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Research Article

Local Gevrey Regularity for Linearized Homogeneous Boltzmann Equation

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The local Gevrey regularity of the solutions of the linearized spatially homogeneous Boltzmann equation has been shown in the non-Maxwellian case with mild singularity.

1. Introduction

This paper focuses on the Gevrey class smoothing property of solutions of the following linear Cauchy problems of the spatially homogeneous Boltzmann equation:

$$\begin{gathered} \frac{\partial f}{\partial t} = Lf = Q(\mu, f) + Q(f, \mu), \quad v \in \mathbb{R}^3,\ t > 0, \quad \mu(v) = (2\pi)^{-3/2} e^{-|v|^2/2}, \\ f|_{t=0} = f_0, \end{gathered} \tag{1.1}$$

where the initial datum $f_0 \not\equiv 0$ satisfies the natural boundedness on mass, energy, and entropy:

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3} f_0(v) \Big\{ 1 + |v|^2 + \log\big(1 + f_0(v)\big) \Big\} dv < +\infty. \tag{1.2}$$

$Q(g, f)$ is the Boltzmann quadratic operator which has the following form:

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{ g(v'_*) f(v') - g(v_*) f(\mathrm{v}) \} d\sigma\, dv_*, \tag{1.3}$$

where $\sigma \in \mathbb{S}^2$ (unit sphere of $\mathbb{R}^3$); the post- and precollisional velocities are given as follows:

$$v' = \frac{v+v_*}{2} + \frac{|v+v_*|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v+v_*|}{2}\sigma. \tag{1.4}$$

The Boltzmann collision cross-section $B(|z|,\sigma)$ is a nonnegative function which depends only on $|z|$ and the scalar product $\langle z/|z|,\sigma\rangle$. To capture its main properties, we usually assume

$$B(|v-v_*|,\sigma) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \left\langle \frac{v-v_*}{|v-v_*|},\sigma\right\rangle,\ \theta\in\left[0,\frac{\pi}{2}\right]. \tag{1.5}$$

μ is called the normalized Maxwellian distribution in (1.1). Notice that $Q(\mu,\mu)\equiv 0$.

Recall that the inverse power law potential $1/\rho^s$, where $s>1$, and ρ denotes the distance between two particles, has the form (1.5) with the corresponding kinetic factors:

$$\begin{aligned} \Phi(|v-v_*|) &\approx |v-v_*|^{1-4/s}, \\ b(\cos\theta) &\approx \frac{K}{\theta^{2+\nu}}, \quad \theta \longrightarrow 0, \end{aligned} \tag{1.6}$$

for a constant $K>0$ and $0<\nu=2/s<2$. The cases $1<s<4$, $s=4$, and $s>4$ correspond to so-called soft, Maxwellian, and hard potentials, respectively.

We will concentrate on the modified hard potentials as follows:

$$\begin{aligned} \Phi(|v-v_*|) &= \left(1+|v-v_*|^2\right)^{\gamma/2}, \quad 0<\gamma<1, \\ b(\cos\theta) &\approx \frac{K}{\theta^{2+\nu}}, \quad \theta\longrightarrow 0,\ 0<\nu<2, \end{aligned} \tag{1.7}$$

where the singularity is called the mild singularity when $0<\nu<1$ and the strong singularity when $1\le\nu<2$. In this paper, we consider only the case of the mild singularity. Before making the discussion, we start by introducing the norms of the weighted function spaces:

$$\|f\|_{L^p_r} = \|\langle|v|\rangle^r f(v)\|_{L^p}, \qquad \|f\|_{H^s_r} = \|\langle|D|\rangle^s\langle|v|\rangle^r f(v)\|_{L^2}, \tag{1.8}$$

where $\langle|v|\rangle = (1+|v|^2)^{1/2}$ and $\langle|D|\rangle$ is the corresponding pseudodifferential operator. And then, we list the definition of the weak solution in the Cauchy problem (1.1); compare [1].

Definition 1.1. For an initial datum $f_0(v) \in L^1_2(\mathbb{R}^3)$, $f(t,v)$ is called a weak solution of the Cauchy problem (1.1) if it satisfies

$$\begin{gathered} f(t,v) \in C\left(\mathbb{R}^+;\mathcal{D}'\left(\mathbb{R}^3\right)\right) \cap L^2\left([0,T];L^1_2\left(\mathbb{R}^3\right)\right) \cap L^\infty\left([0,T];L^1\left(\mathbb{R}^3\right)\right), \qquad f(0,v) = f_0, \\ \int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f(0,v)\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv \\ = \int_0^t d\tau \int_{\mathbb{R}^3} L(f)(\tau,v)\varphi(\tau,v)dv, \end{gathered} \tag{1.9}$$

for any test function $\varphi \in L^\infty([0,T];W^{2,\infty}(\mathbb{R}^3))$.

For the definition of the Gevrey class functions, compare [1–5].

Definition 1.2. Suppose that W is a bounded open set on $\mathbb{R}^3$, for $s \geq 1$, $u \in G^s(W)$ which is the Gevrey class function space with index s, if $u \in C^\infty(W)$ and for any compact subset $U \subset W$, there exists a constant $C = C(U) > 0$ such that for any $k \in \mathbb{N}$,

$$\left\|D^k u\right\|_{L^2(U)} \leq C^{k+1}(k!)^s, \tag{1.10}$$

or equivalently,

$$\left\|\langle|D|\rangle^k u\right\|_{L^2(U)} \leq C^{k+1}(k!)^s, \tag{1.11}$$

where

$$\left\|D^k u\right\|^2_{L^2(U)} = \sum_{|\beta|=k}\left\|D^\beta u\right\|^2_{L^2(U)}, \qquad \langle|D|\rangle = \left(1+|D_v|^2\right)^{1/2}. \tag{1.12}$$

Particularly, $u \in G^s(\mathbb{R}^3)$, that is, $\|D^k u\|_{L^2(\mathbb{R}^3)} \leq C^{k+1}(k!)^s$, is equivalent to the fact that there exists $\epsilon_0 > 0$ such that $e^{\epsilon_0\langle|D|\rangle^{1/s}}u \in L^2(\mathbb{R}^3)$.

