambda_n}}(\Phi(z'), \Phi(w'))J_{\mathbb{C}}\Phi(z')\overline{J_{\mathbb{C}}\Phi(w')}$$

where $J_{\mathbb{C}}\Phi$ is the complex Jacobian of Φ. Once in the unbounded domain, we shall define a type of mixed normed spaces which will help us to obtain our improvement by interpolation.

The paper is organized as follows: section 1 is the introduction above. In section 2, in a more general setting, we define, in Jordan algebra terms, a 'Cayley transform' i.e the isomorphism mapping a bounded domain, the Harish-Chandra realization of the symmetric domain to the tube domain T_Ω over a symmetric cone Ω. This helps us in section 3 to carry the problem from the bounded symmetric domain, to its unbounded realization T_Ω via a transfer principle which generalizes the one done in [1, Section 3]. From there, we will deal with an integral $I(q, \Omega)$ depending on q and the cone Ω. The purpose of this paper is to show how in the case of rank 2, a good approximation of this integral leads us to the estimations with loss on the one hand, and on the other hand, to improve the results of [1] on two different aspects. More precisely, we shall study the problem in the weighted case where we give a new proof of the estimation of the integral $I(q, \Lambda_n)$; moreover, by interpolation, we shall improve the range of q. The last section is devoted to the proof of Theorem 1.1. Note that in higher rank, the difficulty of the computation of $I(q, \Omega)$ increases considerably. This will be addressed in a joint work with A. Bonami and G. Garrigós.

2. Preliminaries

In this section, we start with a background on symmetric cones. Next, we recall the expression of the Cayley transform, that is the isomorphism

mapping a bounded symmetric domain D in its Harish-Chandra realization onto an unbounded tube domain T_Ω over a symmetric cone Ω. We end this section by the recall of the formula for the respective Bergman projections.

2.1 Symmetric cones. We recall some technical results about symmetric cones. Let Ω be an open convex cone in a n-dimensional Euclidean vector space V i.e for x, $y \in \Omega$, and λ, $\mu > 0$, we have $\lambda x \in \Omega$ and $\lambda x + \mu y \in \Omega$. We assume that Ω is *symmetric*, that is, the group $G(\Omega)$ of all transformations of $GL(V)$ which leave invariant Ω acts transitively on Ω, and Ω is self-dual i.e $\Omega = \{\xi \in V : (x|\xi) > 0,\ x \in \overline{\Omega} \setminus \{0\}\}$. Vinberg [9] shows that there is a solvable subgroup H of $G(\Omega)$ acting simply transitively on Ω. That is, every $y \in \Omega$ can be written uniquely as $y = t\mathbf{e}$, with $t \in H$ and a fixed $\mathbf{e} \in \Omega$. This gives the identification $\Omega \equiv H = G/\mathsf{K}$, where K is the maximal compact subgroup $\{g \in G : g\mathbf{e} = \mathbf{e}\} = G \cap O(V)$ and G is the identity component of $G(\Omega)$.

It is well known that for every symmetric cone Ω, its underlying vector space V can be endowed with a multiplication rule which makes it a Euclidean Jordan algebra, with identity element $\mathbf{e}$. For any element $x \in V$, we denote $L(x)$ the linear map of V defined by $L(x)y = xy$. With such multiplication, $\overline{\Omega}$ coincides with the set $\{x^2 : x \in V\}$ of all squares in V. The notions of rank, trace and determinant in Ω are those inherited from Jordan algebra structure of V. (See [7, Chapter II]).

Suppose now that the cone Ω is irreducible and has rank r. Following [7, Chapter IV], we fix a Jordan frame $\{c_1, \ldots, c_r\}$ (that is, a complete system of idempotents), to which we associate a Peirce decomposition of the space V, that is, an orthogonal decomposition $V = \bigoplus_{1 \le i \le j \le r} V_{ij}$ where

$$V_{ii} = \mathbb{R}c_i,\ V_{ij} = \{x \in V : c_i x = c_j x = \frac{1}{2}x\},\ i < j.$$

Then by [7, Theorem VI.3.6], H may be taken as the corresponding solvable Lie group with factors as the semidirect product $H = NA = AN$ of a nilpotent subgroup N (of lower triangular matrices), and an abelian group A (of diagonal matrices). The latter takes the form

$$A = \{P(a) : a = \sum_{i=1}^{r} a_i c_i,\ a_i > 0\}.$$

Still following [7, Chapter VI], we shall denote by $\Delta_1(x), \ldots \Delta_r(x)$ the *principal minors* of $x \in V$, with respect to the Jordan frame $\{c_1, \ldots, c_r\}$. These are invariant functions under the group N,

$$\Delta_k(nx) = \Delta_k(x), \text{ where } n \in N, x \in V, k = 1, \ldots, r$$

and satisfy a homogeneity relation under A,

$$\Delta_k(P(a)x) = a_1^2 \cdots a_k^2 \Delta_k(x), \text{ if } a = a_1c_1 + \cdots + a_rc_r.$$

The determinant function $\Delta(x) = \Delta_r(x)$ is also invariant under K, and satisfies the formula

$$\Delta(gy) = (\det g)^{\frac{r}{n}} \Delta(y);$$

it follows from this formula that an invariant measure in Ω is given by $\Delta^{-\frac{n}{r}}(y)dy$. The Sylvester's Theorem for symmetric cones allows us to write

$$\Omega = \{x \in V : \Delta_k(x) > 0,\ k = 1, \ldots, r\}.$$

We have the following:

Proposition 2.1. [7, Proposition VII.1.2] *For any $x \in \Omega$ the integral*

$$\int_\Omega e^{-(x|\xi)} \Delta^{\mu - \frac{n}{r}}(\xi) d\xi$$

is finite if and only if $\mu > \frac{n}{r} - 1$. In this case,

$$\int_\Omega e^{-(x|\xi)} \Delta^{\mu - \frac{n}{r}}(\xi) d\xi = c_\mu \Delta^{-\mu}(x).$$

2.2 The Cayley transform. Let ω and ω' be two domains in $\mathbb{C}^n$.

Definition 2.2. We say that $F : \omega \to \omega'$ is a *biholomorphic transformation* if F is a holomorphic bijection with its inverse holomorphic. The domains ω and ω' of $\mathbb{C}^n$ are said to be *isomorphic* if there is a biholomorphic transformation that carries ω onto ω'. A biholomorphic transformation $F : \omega \to \omega$ is called an *automorphism* of ω.

We denote by $Aut(\omega)$ the group of all automorphisms of ω.

Definition 2.3. The domain ω is *homogeneous* if the group $Aut(\omega)$ acts transitively on ω; that is, for all $z, w \in \omega$, there is $\Phi \in Aut(\omega)$ such that $z = \Phi(w)$. The domain ω is *symmetric* if it is homogeneous and it exists $z_0 \in \omega$ and an involution $s \in Aut(\omega)$ such that z_0 is an isolated fixed point.

Now, we introduce the tube domain $T_\Omega = V + i\Omega$ of $\mathbb{C}^n$ over the symmetric cone Ω in the n-dimensional vector space V. It is well known that the domain T_Ω is symmetric if and only if Ω is a symmetric cone.

In the sequel, Ω is a symmetric cone of rank r and T_Ω is the associated tube domain. In [10], E. B. Vinberg, S. Gindikin et I. I. Piatetski-Shapiro proved that the tubular domain T_Ω is isomorphic to a bounded symmetric domain, in its Harish-Chandra realization. We denote D this bounded

symmetric domain of $\mathbb{C}^n$. Let us now give, in the language of Jordan algebras, a description of the domain D. This can be found in [7, Chapter X, section 4]. Let $V^{\mathbb{C}}$ be the complexification of V. For $z \in V^{\mathbb{C}}$, we consider the Hermitian bilinear form on $V^{\mathbb{C}}$:

$$(w_1|w_2)_z = ((w_1 \Box \bar{w}_2)z|z)$$

where $w \Box t = L(wt) + [L(w), L(t)]$. We know that this Hermitian form is positive (see [7, pages 198-199]) and the function

$$w \mapsto |w| = \sup_{\|z\| \leq 1} \sqrt{(w|w)_z}$$

is a norm on $V^{\mathbb{C}}$, called the *spectral norm*. The domain D can be defined as the open unit ball for the spectral norm,

$$D = \{w \in V^{\mathbb{C}} : |w| < 1\}$$

or

$$D = \{w \in V^{\mathbb{C}} : I - w \Box \bar{w} >> 0\}$$

where " $>> 0$" means positive definite. It is also relevant to notice that D is a *circular* domain i.e

$$e^{i\theta} z \in D, \ z \in D, \theta \in \mathbb{R}.$$

Theorem 2.4. (Theorem X.4.3 of [7]) *For all $z \in D$ and $w \in T_\Omega$, we put*

$$\Phi(z) = i(e + z)(e - z)^{-1}$$

and

$$\Psi(w) = (w - ie)(w + ie)^{-1}.$$

The map Φ is a biholomorphic transformation from D onto T_Ω with Ψ as its inverse; i.e Φ is a 'Cayley transform' that carries the bounded symmetric domain D onto the unbounded tube domain T_Ω.

Let us remark that for all $z \in D$ and $w \in T_\Omega$, $\Phi(z) = -ie + 2i(e - z)^{-1}$ and $\Psi(w) = e - 2i(w + ie)^{-1}$. It follows that (see for instance [7, page 202])

$$J_{\mathbb{C}}\Phi(z) = (2i)^n \Delta^{-\frac{2n}{r}}(e - z). \tag{1}$$

Similarly,

$$J_{\mathbb{C}}\Psi(w) = (2i)^n \Delta^{-\frac{2n}{r}}(w + ie). \tag{2}$$

2.3 The Bergman kernel. Let $q \geq 1$. We write ω to design domains D and T_Ω. We denote $L^q(\omega)$ (resp. $L^\infty(\omega)$) the space of measurable functions f that satisfy the estimate

$$\|f\|^q_{L^q(\omega)} := \int_\omega |f(z)|^q dv(z) < +\infty \quad (\text{resp. } \|f\|_{L^\infty(\omega)} := \sup_{z\in\omega} |f(z)| < +\infty\ )$$

where dv is the Lebesgue measure of $\mathbb{C}^n$.

We denote by P_ω the unweighted Bergman projection of ω; then

$$P_\omega f(z) = \int_\omega B_\omega(z,w) f(w) dv(w)$$

for all $f \in L^2(\omega, dv)$ where $B_\omega(,)$ is the unweighted Bergman kernel of ω.

Let $\mu > \frac{n}{r} - 1$. For $z' \in D$ and $z \in T_\Omega$, we consider the measures $dA_\mu(z') = \Delta^{\mu-\frac{n}{r}}(e - z'\overline{z'})dv(z')$ and $dV_\mu(z) = \Delta^{\mu-\frac{n}{r}}(\Im m\, z)dv(z)$. These measures coincide with the Lebesgue measure dv of $\mathbb{C}^n$ for $\mu = \frac{n}{r}$. We denote $L^q_\mu(D)$ (resp. $L^q_\mu(T_\Omega)$) the space of measurable functions f that satisfy the estimate

$$\|f\|^q_{L^q_\mu(D)} := \int_D |f(z')|^q dA_\mu(z') < +\infty$$
$$(\text{resp.} \quad \|f\|^q_{L^q_\mu(T_\Omega)} := \int_{T_\Omega} |f(z)|^q dV_\mu(z) < +\infty.)$$

The weighted Bergman space $A^q_\mu(\omega)$ is the closed subspace of $L^q_\mu(\omega)$ consisting of holomorphic functions on ω.

We denote by P_μ (resp. $\mathcal{P}_\mu$) the weighted Bergman projection of T_Ω (resp. D), then

$$P_\mu f(z) = \int_{T_\Omega} B_\mu(z,w) f(w) dV_\mu(w)$$
$$(\text{resp.} \quad \mathcal{P}_\mu f(z') = \int_D \mathbf{B}_\mu(z',w') f(w') dA_\mu(w'))$$

for all $f \in L^2_\mu(T_\Omega)$ (resp. $L^2_\mu(D)$). The weighted Bergman kernel $B_\mu(\cdot,\cdot)$ of T_Ω is given (see [7, page 261] or [3, page 44]) by

$$B_\mu(z,w) = C_\mu \Delta^{-\mu-\frac{n}{r}} \left(\frac{z - \bar{w}}{i}\right)$$

for all $z, w \in T_\Omega$ while an explicit expression of $\mathbf{B}_\mu(,)$ is given in [7, Proposition XIII.1.4] by

$$\mathbf{B}_\mu(z',w') = C_\mu \Delta^{-\mu-\frac{n}{r}}(e - z'\overline{w'})$$

for all $z', w' \in D$.