Notice that $G^1(\mathbb{R}^3)$ is the usual analytic function space. When $0 < s < 1$, we call $G^s(\mathbb{R}^3)$ the ultra-analytic function space, cpmpare [4, 5].

There have been some results about the Gevrey regularity of the solutions for the Boltzmann equation; compare [1, 4, 6–8]. Among them, unique local solutions having the same Gevrey regularity as the initial data are first constructed in [8]. This implies the propagation of the Gevrey regularity. In 2009, Desvillettes et al. improved this result for the nonlinear spatially homogeneous Boltzmann equation, they showed in [6] that, for the Maxwellian molecules model, the Gevrey regularity can propagate globally in time. Other results for the nonlinear case can be found in [4], where the Gevrey regularity of the radially symmetric weak solutions has been proved. Meanwhile, this issue is also considered in [7] for the Maxwellian decay solutions. For the linear case, the best result so far is obtained by the work of Morimoto et al. in [1]; they proved the propagation of Gevrey regularity of the solutions, without any extra assumption for the initial data. We mention that the crucial tools in [1, 6] are the following pseudodifferential operator:

$$G_\delta(t, D_v) = \frac{1}{\delta + e^{-t\langle |D_v| \rangle^{\nu/2}}}, \quad 0 < \nu < 2. \tag{1.13}$$

In the Maxwellian case, this pseudodifferential operator can be used successfully, but it seems unsuitable for the non-Maxwellian model. The difficulty comes from the commutator of the kinetic factor Φ and the pseudodifferential operator (1.13) which lacks of the effective estimations. In this paper, we apply a new method which is based on the mathematical induction to overcome it. Compared with [7], we consider only the local space; however, we discuss this issue by using the much weaker preconditions (actually, we do not need any smooth assumption for the initial data). Concerning the same issue for the other related equations, such as the Landau equation and the Kac equation, compare [2–5].

Now we can state our main result.

Theorem 1.3. *Suppose Φ, b have the forms in (1.7), $0 < \nu < 1$. Let W be a bounded open set of $\mathbb{R}^3$, and $f(t, v)$ be the weak solution of the Cauchy problem (1.1) satisfying*

$$\sup_{t \in (0,T]} \|f(t, \cdot)\|_{L^2(W)} < +\infty. \tag{1.14}$$

Then for any $t \in (0, T]$, there exists a number $s = s(t) > 3$ satisfying $f(t, \cdot) \in G^s(W)$. More precisely, for any fixed $0 < t_0 \le T$ and compact subset $U \subset W$, there exists a constant $C = C(U) > 0$ and a number $s > 3$ such that for any $k \in \mathbb{N}$,

$$\sup_{t \in [t_0, T]} \left\| D^k f(t, \cdot) \right\|_{L^2(U)} \le C^{k+1} (k!)^s. \tag{1.15}$$

From Theorem 1.3, we have the following remark.

Remark 1.4. Suppose that Φ, b have the forms in (1.7), $0 < \nu < 1$. If the weak solution $f(t, v)$ satisfies that

$$\sup_{t \in (0,T]} \|f(t, \cdot)\|_{L^2(\mathbb{R}^3)} < +\infty, \tag{1.16}$$

then for any $t \in (0, T]$, any bounded open set $U \subset \mathbb{R}^3$, there exists a constant $s = s(t)$ satisfying $f(t, \cdot) \in G^s(U)$.

2. Useful Lammas for the Main Result

In order to gain the main result, we need to prove the following lemmas in this section.

Lemma 2.1. *Suppose* $\Phi(v) = \langle |v| \rangle^{\gamma} = (1+|v|^2)^{\gamma/2}$ *where* $\gamma \in (0,1), v \in \mathbb{R}^n$*, and* $n \in \mathbb{N}$*. Then the* k*th order derivative of* Φ *satisfies*

$$\left|\Phi^{(k)}(v)\right| \le 4^k k! \Phi(v) \langle |v| \rangle^{-k}. \tag{2.1}$$

Proof. Without loss of generality, we only consider the case of $n = 1$; the other cases are similar. By direct calculation, we have

$$\begin{aligned} \Phi^{(2m)}(v) &= \sum_{i=0}^{m} C_{i,2m} \gamma(\gamma-2)\cdots(\gamma-2i-2m+2)\left(1+v^2\right)^{\gamma/2-i-m} v^{2i}, \\ \Phi^{(2m+1)}(v) &= \sum_{i=0}^{m} A_{i,2m+1} \gamma(\gamma-2)\cdots(\gamma-2i-2m)\left(1+v^2\right)^{\gamma/2-i-m-1} v^{2i+1}. \end{aligned} \tag{2.2}$$

In addition,

$$\begin{aligned} C_{i,2m} + 2(i+1)C_{i+1,2m} &= A_{i,2m+1}, \\ (2i+1)A_{i,2m+1} + A_{i-1,2m+1} &= C_{i,2m+2}. \end{aligned} \tag{2.3}$$

Thus we obtain

$$C_{i,2m+2} = C_{i-1,2m} + (4i+1)C_{i,2m} + (2i+1)(2i+2)C_{i+1,2m} \tag{2.4}$$

and then we will prove the following inequality:

$$|C_{i,2m}| \le 2^{2m} \left|(\gamma-2i-2m)\cdots(\gamma-4m+2)\right|. \tag{2.5}$$

The inequality is obviously true for $m = 1$. Suppose it is valid for $1 \le m \le M$, then

$$\begin{aligned} |C_{i,2M+2}| &= |C_{i-1,2M} + (4i+1)C_{i,2M} + (2i+1)(2i+2)C_{i+1,2M}| \\ &\le 2^{2M}\left[(4i+1)\left|\gamma-2i-2M\right| + (\gamma-2i+2-2M)\left|\gamma-2i-2M\right| \right. \\ &\qquad \left. + (2i+1)(2i+2)\right]\left|(\gamma-2i-2-2M)\cdots(\gamma-4M+2)\right| \\ &\le 2^{2(M+1)}\left|(\gamma-2i-2-2M)\cdots(\gamma-4M)(\gamma-4M-2)\right| \end{aligned} \tag{2.6}$$

which proves (2.5) by induction. Therefore, we have

$$\begin{aligned}\left|\Phi^{(2m)}(v)\right| &\leq 2^{2m}\left|\gamma(\gamma-2)\cdots(\gamma-4m+2)\right|\sum_{i=0}^{m}\left(1+v^2\right)^{\gamma/2-i-m}v^{2i}\\ &\leq (m+1)2^{4m-1}(2m-1)!\Phi(v)\langle|v|\rangle^{-2m}\\ &\leq 4^{2m}(2m)!\Phi(v)\langle|v|\rangle^{-2m}.\end{aligned}\tag{2.7}$$