We denote by $\mathcal{P}_\mu^+$ (resp. P_μ^+) the positive Bergman operator defined on $L^2_\mu(D)$ (resp. $L^2_\mu(T_\Omega)$) by

$$\mathcal{P}_\mu^+ f(z') = \int_D |\mathbf{B}_\mu(z', w')| f(w') dA_\mu(w')$$

(resp.

$$P_\mu^+ f(z) = \int_{T_\Omega} |B_\mu(z, w)| f(w) dV_\mu(w)).$$

Let z', $w' \in D$; remark that

$$\Phi(z') - \overline{\Phi(w')} \quad = \quad 2i(e - z')^{-1}(e - z'\overline{w'})(e - \overline{w'})^{-1}.$$

We deduce the formula of change of variable between the weighted Bergman kernels $B_\mu(,)$ and $\mathbf{B}_\mu(,)$:

$$\mathbf{B}_\mu(z', w') = 2^{r\mu+n} B_\mu(\Phi(z'), \Phi(w')) \Delta^{-\mu-\frac{n}{r}}(e - z') \Delta^{-\mu-\frac{n}{r}}(e - \overline{w'}). \tag{3}$$

We have also this identity:

$$\Delta^{\mu-\frac{n}{r}}(e - z'\overline{z'}) = \Delta^{\mu-\frac{n}{r}}(\Im m\, \Phi(z')) |\Delta^{2\mu-2\frac{n}{r}}(e - z')|. \tag{4}$$

We recall these results:

Lemma 2.5. [3, Lemma 3.20] *Let $\alpha \in \mathbb{R}$. The integral*

$$J_\alpha(y) = \int_V \left| \Delta^{-\alpha}\left(\frac{x + iy}{i}\right)\right| dx \qquad (y \in \Omega) \tag{5}$$

converges if and only if $\alpha > \frac{2n}{r} - 1$. In this case,

$$J_\alpha(y) = c_\alpha \Delta^{-\alpha+\frac{n}{r}}(y).$$

Lemma 2.6. [3, Page 56] *Let $\alpha \in \mathbb{R}$ and $0 < \lambda < \frac{1}{4}$. There is a constant C_α such that for all $y \in \Omega$, $|y| < \lambda$,*

$$\int_{\{x \in V : |x| < 1\}} \left| \Delta^{-\alpha}\left(\frac{x + iy}{i}\right)\right| dx \geq C_\alpha \Delta^{-\alpha+\frac{n}{r}}(y).$$

3. The transfer principle

In this section, we describe how we carry the problem from the bounded symmetric domain D to the unbounded tube domain T_Ω. We shall study

separately the positive Bergman operator and the Bergman projection. We shall use the method of [1]. We have $\Phi(0) = ie$ and Φ (resp. Ψ) is holomorphic in $\mathbb{C}^n \setminus \Sigma'$ (resp. $\mathbb{C}^n \setminus \Sigma$) where

$$\Sigma' = \{z' \in \mathbb{C}^n : \Delta(e - z') = 0\} \qquad (\text{resp.} \qquad \Sigma = \{z \in \mathbb{C}^n : \Delta(z + ie) = 0\}).$$

Lemma 3.1. *Let $\mathcal{C}'$ (resp. $\mathcal{C}$) a compact set of $\mathbb{C}^n$ such that $\mathcal{C}' \cap \Sigma' = \emptyset$ (resp. $\mathcal{C} \cap \Sigma = \emptyset$). Then there are two positive constants ${c'}_1 > 0$ and ${c'}_2 > 0$ (resp. $c_1 > 0$ et $c_2 > 0$) such that*

$$ {c'}_1 \leq |\Delta(e - z')| \leq {c'}_2 \qquad (\text{resp.} \qquad c_1 \leq |\Delta(z + ie)| \leq c_2)$$

for all $z' \in \mathcal{C}'$ (resp. $z \in \mathcal{C}$).

Proof. The function $z' \mapsto |\Delta(e - z')|$ (resp. $z \mapsto |\Delta(z + ie)|$) is strictly positive and continuous on the compact set $\mathcal{C}'$ (resp. $\mathcal{C}$). □

Lemma 3.2. *For all $z', w' \in \overline{D}$, there is a real number $\theta = \theta(z', w')$ and two bounded open neighborhoods $\mathcal{O}^1(e^{i\theta}z')$ and $\mathcal{O}^2(e^{i\theta}w')$ of $e^{i\theta}z'$ and $e^{i\theta}w'$ respectively such that $\overline{\mathcal{O}^1(e^{i\theta}z')} \cap \Sigma' = \emptyset$ and $\overline{\mathcal{O}^2(e^{i\theta}w')} \cap \Sigma' = \emptyset$.*

Proof. See [1, Lemma 3.3]. □

Definition 3.3. Let D be a compact set of $\mathbb{C}^n$ contained in ω. We define by $L^\infty_{\mathsf{D}}(\omega)$ the space of functions $f \in L^\infty(\omega)$ with their support inside the compact set D.

3.1 The case of the positive Bergman operator $\mathcal{P}^+_\mu$. We prove the following:

Proposition 3.4. *The positive Bergman operator $\mathcal{P}^+_\mu$ is bounded from $L^\infty(D, dv)$ to $L^q(D, dv)$ if and only if the following integral:*

$$I(q, \Omega) = \int_{\{t \in \Omega : |t| \leq 1\}} \left(\int_{\{y \in \Omega : |y| \leq 1\}} \Delta^{-\mu}(y + t) \Delta^{\mu - \frac{n}{r}}(y) dy \right)^q \Delta^{\mu - \frac{n}{r}}(t) dt$$

is finite.