The case of $(2m+1)$th order derivative is similar. This completes the proof of Lemma 2.1. □

Setting $M_N(\xi)=(1+|\xi|^2)^{Nt/2}$ for any $\xi\in\mathbb{R}^3$ and $N\in\mathbb{N}$, by using the similar technique of Lemma 2.1, we conclude the following.

Remark 2.2. For $t\in(0,1]$,

$$\begin{aligned}\left|\partial_\xi^k M_N(\xi)\right| &\leq 4^k\langle|\xi|\rangle^{Nt-k}|N(N-1)\cdots(N-k+1)|\\ &\leq 4^k\langle|\xi|\rangle^{(N-k)t}|N(N-1)\cdots(N-k+1)|,\end{aligned}\tag{2.8}$$

where $k\in\mathbb{N}, 1\leq k\leq N$.

Lemma 2.3. *There exists a constant C such that for any $k\in\mathbb{N}$,*

$$\left|\partial_v^k\mu(v)\right|\leq C^k\cdot k!\cdot\max\left(1,|v|^k\right)\cdot\mu(v),\tag{2.9}$$

where μ is the absolute Maxwellian distribution in (1.1).

Proof. Without loss of generality, we also only consider the case in the real space $\mathbb{R}^1$. Putting

$$\begin{aligned}\partial_v^k\mu(v)&=\partial_v^k\left(e^{-v^2/2}\right)=\sum_{j=0}^{k}a'_{j,k}v^je^{-v^2/2},\\ \partial_v^k\mu^{-1}(v)&=\partial_v^k\left(e^{v^2/2}\right)=\sum_{j=0}^{k}a_{j,k}v^je^{v^2/2}.\end{aligned}\tag{2.10}$$

Evidently,

$$\begin{gathered}0\leq\left|a'_{j,k}\right|\leq a_{j,k},\\ \left|a'_{k,k}\right|=a_{k,k}\equiv1,\\ \sum_{j=0}^{1}a_{j,1}=1\leq 8^1\cdot 1!,\\ a_{j,k+1}=a_{j-1,k}+(j+1)a_{j+1,k}.\end{gathered}\tag{2.11}$$

Therefore, fixed a number $m \geq 0$, together with the following assumption (F_m):

$$\sum_j a_{j,m} \leq 8^m \cdot m!, \tag{2.12}$$

we can obtain (F_{m+1})

$$\begin{aligned}\sum_j a_{j,m+1} &= \sum_j \left(a_{j-1,m} + (j+1)a_{j+1,m}\right) \leq (m+2) \cdot \left(\sum_j a_{j-1,m} + \sum_j a_{j+1,m}\right) \\ &\leq 2(m+2) \cdot 8^m \cdot m! \leq 8^{m+1} \cdot (m+1)!.\end{aligned} \tag{2.13}$$

This completes the proof of Lemma 2.3 by induction. □

Setting

$$H^*(v) = \left(1 + |v_*|^2\right)^4 \mu^*(v) = \left(1 + |v_*|^2\right)^4 \cdot \mu(v + v_*), \tag{2.14}$$

where v is belong to a bounded set U. Then we state Lemma 2.4 as below.

Lemma 2.4. *There exists a constant $C = C(U) > 0$, which satisfies that for any $k \in \mathbb{N}$,*

$$\sup_{v_*} \left|\partial_v^k H^*(v)\right| \leq C^k \cdot (k!)^2. \tag{2.15}$$

Proof. Since $e^{-(v+v_*)^2/4} \leq e^{v^2/4} \cdot e^{-v_*^2/8}$, and the fact that when $|v| \geq 1$,

$$\left|v^k e^{-v^2/4}\right| = \frac{|v|^k}{\left(\sum_{n=0}^{+\infty} |v|^{2n}/2^{2n} \cdot n!\right)} \leq \frac{|v|^k}{\left(|v|^k/2^k \cdot (k/2)!\right)} \leq 2^k \cdot k!, \tag{2.16}$$

by using Lemma 2.3, we have

$$\begin{aligned}\left|\partial_v^k H^*(v)\right| &= \left|\left(1 + |v_*|^2\right)^4 \partial_v^k \mu^*(v)\right| \\ &\leq \left(1 + |v_*|^2\right)^4 \cdot C^k \cdot k! \cdot \max\left(1, |v + v_*|^k\right) \cdot \mu(v + v_*) \\ &\leq \left(1 + |v_*|^2\right)^4 \cdot C^k \cdot k! \cdot \left[\max\left(1, |v + v_*|^k\right) \cdot e^{-(v+v_*)^2/4}\right] \cdot e^{-(v+v_*)^2/4} \\ &\leq \left[\left(1 + |v_*|^2\right)^4 e^{-v_*^2/8}\right] \cdot C^k \cdot k! \cdot \left[\max\left(1, |v + v_*|^k\right) \cdot e^{-(v+v_*)^2/4}\right] \cdot e^{v^2/4} \\ &\leq \left(C^k \cdot k!\right)^2 \leq [C(U)]^k \cdot (k!)^2.\end{aligned} \tag{2.17}$$

This completes the proof of Lemma 2.4. □

By applying the Cauchy integral theorem, we will prove the helpful estimates as follows.

Lemma 2.5. *Suppose the Fourier transform for v_*,*

$$\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) = h(v,\xi)\widehat{\mu}(\xi), \tag{2.18}$$

where μ is the absolute Maxwellian distribution in (1.1)*. Then we have*

$$h(v,\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\left[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\right]^{\gamma/2}dv_*. \tag{2.19}$$

Proof. First we consider the case of $n=1$,

$$\begin{aligned}\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) &= \int_{\mathbb{R}^1}\left(1+|v-v_*|^2\right)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}dv_* \\ &= (2\pi)^{-3/2}e^{-\xi^2/2}\int_{\mathbb{R}^1}\left[1+(v-v_*)^2\right]^{\gamma/2}e^{-(v_*+i\xi)^2/2}dv_* \\ &= (2\pi)^{-3/2}e^{-\xi^2/2}\int_C e^{-z^2/2}\left[1+(v-z+i\xi)^2\right]^{\gamma/2}dz,\end{aligned} \tag{2.20}$$

where $z=v_*+i\xi$, and C denotes the curve: $v_*+i\xi,\ -\infty<v_*<\infty$. By Cauchy integral theorem [9], it follows that

$$\int_C e^{-z^2/2}\left[1+(v-z+i\xi)^2\right]^{\gamma/2}dz = \int_{\mathbb{R}^1} e^{-|v_*|^2/2}\left[1+(v-v_*+i\xi)^2\right]^{\gamma/2}dv_*. \tag{2.21}$$

Now we turn to consider the case of $n=3$. Letting $v=(v_1,v_2,v_3)$, and $v_*=(v_{*1},v_{*2},v_{*3})$ and using the previous result, we have

$$\begin{aligned}&\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) \\ &\quad= \int_{\mathbb{R}^3}\left(1+|v-v_*|^2\right)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}dv_* \\ &\quad= (2\pi)^{-3/2}\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^1}\left[1+(v_1-v_{*1})^2+(v_2-v_{*2})^2+(v_3-v_{*3})^2\right]^{\gamma/2}e^{-v_{*1}^2/2-iv_{*1}\xi_1}dv_{*1}\right) \\ &\quad\quad\times e^{-(v_{*2}^2+v_{*3}^2)/2-i(v_{*2}\xi_2+v_{*3}\xi_3)}dv_{*2}dv_{*3}\end{aligned}$$

$$
\begin{aligned}
&= (2\pi)^{-3/2} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^1} \left[1 + (v_1 - v_{*1} + i\xi_1)^2 + (v_2 - v_{*2})^2 + (v_3 - v_{*3})^2 \right]^{\gamma/2} e^{-v_{*1}^2/2 - \xi_1^2/2} dv_{*1} \right) \\
&\quad \times e^{-(v_{*2}^2 + v_{*3}^2)/2 - i(v_{*2}\xi_2 + v_{*3}\xi_3)} dv_{*2} dv_{*3} \\
&= (2\pi)^{-3/2} e^{-|\xi|^2/2} \int_{\mathbb{R}^3} e^{-|v_*|^2/2} \left[1 + \sum_{j=1}^{3} (v_j - v_{*j} + i\xi_j)^2 \right]^{\gamma/2} dv_* \\
&= (2\pi)^{-3/2} \widehat{\mu}(\xi) \int_{\mathbb{R}^3} e^{-|v_*|^2/2} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*) \cdot \xi \right]^{\gamma/2} dv_*.
\end{aligned}
\tag{2.22}
$$

Thus we conclude the result of Lemma 2.5. □

Lemma 2.6. *For the expression of $h(v,\xi)$ in Lemma 2.5, we have*

$$
\begin{gathered}
|h(v,\xi)| \leq C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{\gamma}, \\
\left| \nabla_{\xi}^2 h(v,\xi) \right| \leq C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{\gamma}, \\
|h(v,\xi^+) - h(v,\xi)| \leq C \cdot \langle |v| \rangle^{\gamma} \langle |\xi| \rangle^{1+\gamma} \sin\frac{\theta}{2}, \quad \theta = \arccos \left\langle \frac{\xi}{|\xi|}, \sigma \right\rangle,
\end{gathered}
\tag{2.23}
$$

where $\xi^+ = (\xi + |\xi|\sigma)/2$, and C is a constant independent of v and ξ.

Proof. The first inequality is obvious. To prove the third one, set $\xi = (\xi_1, \xi_2, \xi_3)$. Since

$$
\begin{aligned}
h(v,\xi)\widehat{\mu}(\xi) &= \mathcal{F}\left(\Phi(|v - v_*|)\mu(v_*) \right)(\xi) \\
&= \int_{\mathbb{R}^3} \left(1 + |v - v_*|^2 \right)^{\gamma/2} (2\pi)^{-3/2} e^{-|v_*|^2/2 - iv_* \cdot \xi} dv_*,
\end{aligned}
\tag{2.24}
$$

proceeding as in the proof of Lemma 2.5, we can get

$$
\begin{aligned}
\partial_{\xi_i}\left(h(v,\xi)\widehat{\mu}(\xi) \right) &= \widehat{\mu}(\xi)\left[\partial_{\xi_i} h(v,\xi) - \xi_i h(v,\xi) \right] \\
&= \int_{\mathbb{R}^3} \left(1 + |v - v_*|^2 \right)^{\gamma/2} (2\pi)^{-3/2} e^{-|v_*|^2/2 - iv_* \cdot \xi} (-iv_{*i}) dv_* \\
&= (2\pi)^{-3/2} \widehat{\mu}(\xi) \int_{\mathbb{R}^3} \left[1 + |v - v_*|^2 - |\xi|^2 + 2i(v - v_*) \cdot \xi \right]^{\gamma/2} \\
&\quad \cdot e^{-|v_*|^2/2} \cdot (-\xi_i - iv_{*i}) dv_*.
\end{aligned}
\tag{2.25}
$$

Therefore,

$$\begin{aligned}\partial_{\xi_i} h(v,\xi) &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\Big[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\Big]^{\gamma/2}\\ &\quad\cdot(-\xi_i - iv_{*i})dv_* + \xi_i h(v,\xi)\\ &= (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-|v_*|^2/2}\Big[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\Big]^{\gamma/2}\\ &\quad\cdot(-iv_{*i})dv_*\end{aligned} \tag{2.26}$$

which implies that

$$\left|\nabla_\xi h(v,\xi)\right| \le C\cdot\langle|v|\rangle^{\gamma}\langle|\xi|\rangle^{\gamma}. \tag{2.27}$$

By the mean value theorem of differentials, we have

$$\begin{aligned}|h(v,\xi^+)-h(v,\xi)| &\le C\cdot\left|\nabla_\xi h(v,\eta)\right|\cdot|\xi^+-\xi|\\ &\le C'\cdot\langle|v|\rangle^{\gamma}\langle|\eta|\rangle^{\gamma}|\xi^+-\xi| \le C''\cdot\langle|v|\rangle^{\gamma}\langle|\xi|\rangle^{1+\gamma}\sin\frac{\theta}{2},\end{aligned} \tag{2.28}$$

where $\theta = \arccos\langle\xi/|\xi|,\sigma\rangle$. Thus the third inequality has been obtained.