We have this lemma:

Lemma 3.5. *The positive operator $\mathcal{P}^+_\mu$ is bounded from $L^\infty(D)$ to $L^q_\mu(D)$ if and only if the following estimation holds*

$$\text{(E)} \qquad \int_D \left(\int_D |\mathbf{B}_\mu(z', w')| dA_\mu(w') \right)^q dA_\mu(z') < +\infty.$$

Proof. Let $f \in L^\infty(D)$,

$$\|\mathcal{P}_\mu^+ f\|^q_{L^q_\mu(D)} \quad \leq \quad \|f\|^q_{L^\infty(D)} \int_D \left(\int_D |\mathbf{B}_\mu(z', w')| dA_\mu(w') \right)^q dA_\mu(z');$$

it follows that if the integral

$$\int_D \left(\int_D |\mathbf{B}_\mu(z', w')| dA_\mu(w') \right)^q dA_\mu(z')$$

is finite then $\mathcal{P}_\mu^+$ is bounded from $L^\infty(D)$ to $L^q_\mu(D)$.

Conversely, if $\mathcal{P}_\mu^+$ is bounded from $L^\infty(D)$ to $L^q_\mu(D)$, then for $f = 1 \in L^\infty(D)$ one has

$$\int_D \left(\int_D |\mathbf{B}_\mu(z', w')| dA_\mu(w') \right)^q dA_\mu(z') \quad = \quad \|\mathcal{P}_\mu^+ f\|^q_{A^q_\mu(D)} < +\infty.$$

□

Let us remark now that the estimation (E) is equivalent to the the following: for all compact set $\mathcal{C}'$ of $\mathbb{C}^n$ such that $\mathcal{C}' \cap \Sigma' = \emptyset$ and the interior of $\mathcal{C}' \cap D$ is nonempty

$$\text{(E}'\text{)} \qquad \int_{\mathcal{C}' \cap D} \left(\int_{\mathcal{C}' \cap D} |\mathbf{B}_\mu(z', w')| dA_\mu(w') \right)^q dA_\mu(z') < +\infty.$$

In fact, if the estimation (E) holds, so is (E′) because $\mathcal{C}' \cap D \subset D$.

Conversely, assume that estimation (E′) holds. For $(z', w') \in \overline{D} \times \overline{D}$, let us denote $\mathcal{O}(z', w')$ an open neighbourhood of (z', w'). Since $\overline{D} \times \overline{D}$ is compact, its open covering $\left\{ \mathcal{O}(z', w') : (z', w') \in \overline{D} \times \overline{D} \right\}$ has a finite subcover $\{\mathcal{O}_j(z', w') : j = 1, ..., N\}$. Now, let us choose $\mathcal{O}(z', w')$ on the form

$$\mathcal{O}(z', w') = \mathcal{O}^1(z') \times \mathcal{O}^2(w')$$

where $\mathcal{O}^1(z')$ and $\mathcal{O}^2(w')$ are open neighbourhoods of z' and w' respectively. So, for $j = 1, \ldots, N$,

$$\mathcal{O}_j(z', w') = \mathcal{O}^1_j(z') \times \mathcal{O}^2_j(w').$$

According to Lemma 3.2, there exists $\theta = \theta(z', w')$ and two bounded open neighbourhoods $\mathcal{O}^1(e^{i\theta} z')$ and $\mathcal{O}^2(e^{i\theta} w')$ such that none of their respective closure intersects Σ'. It follows that

$$(6) \qquad K'_1 = \cup_{j=1}^N \left[\overline{\mathcal{O}^1_j(e^{i\theta_j} z')} \cup \overline{\mathcal{O}^2_j(e^{i\theta_j} w')} \right]$$

is a compact set of $\mathbb{C}^n$ such that $K'_1 \cap \Sigma' = \emptyset$.

From our hypothesis on (E′),

$$\int_{K_1'\cap D}\left(\int_{K_1'\cap D}|\mathbf{B}_\mu(z',w')|dA_\mu(w')\right)^q dA_\mu(z')<+\infty. \tag{7}$$

We need these two lemmas:

Lemma 3.6. *The estimation (7) is equivalent to the fact that the positive operator $\mathcal{P}_\mu^+$ is bounded from $L^\infty_{K_1'}(D)$ to $L^q_\mu(K_1'\cap D)$.*

Proof. See Lemma 3.5. □

Lemma 3.7. *If the positive operator $\mathcal{P}_\mu^+$ is bounded from $L^\infty_{K_1'}(D)$ to $L^q_\mu(K_1'\cap D)$ then $\mathcal{P}_\mu^+$ is bounded from $L^\infty(D)$ to $L^q_\mu(D)$.*

Proof. Let $f\in L^\infty(D)$,

$$\|\mathcal{P}_\mu^+ f\|_{L^q_\mu(D)}=\sup_{\|g\|_{L^{q'}_\mu(D)}\le 1}|\langle\mathcal{P}_\mu^+ f,g\rangle_{(L^q_\mu,L^{q'}_\mu)}|;$$

we have

$$|\langle\mathcal{P}_\mu^+ f,g\rangle_{(L^q_\mu,L^{q'}_\mu)}| \quad\le\quad \sum_{j=1}^N T_j,$$

where

$$\begin{aligned}T_j &= \int\int_{\overline{\mathcal{O}_j^1(e^{i\theta_j}z')}\times\overline{\mathcal{O}_j^2(e^{i\theta_j}w')}}|\mathbf{B}_\mu(z',w')||f(e^{-i\theta_j}w')||g(e^{-i\theta_j}z')|dA_\mu(z')dA_\mu(w')\\ &\le \int_{K_1'\cap D}\int_{K_1'\cap D}|\mathbf{B}_\mu(z',w')||f(e^{-i\theta_j}w')||g(e^{-i\theta_j}z')|dA_\mu(z')dA_\mu(w').\end{aligned}$$

For all $(z',w')\in D\times D$, we put

$$F_j(w')=|f(e^{-i\theta_j}w')|\chi_{K_1'}(w'),\qquad\text{and}\qquad G_j(z')=|g(e^{-i\theta_j}z')|\chi_{K_1'}(z');$$

then $F_j\in L^\infty_{K_1'}(D)$ and $\|F_j\|_{L^\infty_{K_1'}(D)}\le\|f\|_{L^\infty(D)}$. Similarly, $G_j\in L^{q'}_\mu(K_1'\cap D)$ and $\|G_j\|_{L^{q'}_\mu(K_1'\cap D)}\le\|g\|_{L^{q'}_\mu(D)}\le 1$. Hence,

$$|\langle\mathcal{P}_\mu^+ f,g\rangle_{(L^q_\mu,L^{q'}_\mu)}|\le\sum_{j=1}^N|\langle\mathcal{P}_\mu^+ F_j,G_j\rangle_{(L^q_\mu,L^{q'}_\mu)}|$$

and we conclude by our hypothesis that

$$\|\mathcal{P}_\mu^+ f\|_{L^q_\mu(D)} \quad \leq \quad \sum_{j=1}^N \|\mathcal{P}_\mu^+ F_j\|_{L^q_\mu(K_1'\cap D)} \leq C\sum_{j=1}^N \|F_j\|_{L^\infty_{K_1'}(D)} \leq CN\|f\|_{L^\infty(D)}.$$

□

If the estimation (E') holds, so is (7); according to Lemma 3.6, the positive Bergman operator $\mathcal{P}_\mu^+ : L^\infty_{K_1'}(D) \to L^q_\mu(K_1'\cap D)$ is bounded; then by Lemma 3.7, the operator $\mathcal{P}_\mu^+ : L^\infty(D) \to L^q_\mu(D)$ is bounded and we conclude via Lemma 3.5 that the estimation (E) holds.

Now we carry the problem to the unbounded domain T_Ω. Let $\mathcal{C}'$ a compact set $\mathbb{C}^n$ such that $\mathcal{C}' \cap \Sigma' = \emptyset$ and the interior of $\mathcal{C}' \cap D$ is nonempty. Using (3), (4), Lemma 3.1 and the change of variables $z' = \Psi(z),\ w' = \Psi(w)$, we have

$$\int_{\mathcal{C}'\cap D} \left(\int_{\mathcal{C}'\cap D} |\mathbf{B}_\mu(z',w')| dA_\mu(w')\right)^q dA_\mu(z') \sim Q_1$$

with

$$Q_1 = \int_{\Phi(\mathcal{C}')\cap T_\Omega} \left(\int_{\Phi(\mathcal{C}')\cap T_\Omega} |B_\mu(z,w)| dV_\mu(w)\right)^q dV_\mu(z).$$

Since the compact set $\Phi(\mathcal{C}')$ is bounded, there exists $m \in \mathbb{R}^{*+}$ such that $m \geq \max\{|z|,\ z \in \Phi(\mathcal{C}') \cap T_\Omega\}$. So, using this change of variables $z = m\zeta$ and $w = m\xi$, we obtain $Q_1 \sim Q_2$, where

$$Q_2 = \int_{\mathbb{B}_n\cap T_\Omega} \left(\int_{\mathbb{B}_n\cap T_\Omega} \left|\Delta^{-\mu-\frac{n}{r}}\left(\frac{z-\bar{w}}{i}\right)\right| dV_\mu(w)\right)^q dV_\mu(z)$$

and $\mathbb{B}_n$ is the closed unit ball of $\mathbb{C}^n$ i.e $\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| \leq 1\}$.

If we put

$$\mathcal{K} = \{z = x+iy \in T_\Omega : |x| \leq 1 \text{ et } |y| \leq 1\}\,; \tag{8}$$

then the estimation (E') is equivalent to

$$\int_{\mathcal{K}} \left(\int_{\mathcal{K}} \left|\Delta^{-\mu-\frac{n}{r}}\left(\frac{z-\bar{w}}{i}\right)\right| dV_\mu(w)\right)^q dV_\mu(z) < +\infty. \tag{E$'_1$}$$

Writing $w = x+iy,\ z = s+it$ and using Lemma 2.5 and Lemma 2.6, we conclude that the estimation (E) becomes on the cone Ω

$$(\mathrm{E}'') \qquad \int_{\{t\in\Omega:\ |t|\le 1\}} \left(\int_{\{y\in\Omega:\ |y|\le 1\}} \Delta^{-\mu}(y+t)\Delta^{\mu-\frac{n}{r}}(y)dy \right)^q \Delta^{\mu-\frac{n}{r}}(t)dt < +\infty.$$

This ends the proof of Proposition 3.4.

3.2 The case of the Bergman projection. Here we describe the transfer process for the Bergman projections P_μ and $\mathcal{P}_\mu$. We prove the following two lemmas:

Lemma 3.8. *Let $\mathcal{C}'$ be a compact set of $\mathbb{C}^n$ such that $\mathcal{C}' \cap \Sigma' = \emptyset$ and the interior of $\mathcal{C}' \cap D$ is nonempty. Then $P_\mu : L^\infty_{\Phi(\mathcal{C}')}(T_\Omega) \to L^q_\mu(\Phi(\mathcal{C}') \cap T_\Omega)$ is bounded if and only if $\mathcal{P}_\mu : L^\infty_{\mathcal{C}'}(D) \to L^q_\mu(\mathcal{C}' \cap D)$ is bounded.*

Proof. Let $f \in L^\infty(D)$ with its support in the compact set $\mathcal{C}'$. If we put

$$g(w) = (f \circ \Psi)(w)\overline{(2i)^{-r\mu-n}\Delta^{\mu+\frac{n}{r}}(w+ie)}|\Delta^{-2\mu-\frac{2n}{r}}(w+ie)|\chi_{\Phi(\mathcal{C}')}(w),$$

then $g \in L^\infty_{\Phi(\mathcal{C}')}(T_\Omega)$ and

$$(9) \qquad \|g\|_{L^\infty_{\Phi(\mathcal{C}')}(T_\Omega)} \quad \sim \quad \|f\|_{L^\infty_{\mathcal{C}'}(D)} < +\infty.$$

So in view of Lemma 3.1, we have:

$$(10) \qquad \|\mathcal{P}_\mu f\|^q_{L^q_\mu(\mathcal{C}'\cap D)} \quad \sim \quad \|P_\mu g\|^q_{L^q_\mu(\Phi(\mathcal{C}')\cap T_\Omega)}.$$

The conclusion follows from (9) and (10). □

Lemma 3.9. *Assume that for any compact set $\mathcal{C}'$ of $\mathbb{C}^n$ such that $\mathcal{C}' \cap \Sigma' = \emptyset$, the weighted Bergman projection $\mathcal{P}_\mu : L^\infty_{\mathcal{C}'}(D) \to L^q_\mu(\mathcal{C}' \cap D)$ is bounded. Then $\mathcal{P}_\mu : L^\infty(D) \to L^q_\mu(D)$ is bounded.*