Finally, the above way can also be used in estimating the second one. Similarly,

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\big(h(v,\xi)\widehat{\mu}(\xi)\big) &= \partial_{\xi_j}\big(\partial_{\xi_i}\big(h(v,\xi)\widehat{\mu}(\xi)\big)\big)\\ &= -\int_{\mathbb{R}^3}\Big(1+|v-v_*|^2\Big)^{\gamma/2}(2\pi)^{-3/2}e^{-|v_*|^2/2-iv_*\cdot\xi}v_{*i}v_{*j}dv_*\\ &= -(2\pi)^{-3/2}\widehat{\mu}(\xi)\int_{\mathbb{R}^3}\Big[1+|v-v_*|^2-|\xi|^2+2i(v-v_*)\cdot\xi\Big]^{\gamma/2}\\ &\quad\cdot e^{-|v_*|^2/2}(v_{*i}-i\xi_i)(v_{*j}-i\xi_j)dv_*.\end{aligned} \tag{2.29}$$

On the other hand,

$$\begin{aligned}\partial^2_{\xi_i\xi_j}\big(h(v,\xi)\widehat{\mu}(\xi)\big) &= \partial_{\xi_j}\left\{\widehat{\mu}(\xi)\left[\partial_{\xi_i}h(v,\xi)-\xi_i h(v,\xi)\right]\right\}\\ &= \widehat{\mu}(\xi)\Big[\xi_j\xi_i h(v,\xi)+\partial^2_{\xi_j\xi_i}h(v,\xi)-\xi_j\partial_{\xi_i}h(v,\xi)-\xi_i\partial_{\xi_j}h(v,\xi)\Big].\end{aligned} \tag{2.30}$$

Combining with the above expressions of $h(v,\xi)$ and $\partial_{\xi_i} h(v,\xi)$, we get

$$\begin{aligned}
\partial^2_{\xi_j\xi_i} h(v,\xi) = {} & \xi_i \partial_{\xi_j} h(v,\xi) + \xi_j \partial_{\xi_i} h(v,\xi) - \xi_j\xi_i h(v,\xi) \\
& - (2\pi)^{-3/2} \int_{\mathbb{R}^3} \left[1 + |v-v_*|^2 - |\xi|^2 + 2i(v-v_*)\cdot\xi\right]^{\gamma/2} \\
& \cdot e^{-|v_*|^2/2} (v_{*i} - i\xi_i)(v_{*j} - i\xi_j)\, dv_* \\
= {} & -(2\pi)^{-3/2} \int_{\mathbb{R}^3} \left[1 + |v-v_*|^2 - |\xi|^2 + 2i(v-v_*)\cdot\xi\right]^{\gamma/2} \\
& \cdot e^{-|v_*|^2/2} v_{*i} v_{*j}\, dv_*.
\end{aligned} \tag{2.31}$$

Therefore,

$$\left|\nabla^2_\xi h(v,\xi)\right| \le C \cdot \langle |v| \rangle^\gamma \langle |\xi| \rangle^\gamma. \tag{2.32}$$

This completes the proof of the second inequality. □

Lemma 2.7. *Suppose that* $0 < \nu < 1$ *in* (1.7). *Then for any* $r > 0$, $f \in L^1_{2+\gamma}(\mathbb{R}^3) \cap H^{+\infty}(\mathbb{R}^3)$, *there exists a constant C independent of r satisfying*

$$I_0(\tau) = \left(Q(f,\mu), \langle |D| \rangle^r f\right)_{L^2} \le C \|f\|_{L^1_{2+\gamma}} \|f\|_{L^1} (r+3)!. \tag{2.33}$$

Proof. Let $\xi^\pm = (\xi \pm |\xi|\sigma)/2$, from Bobylev's formula (see [10]), we have

$$\begin{aligned}
I_0(\tau) = {} & \int_{\mathbb{R}^3} \langle |\xi| \rangle^r \overline{\widehat{f}(\xi)} \left[\iint_{\mathbb{R}^6} \int_{\mathbb{S}^2} \Phi(|v-v_*|) b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) \mu(v_*) f(v) \right. \\
& \left. \times \left(e^{-i(v_*\cdot\xi^+ + v\cdot\xi^-)} - e^{-iv_*\cdot\xi}\right) d\sigma\, dv\, dv_* \right] d\xi \\
= {} & \int_{\mathbb{R}^3} \langle |\xi| \rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[\mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi^+) b f(v) e^{-iv\cdot\xi^-} \right. \\
& \left. - \mathcal{F}\left(\Phi(|v-v_*|)\mu(v_*)\right)(\xi) b f(v) \right] d\sigma\, dv\, d\xi \\
= {} & \int_{\mathbb{R}^3} \langle |\xi| \rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[h(v,\xi^+) - h(v,\xi)\right] \widehat{\mu}(\xi^+) b f(v) e^{-iv\cdot\xi^-}\, d\sigma\, dv\, d\xi \\
& + \int_{\mathbb{R}^3} \langle |\xi| \rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} h(v,\xi) \left[\widehat{\mu}(\xi^+) - \widehat{\mu}(\xi)\right] b f(v) e^{-iv\cdot\xi^-}\, d\sigma\, dv\, d\xi \\
& + \int_{\mathbb{R}^3} \langle |\xi| \rangle^r \overline{\widehat{f}(\xi)} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} h(v,\xi) \widehat{\mu}(\xi) b f(v) \left[e^{-iv\cdot\xi^-} - e^0\right] d\sigma\, dv\, d\xi \\
= {} & I_{01} + I_{02} + I_{03}.
\end{aligned} \tag{2.34}$$