Proof. As we saw previously,

$$\left\{\mathcal{O}^1(z') \times \mathcal{O}^2(w') : (z', w') \in \overline{D} \times \overline{D}\right\}$$

is an open covering of the compact set $\overline{D} \times \overline{D}$ with

$$\left\{\mathcal{O}^1_j(z') \times \mathcal{O}^2_j(w'),\ \ j = 1, \ldots, N\right\}$$

as a finite subcover; we can write

$$\overline{D} \times \overline{D} = \cup_{j=1}^N (B^1_j \times B^2_j)$$

with

$$B^1_j = \mathcal{O}^1_j(z') \cap \overline{D} \text{ and } B^2_j = \mathcal{O}^2_j(w') \cap \overline{D},\ j = 1, \ldots, N.$$

Now, we put

$$\begin{aligned} C_1^1 \times C_1^2 &= B_1^1 \times B_1^2; \\ C_j^1 \times C_j^2 &= (B_j^1 \times B_j^2) \setminus \left[\cup_{k=1}^{j-1}(B_k^1 \times B_k^2)\right] \quad \text{for j=2,}\ldots\text{, N.} \end{aligned}$$

Then the family

$$\left\{C_j^1 \times C_j^2, \qquad j = 1, \ldots, N\right\}$$

is a disjoint covering of $\overline{D} \times \overline{D}$.

Let us do this remark:

Remark 3.10. It exists $j_0 \in \{1, \ldots, N\}$ such that

$$\frac{1}{N}\left|\int\int_{\overline{D}\times\overline{D}} \mathbf{B}_\mu(z', w') f(w')\overline{g(z')} dA_\mu(w') dA_\mu(z')\right|$$
$$\leq \left|\int\int_{\left(C_{j_0}^1 \times C_{j_0}^2\right)} \mathbf{B}_\mu(z', w') f(w')\overline{g(z')} dA_\mu(w') dA_\mu(z')\right|.$$

Let $f \in L^\infty(D)$. According to Remark 3.10, for $g \in L_\mu^{q'}$,

$$|\langle \mathcal{P}_\mu f, g\rangle| \quad \leq \quad N\left|\int_{K_1' \cap D} \mathcal{P}_\mu F(z')\overline{G(z')} dA_\mu(z')\right|$$

where

$$F(w') = f(e^{-i\theta_{j_0}} w')\chi_{e^{i\theta_{j_0}} C_{j_0}^2}(w'), \qquad G(z') = g(e^{-i\theta_{j_0}} z')\chi_{e^{i\theta_{j_0}} C_{j_0}^1}(z')$$

and K_1' is the compact set given in (6).

Moreover,

$$\sup_{w' \in D} |F(w')| \leq \|f\|_{L^\infty(D)} \quad \text{and} \quad \|G\|_{L_\mu^{q'}(K_1' \cap D)} \leq \|g\|_{L_\mu^{q'}(D)} \leq 1.$$

So $F \in L^\infty(D)$ with its support in the compact set K_1' and $G \in L_\mu^{q'}(K_1' \cap D)$. We conclude thanks to our hypothesis that

$$\|\mathcal{P}_\mu f\|_{L_\mu^q(D)} \quad \leq \quad N\|\mathcal{P}_\mu F\|_{L_\mu^q(K_1' \cap D)} \leq NC\|F\|_{L_{K_1'}^\infty(D)} \leq NC\|f\|_{L^\infty(D)}.$$

□

3.3 Estimation of the integral $I(q, \Lambda_n)$. As announced above, we restrict ourselves in the case where the rank is equal to 2. In the sequel, we write $K = \{y \in \Lambda_n : \ |y| \leq 1\}$.

From the transfer principle, the boundedeness of the positive Bergman operator $\mathcal{P}^+$ from $L^\infty(\mathcal{B}_n)$ to $L^q(\mathcal{B}_n)$ turns now to the determination of a necessary and sufficient condition on q so that the integral $I(q,\Lambda_n)$ is finite. We have the following:

Theorem 3.11. *The integral*

$$I(q,\Lambda_n)=\int_K\left(\int_K \Delta^{-\mu}(y+t)\Delta^{\mu-\frac{n}{2}}(y)dy\right)^q \Delta^{\mu-\frac{n}{2}}(t)dt$$

is finite if and only if $q<\frac{2\mu}{\frac{n}{2}-1}$.

The proof of this theorem is based on the estimation of the following integral

$$I(t)=\int_K \Delta^{-\mu}(y+t)\Delta^{\mu-\frac{n}{2}}(y)dy,\ t\in K. \tag{11}$$

The case $\mu=\frac{n}{2}$ has been done in [1]. We shall give a simplified proof based on invariance properties due to the geometry of the cone. Let us observe that if $t=(t_1,t_2,t')\in K$, then $\tau=(t_1,v,0,\dots,0)\in K$ where $v=\sqrt{t_2^2+\cdots+t_n^2}$; since $|t|=|\tau|$, there is a rotation R leaving invariant the axis (Oy_1) which transforms t in τ. Since $R\in SO(1,n-1)$, where $SO(1,n-1)$ is the group of $n\times n$ matrices of determinant 1 which preserve the Lorentz form $\Delta(y)=y_1^2-y_2^2-\cdots-y_n^2$, it follows that

$$\begin{aligned} I(R(t)) &= \int_K \Delta^{-\mu}(y+R(t))\Delta^{\mu-\frac{n}{2}}(y)dy \\ &= \int_K \Delta^{-\mu}(R(y+t))\Delta^{\mu-\frac{n}{2}}(R(y))dy \\ &= I(t). \end{aligned}$$

It is then sufficient to take $t=(t_1,t_2,0,\dots,0)\in K$ to evaluate $I(t)$.

Proposition 3.12. *Let $t\in K$. There are two positive constants c_1 and c_2 such that*

$$c_1\left(t_1+\sqrt{t_2^2+\cdots+t_n^2}\right)^{-\frac{n}{2}+1}\le I(t)$$

and

$$I(t)\le c_2\left(t_1+\sqrt{t_2^2+\cdots+t_n^2}\right)^{-\frac{n}{2}+1}\log\frac{1}{t_1-\sqrt{t_2^2+\cdots+t_n^2}}.$$

Proof. Let $t = (t_1, t_2, 0, \dots, 0) \in K$. We use the spherical coordinates for which $\chi_1(y) = \bar{y_1} = y_1 + y_2$, $\chi_2(y) = \bar{y_2} - \frac{|y'|^2}{\bar{y_1}}$, with $\bar{y_2} = y_1 - y_2$. Then $\Delta(y) = \chi_1(y)\chi_2(y)$.

Consider the sets $E_1 = \{y = (y_1, y_2, y') \in \mathbb{R}^n : 0 < \bar{y_1} \le 1/8,\ |y'|^2 < \bar{y_1},\ 0 < \bar{y_2} - \frac{|y'|^2}{\bar{y_1}} < 1/8\}$ and $E_2 = \{y = (y_1, y_2, y') \in \mathbb{R}^n : 0 < \bar{y_1} \le 2,\ |y'|^2 < 1,\ 0 < \bar{y_2} - \frac{|y'|^2}{\bar{y_1}} < 1\}$; then $E_1 \subset K \subset E_2$ so that

$$\int_{E_1} \Delta^{-\mu}(y+t)\Delta^{\mu-\frac{n}{2}}(y)dy \le I(t) \le \int_{E_2} \Delta^{-\mu}(y+t)\Delta^{\mu-\frac{n}{2}}(y)dy.$$

By using the change of variables

$$\bar{y_2} - \frac{|y'|^2}{\bar{y_1}} = \bar{t_2}s, \quad y' = \left(\frac{\bar{y_1}(\bar{y_1} + \bar{t_1})\bar{t_2}}{\bar{t_1}}\right)^{\frac{1}{2}} u',$$

we get

$$\int_0^{1/8} (\bar{y_1}+\bar{t_1})^{-\mu}(\bar{y_1})^{\mu-\frac{n}{2}} L_1(t, \bar{y_1})d\bar{y_1} \le I(t) \le \int_0^2 (\bar{y_1}+\bar{t_1})^{-\mu}(\bar{y_1})^{\mu-\frac{n}{2}} L_2(t, \bar{y_1})d\bar{y_1}$$

where for $j = 1, 2$,

$$L_j(t, \bar{y_1}) \quad = \quad \left[\frac{\bar{y_1}(\bar{y_1} + \bar{t_1})}{\bar{t_1}}\right]^{\frac{n}{2}-1} \times R_j(\bar{t_1}, \bar{t_2})$$

with

$$\begin{aligned} R_1(\bar{t_1}, \bar{t_2}) &= \int_{0<s<1/8\bar{t_2}} \int_{|u'|^2 \le \frac{\bar{t_1}}{\bar{t_2}(\bar{y_1}+\bar{t_1})}} (1 + s + |u'|^2)^{-\mu} du' s^{\mu-\frac{n}{2}} ds \\ &\ge \int_{0<s<\frac{1}{16}} \int_{|u'|^2 \le \frac{1}{4}} (1 + s + |u'|^2)^{-\mu} du' s^{\mu-\frac{n}{2}} ds \ge C_1 \end{aligned}$$

(since for $t \in K$, we can choose $\bar{t_2} < \bar{t_1} < 2$, so that $\frac{\bar{t_1}}{\bar{t_2}(\bar{y_1}+\bar{t_1})} > 1/4$); and

$$\begin{aligned} R_2(\bar{t_1}, \bar{t_2}) &= \int_{0<s<1/\bar{t_2}} \int_{|u'|^2 \le \bar{t_1}/\bar{t_2}(\bar{y_1}+\bar{t_1})} (1 + s + |u'|^2)^{-\mu} du' s^{\mu-\frac{n}{2}} ds \\ &\le \int_{0<s<1/\bar{t_2}} \int_{u' \in \mathbb{R}^{n-2}} (1 + s + |u'|^2)^{-\mu} du' s^{\mu-\frac{n}{2}} ds \\ &\le C \int_{0<s<1/\bar{t_2}} (1+s)^{-\mu+\frac{n}{2}-1} s^{\mu-\frac{n}{2}} ds \\ &\le C_2 \log \frac{1}{\bar{t_2}}. \end{aligned}$$

It follows that

$$C_1 \bar{t_1}^{-\frac{n}{2}+1} \times \int_0^{1/8} (\bar{y}_1 + \bar{t}_1)^{-\mu+\frac{n}{2}-1} (\bar{y}_1)^{\mu-1} d\bar{y}_1 \leq I(t)$$

and

$$I(t) \leq C_2 \bar{t_1}^{-\frac{n}{2}+1} \log \frac{1}{\bar{t}_2} \times \int_0^{2} (\bar{y}_1 + \bar{t}_1)^{-\mu+\frac{n}{2}-1} (\bar{y}_1)^{\mu-1} d\bar{y}_1$$

i.e

$$c_1 \bar{t_1}^{-\frac{n}{2}+1} \leq I(t) \leq c_2 \bar{t_1}^{-\frac{n}{2}+1} \log \frac{1}{\bar{t}_2}$$

i.e

$$c_1 (t_1 + t_2)^{-\frac{n}{2}+1} \leq I(t) \leq c_2 (t_1 + t_2)^{-\frac{n}{2}+1} \log \frac{1}{t_1 - t_2}.$$

□

Proof of Theorem 3.11. Using the polar coordinates: $v^2 = t_2^2 + \cdots + t_n^2$, we have $dt = v^{n-2} dt_1 dv d\theta$ with $0 < v < t_1 < 1$. It follows from Proposition 3.12 that

$$\begin{aligned} I(q, \Lambda_n) &\geq c_q \int_0^1 \int_0^{t_1} (t_1 + v)^{(-n/2+1)q} (t_1^2 - v^2)^{\mu - \frac{n}{2}} v^{n-2} dv dt_1 \\ &\geq c'_q \int_0^1 t_1^{(-n/2+1)q+2\mu-1} dt_1 \end{aligned}$$

and the last integral above is convergent only if $q < \frac{2\mu}{\frac{n}{2}-1}$.