In [1], it is shown that

$$\left|\widehat{\mu}(\xi^+) - \widehat{\mu}(\xi)\right| \leq \widehat{\mu}(\xi^+)|\xi|^2 \sin^2\frac{\theta}{2}, \qquad e^{-|\xi|^2/2} = \widehat{\mu}(\xi) \leq \widehat{\mu}(\xi^+) \leq e^{-|\xi|^2/4},$$
$$\langle|\xi|\rangle^{r+\gamma+2} \leq (r+3)! e^{\langle|\xi|\rangle^{(r+\gamma+2)/(r+3)}} \leq (r+3)! e^{\langle|\xi|\rangle}. \tag{2.35}$$

Together with Lemma 2.6, we have

$$\begin{aligned}
|I_{02}| &\leq C \cdot \|f\|_{L^1_{1+\gamma}} \int_{\mathbb{R}^3} \langle|\xi|\rangle^{r+\gamma+2} e^{-|\xi|^2/4} \overline{\widehat{f}(\xi)} d\xi \\
&\leq C \cdot \|f\|_{L^1_{1+\gamma}} \|f\|_{L^1} \int_{\mathbb{R}^3} \langle|\xi|\rangle^{r+\gamma+2} e^{-|\xi|^2/4} d\xi \\
&\leq C \cdot \|f\|_{L^1_{1+\gamma}} \|f\|_{L^1} (r+3)! \int_{\mathbb{R}^3} e^{\langle|\xi|\rangle - |\xi|^2/4} d\xi \\
&\leq C' \|f\|_{L^1_{1+\gamma}} \|f\|_{L^1} (r+3)!.
\end{aligned} \tag{2.36}$$

Now we turn to estimate the terms in I_{01} and I_{03}. For the case $0 < \nu < 1$ in (1.7), it is easy to see that

$$\left|e^{-iv\cdot\xi^-} - e^0\right| = \left|-2\sin\frac{v\cdot\xi^-}{2}\left(\sin\frac{v\cdot\xi^-}{2} + i\cos\frac{v\cdot\xi^-}{2}\right)\right| \leq C|v|\left|\xi^-\right| \leq C|v||\xi| \sin\frac{\theta}{2}. \tag{2.37}$$

Therefore, applying the above estimates and Lemma 2.6, we also conclude that

$$|I_{0i}| \leq C' \|f\|_{L^1_{1+\gamma}} \|f\|_{L^1} (r+3)! \tag{2.38}$$

for any $i \in \{1, 3\}$. This completes the proof of Lemma 2.7. □

3. Related Analysis

Let f be the weak solution of the Cauchy problem (1.1). For any $k \in \mathbb{N}$, the compact support

$$\operatorname{supp}(M_k f) \subseteq \operatorname{supp}(f), \tag{3.1}$$

which implies that for any compact subset $U \subset W$,

$$f^* = \begin{cases} f, & \text{if } v \in U, \\ 0, & \text{if } v \notin U, \end{cases} \tag{3.2}$$

is also a weak solution of the following equation

$$\left(\frac{\partial f}{\partial t}, M_k^2 f\right)_{L^2(\mathbb{R}^3)} = \left(Q(\mu, f) + Q(f, \mu), M_k^2 f\right)_{L^2(\mathbb{R}^3)}. \tag{3.3}$$

Since Theorem 1.3 is mainly concerned with the Gevrey smoothness property of the solution f on W, we need only to study the solution of the above equation on any fixed compact subset of W. That is, we can suppose that f has compact support in U for any $t \in [0, T]$,

$$\operatorname{supp}(f) \subseteq U, \qquad f(U^c) \equiv 0. \tag{3.4}$$

Thus, for any $p \geq 0$,

$$\begin{gathered} \|f\|_{L^1_p(\mathbb{R}^3)} \leq O(1)\|f\|_{L^1(U)} < +\infty, \\ \|f\|_{H^p(\mathbb{R}^3)} = \|f\|_{H^p(U)}. \end{gathered} \tag{3.5}$$

Together with Lemma 2.6, we can get the fact that $f \in H^{+\infty}(\mathbb{R}^3)$. This proof is similar as the proof of [11, Theorem 1.1] and hence omitted. Clearly, $\|f\|_{H^r(\mathbb{R}^3)} = \|f\|_{H^r(U)}$. Moreover, without loss of generality, we restrict $T \leq 1$, then for any $k \in \mathbb{N}$, it is assumed that

$$(E_k)\text{: for any } i \in [0, k-1], \quad \sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{it}} \leq C_0^{i+1}(i!)^s, \tag{3.6}$$

where C_0 is a sufficiently large constant satisfying

$$C_0 \geq 16^6 \max\left(\sup_{t\in(0,T]} \|f\|_{L^i},\ i = 1, 2\right). \tag{3.7}$$

In the following discussion, we will use C and C_i, $i \in \mathbb{N}$ to denote the positive constants independent of k and t. Let $M_k(D_v) = \langle |D_v| \rangle^{kt}$ and $\Phi^*(v) = \langle |v - v_*| \rangle^{\gamma}$. In order to prove Theorem 1.3, we need the propositions as below.

Proposition 3.1. *One has*

$$\sup_{t\in(0,T]} \left\|[M_k(D_v),\Phi^*]f(t,v)\right\|_{L^2} \leq C\cdot C_0^{k+1}(k!)^s. \tag{3.8}$$

Proposition 3.2. *One has*

$$\sup_{t\in(0,T]} \left\|\nabla_v[M_k(D_v),\Phi^*]f(t,v)\right\|_{L^2} \leq C\Big\{(k+1)\left\|M_k f(t,v)\right\|_{L^2} + C_0^{k+1}(k!)^s\Big\}, \tag{3.9}$$

$$\sup_{t\in(0,T]} \left\|[M_k(D_v),H^*]f(t,v)\right\|_{L^2} \leq C\cdot C_0^{k+1}(k!)^s, \tag{3.10}$$

where H^* is the function which has the form (2.14).

The proof of the above propositions will be given in Section 5.