Moreover, assume that $q < \frac{2\mu}{\frac{n}{2}-1}$. For all $\varepsilon > 0$, there is c_ε such that $\log \frac{1}{t_1 - v} \leq c_\varepsilon (t_1 - v)^{-\varepsilon}$. Therefore from Proposition 3.12,

$$\begin{aligned} I(q, \Lambda_n) &\leq c_q \int_0^1 \int_0^{t_1} (t_1 + v)^{(-n/2+1)q} (t_1 - v)^{-\varepsilon q} (t_1^2 - v^2)^{\mu - \frac{n}{2}} v^{n-2} dv dt_1 \\ &\leq c'_{q\varepsilon} \int_0^1 t_1^{-\varepsilon q + (-n/2+1)q + 2\mu - 1} dt_1 \end{aligned}$$

and if we choose $\varepsilon < \min\{\frac{\mu - \frac{n}{2} + 1}{q}, -\frac{n}{2} + 1 + \frac{2\mu}{q}\}$, then $I(q, \Lambda_n)$ is finite. □

4. Proof of the main result

We denote by $T_K = \mathbb{R}^n + iK$ the tubular domain over the compact set K of $\mathbb{R}^n$. We introduce mixed normed spaces in the unbounded domain T_{Λ_n} which are useful for us to draw the conclusion of this paper. We recall that $\mathcal{K} = K_0 + iK$ with $K_0 = \{x \in \mathbb{R}^n : |x| \leq 1\}$. (cf. (8)).

Definition 4.1. We say that a function f belongs to $L^{\infty,q}_\mu(T_K)$ if

$$\|f\|_{L^{\infty,q}_\mu(T_K)} := \left(\int_K \sup_{x\in\mathbb{R}^n} |f(x+iy)|^q \Delta^{\mu-\frac{n}{2}}(y)dy\right)^{\frac{1}{q}} < +\infty.$$

We say that a function f belongs to $L^{p,q}_\mu(T_{\Lambda_n})$ if

$$\|f\|_{L^{p,q}_\mu(T_{\Lambda_n})} := \left(\int_{\Lambda_n} \left(\int_{\mathbb{R}^n} |f(x+iy)|^p dx\right)^{\frac{q}{p}} \Delta^{\mu-\frac{n}{2}}(y)dy\right)^{\frac{1}{q}} < +\infty.$$

Theorem 4.2. *The positive Bergman operator P^+_μ is bounded from $L^\infty_{\mathcal{K}}(T_{\Lambda_n})$ to $L^{\infty,q}_\mu(T_K)$ whenever $q < \frac{2\mu}{\frac{n}{2}-1}$. Moreover, the Bergman projection P_μ is bounded from $L^\infty_{\mathcal{K}}(T_{\Lambda_n})$ to $L^{\infty,q}_\mu(T_K)$ for $q < \frac{2\mu}{\frac{n}{2}-1}$.*

Proof. Assume $q < \frac{2\mu}{\frac{n}{2}-1}$. Let $f \in L^\infty(T_{\Lambda_n})$ with its support in $\mathcal{K}$. Then from Lemma 2.6 and (11),

$$|P^+_\mu f(x+iy)| \quad \leq \quad C_\mu \|f\|_{L^\infty_{\mathcal{K}}(T_{\Lambda_n})} I(y)$$

Hence,

$$\|P^+_\mu f\|^q_{L^{\infty,q}_\mu(T_K)} \quad \leq \quad C^q_\mu \|f\|^q_{L^\infty_{\mathcal{K}}(T_{\Lambda_n})} \times I(q,\Lambda_n) \leq C'_{\mu q} \|f\|^q_{L^\infty_{\mathcal{K}}(T_{\Lambda_n})}$$

since from Theorem 3.11, the integral $I(q,\Lambda_n)$ is finite whenever $q < \frac{2\mu}{\frac{n}{2}-1}$. □

We have the following

Theorem 4.3. [2, Theorem 1.1] *Let $q_\mu = 1 + \frac{\mu}{\frac{n}{2}-1}$. The Bergman projection P_μ is bounded from $L^{2,q_0}_\mu(T_{\Lambda_n})$ to $L^{2,q_0}_\mu(T_{\Lambda_n})$ if and only if $\frac{2q_\mu}{2q_\mu-1} < q_0 < 2q_\mu$.*

The following theorem gives an improvement of the index obtained in Theorem 4.2.

Theorem 4.4. *The Bergman projection P_μ is bounded from $L^\infty_{\mathcal{K}}(T_{\Lambda_n})$ in $L^q(\mathcal{K})$ whenever $2 < q < \frac{2\mu}{\frac{n}{2}-1} + \frac{2}{1+\frac{\mu}{\frac{n}{2}-1}}$.*

Proof. First observe that

$$L^{\infty}_{\mathcal{K}}(T_{\Lambda_n}) \subset L^{2,q_0}_{\mu}(T_{\Lambda_n}) \subset L^{2,q_0}_{\mu}(T_K) \subset L^{2,q_0}_{\mu}(\mathcal{K}).$$

Hence, from Theorem 4.3,

$$P_\mu : L^{\infty}_{\mathcal{K}}(T_{\Lambda_n}) \to L^{2,q_0}_{\mu}(T_K) \text{ is bounded whenever } \frac{2q_\mu}{2q_\mu - 1} < q_0 < 2q_\mu.$$

Moreover, according to Theorem 4.2,

$$P_\mu : L^{\infty}_{\mathcal{K}}(T_{\Lambda_n}) \to L^{\infty,q_1}_{\mu}(T_K) \text{ is bounded whenever } q_1 < \frac{2\mu}{\frac{n}{2} - 1}.$$

Therefore we conclude by interpolation that

$$P_\mu : L^{\infty}_{\mathcal{K}}(T_{\Lambda_n}) \to L^{p,q}_{\mu}(T_K) \qquad \text{is bounded}$$

where $\frac{1}{p} = \frac{1-\theta}{\infty} + \frac{\theta}{2}$; $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_0}$ and $0 < \theta < 1$. It follows that $\theta = \frac{2}{p}$ i.e $p > 2$ and for $p = q > 2$ we get the answer. □

Proof of Theorem 1.1. Assume $2 < q < \frac{2n}{n-2} + \frac{n-2}{n-1}$. From Theorem 4.4, $P : L^{\infty}_{\mathcal{K}}(T_{\Lambda_n}) \to L^q(\mathcal{K})$ is bounded. It follows from Lemma 3.8 and Lemma 3.9 that $\mathcal{P} : L^{\infty}(\mathcal{B}_n) \to A^q(\mathcal{B}_n)$ is bounded. Moreover, from Theorem 3.11, we have that $\mathcal{P} : L^{\infty}(\mathcal{B}_n) \to A^q(\mathcal{B}_n)$ is bounded for $q < \frac{2n}{n-2}$. Thus, we conclude that the result holds for $q < \frac{2n}{n-2} + \frac{n-2}{n-1}$. □

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