4. Proof of Theorem 1.3

Now we will prove the main result in this section. For any $t\in(0,T]$, we state the following identity from [11]:

$$\Big(Q(\mu,f),M_k^2 f\Big)_{L^2} - \left(Q(\mu,M_k f),M_k f\right)_{L^2} = I_1+I_2+I_3, \tag{4.1}$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)(M_k(\xi)-M_k(\xi^+))\widehat{\Phi^* f}(\xi^+)e^{-iv_*\xi^-}\overline{M_k(\xi)\widehat{f}(\xi)}d\sigma\, dv_* d\xi,\\
I_2 &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)\{[M_k,\Phi^*]f(v')\cdot M_k f(v') - [M_k,\Phi^*]f(v)\cdot M_k f(v)\}d\sigma\, dv_* dv,\\
I_3 &= \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(\cos\theta)\mu(v_*)\left([M_k,\Phi^*]f(v) - [M_k,\Phi^*]f(v')\right)M_k f(v')d\sigma\, dv_* dv.
\end{aligned} \tag{4.2}$$

Our purpose is to obtain the estimations of I_1, I_2 and I_3. Setting $\eta=|\xi|^2$ and $\eta^+=|\xi^+|^2$, since $|\xi^+|=|\xi|\cos(\theta/2)$ and $|\xi^+|^2-|\xi|^2=|\xi|^2\sin^2(\theta/2)$, applying the mean value theorem and the fact that $0<t\leq T\leq 1$, we have

$$\begin{aligned}
|M_k(\xi)-M_k(\xi^+)| &= \left|(1+\eta)^{kt/2}-(1+\eta^+)^{kt/2}\right|\\
&\leq C\cdot kt\cdot(\eta-\eta^+)(1+\eta_0)^{kt/2-1}\\
&\leq C\cdot kt\cdot\sin^2\frac{\theta}{2}(1+\eta)^{kt/2}\\
&\leq C'\cdot k\cdot\sin^2\theta M_k(\xi),
\end{aligned} \tag{4.3}$$

where η_0 is a number between η an η^+. Therefore,

$$
\begin{aligned}
|I_1| &\leq C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\sin^2\theta M_k(\xi)\left|\iint_{\mathbb{R}^6}\Phi(|v-v_*|)\mu(v_*)f(v)e^{-iv_*\cdot\xi^- - iv\cdot\xi^+}dv_*dv\right| \\
&\quad\times\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\, d\xi \\
&= C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\sin^2\theta M_k(\xi)\left|\iint_{\mathbb{R}^6}\Phi(|v_*|)\mu(v_*+v)f(v)e^{-iv_*\cdot\xi^- - iv\cdot\xi}dv_*dv\right| \\
&\quad\times\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\, d\xi \\
&\leq C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|M_k(\xi)\widehat{H^*f}(\xi)\right|\cdot\left|kM_k(\xi)\widehat{f}(\xi)\right|d\sigma\, dv_*d\xi \\
&\leq C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|M_k(\xi)\widehat{H^*f}(\xi)\right|^2 d\sigma\, dv_*d\xi \\
&\quad+C\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\sin^2\theta\langle|v_*|\rangle^{\gamma-8}\cdot\left|kM_k(\xi)\widehat{f}(\xi)\right|^2 d\sigma\, dv_*d\xi = I_{11}+I_{12}.
\end{aligned}
\tag{4.4}
$$

Here H^* is the function which has the form (2.14). It is clear that

$$
\begin{aligned}
I_{12} &\leq C'k^2\|M_kf\|_{L^2}^2, \\
I_{11} &\leq C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\cdot\|M_k(D_v)H^*f\|_{L^2}^2dv_* \\
&\leq C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\|[M_k,H^*]f\|_{L^2}^2dv_* \\
&\quad+C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma-8}\|H^*M_kf\|_{L^2}^2dv_* \\
&= I_{111}+I_{112}.
\end{aligned}
\tag{4.5}
$$

By the hypothesis (3.4), f has compact support in $\overline{U}$, we obtain

$$
\begin{aligned}
I_{112} &\leq C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma+8}\cdot\|\mu(v+v_*)M_kf\|_{L^2}^2dv_* \\
&\leq C\int_{\mathbb{R}^3}\langle|v_*|\rangle^{\gamma+8}e^{-|v_*|^2/4}\cdot\left\|e^{|v|^2/2}M_kf\right\|_{L^2}^2dv_* \\
&\leq C'\|M_kf\|_{L^2}^2.
\end{aligned}
\tag{4.6}
$$

Here we use the fact that $e^{-|v+v_*|^2/2} \leq e^{|v|^2/2} \cdot e^{-|v_*|^2/4}$. By (3.10) of Proposition 3.2, we get

$$I_{111} \leq C' \cdot \left[C_0^{k+1}(k!)^s\right]^2. \tag{4.7}$$

This, together with (4.5)-(4.6), implies

$$|I_1| \leq C_1 \cdot \left(k^2\|M_k f\|_{L^2}^2 + \left[C_0^{k+1}(k!)^s\right]^2\right). \tag{4.8}$$

The cancellation lemma gives (cf. [10, 11])

$$I_2 = S\int_{\mathbb{R}^6} \mu(v_*)[M_k, \Phi^*]f(v) \cdot M_k f(v) dv\, dv_*, \tag{4.9}$$

where S is a constant function. Therefore,

$$\begin{aligned} |I_2| &\leq C\|\mu\|_{L^1} \cdot \|[M_k, \Phi^*]f\|_{L^2} \cdot \|M_k f\|_{L^2} \\ &\leq C \cdot C_0^{k+1}(k!)^s\|\mu\|_{L^1} \cdot \|M_k f\|_{L^2} \\ &\leq C_2\left\{\left[C_0^{k+1}(k!)^s\right]^2 + \|M_k f\|_{L^2}^2\right\}. \end{aligned} \tag{4.10}$$

Since $|v' - v| \leq C\langle|v'|\rangle\langle|v_*|\rangle \sin(\theta/2)$, by using (3.4), Proposition 3.2, and the change of variables

$$v \longrightarrow z = v' + \tau(v - v') \tag{4.11}$$

whose Jacobian is bounded uniformly for v_*, σ, τ (see [11]), we have

$$\begin{aligned} |I_3| \leq C\int_0^1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} & b(\cos\theta)|v' - v| \cdot |M_k f(v')| \cdot |\nabla_v[M_k, \Phi^*]f(v' + \tau(v - v'))| \\ &\times |\mu(v_*)| d\sigma\, dv\, dv_* d\tau \\ \leq C'\int_0^1\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} & b(\cos\theta)\sin\frac{\theta}{2}|M_k f(v')| \cdot |\nabla_v[M_k, \Phi^*]f(v' + \tau(v - v'))| \\ &\times \langle|v_*|\rangle|\mu(v_*)| d\sigma\, dv\, dv_* d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq C' \int_0^1 \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(\cos\theta) \sin\frac{\theta}{2} \langle |v_*| \rangle \mu(v_*) \\
&\qquad\qquad \times \left\{ \left| M_k f(v') \right|^2 + \left| \nabla_v [M_k, \Phi^*] f(v' + \tau(v - v')) \right|^2 \right\} d\sigma dv dv_* d\tau \\
&\leq C'' \left(\| M_k f \|_{L^2}^2 + \| \nabla_v [M_k, \Phi^*] f \|_{L^2}^2 \right) \\
&\leq C_3 \left\{ \left[C_0^{k+1} (k!)^s \right]^2 + k^2 \| M_k f \|_{L^2}^2 \right\}.
\end{aligned}
\tag{4.12}
$$

Combining (4.8), (4.10), and (4.12), we obtain

$$
\left(Q(\mu, f), M_k^2 f \right)_{L^2} - \left(Q(\mu, M_k f), M_k f \right)_{L^2} \leq C_4 \left\{ \left[C_0^{k+1} (k!)^s \right]^2 + k^2 \| M_k f \|_{L^2}^2 \right\}. \tag{4.13}
$$

Moreover, by [11, Lemma 2.2] and [11, page 467], we have

$$
\begin{gathered}
\left\| \langle \cdot \rangle^{\gamma/2} M_k f \right\|_{H^{\nu/2}}^2 = O(1) \left\| \langle |D| \rangle^{\nu/2} M_k f \right\|_{L^2_{\gamma/2}}^2, \\
\left(Q(\mu, M_k f), M_k f \right)_{L^2} \leq -C_{\mu,1} \left\| \langle \cdot \rangle^{\gamma/2} M_k f \right\|_{H^{\nu/2}}^2 + C_{\mu,2} \| M_k f \|_{L^2_{\gamma/2}}^2,
\end{gathered}
\tag{4.14}
$$

where $C_{\mu,1}$ and $C_{\mu,2}$ are the constants depending only on μ. Therefore, by (3.4) and (4.14), we get

$$
\left(Q(\mu, M_k f), M_k f \right)_{L^2} \leq -C_5 \| M_k f \|_{H^{\nu/2}}^2 + C_6 \| M_k f \|_{L^2}^2. \tag{4.15}
$$

Together with (4.13), we thus have

$$
\left(Q(\mu, f), M_k^2 f \right)_{L^2} \leq C_7 \cdot \left[C_0^{k+1} (k!)^s \right]^2 + C_8 \cdot k^2 \| M_k f \|_{L^2}^2 - C_5 \| M_k f \|_{H^{\nu/2}}^2. \tag{4.16}
$$

Let $M_k^2 f$ be the test function in the Cauchy problem (1.1), for any $t \in (0, T]$, we have

$$
\begin{aligned}
\| M_k f(t, v) \|_{L^2}^2 &= \| f_0(v) \|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} L(f)(\tau, v) M_k^2 f(\tau, v) dv \, d\tau \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} f(\tau, v) \left(\partial_\tau M_k^2(\tau) \right) f(\tau, v) dv \, d\tau \\
&= 2 \int_0^t \left\{ \left(Q(\mu, f), M_k^2 f \right)_{L^2} + \left(Q(f, \mu), M_k^2 f \right)_{L^2} \right\} d\tau \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} f(\tau, v) \left(\partial_\tau M_k^2(\tau) \right) f(\tau, v) dv \, d\tau + \| f_0(v) \|_{L^2}^2.
\end{aligned}
\tag{4.17}
$$

Since

$$O(1)(k!)^{2s} \geq 2^{2(k+2)}[(k+2)!]^2 \geq (2k+3)!,$$
$$\partial_t M_k^2(t,\xi) = 2k M_k^2(t,\xi)\log\langle\xi\rangle, \tag{4.18}$$

by Lemma 2.7 and (4.16), it holds that

$$\begin{aligned}
&\|M_k f(t,v)\|_{L^2}^2 + C_5 \int_0^t \|M_k f\|_{H^{\nu/2}}^2 d\tau \\
&\quad \leq 2\mathrm{k} \int_0^t \left\| (\log\langle D_v\rangle)^{1/2} (M_k f)(\tau) \right\|_{L^2}^2 d\tau \\
&\qquad + C_8 \int_0^t k^2 \|M_k f\|_{L^2}^2 d\tau + C_9 \left[C_0^{k+1}(k!)^s\right]^2 + \|f_0(v)\|_{L^2}^2.
\end{aligned} \tag{4.19}$$

The Young's inequality gives

$$\begin{aligned}
C_8 k^2 &\leq \left[\frac{4C_8}{C_5\cdot(\nu+2)}\right]^{2/\nu} \cdot \frac{C_8\nu}{\nu+2} \cdot k^{2+4/\nu}\langle|\xi|\rangle^{-2} + \frac{C_5}{2}\langle|\xi|\rangle^{\nu}, \\
2k\log\langle|\xi|\rangle &= \frac{4k}{\nu}\log\langle|\xi|\rangle^{\nu/2} \\
&\leq \frac{4k}{\nu}\cdot\langle|\xi|\rangle^{\nu/2} \\
&\leq \left[\frac{4\cdot(4+\nu)}{C_8\cdot\nu(\nu+2)}\right]^{(4+\nu)/\nu} \frac{2}{\nu+2}\cdot k^{2+4/\nu}\langle|\xi|\rangle^{-2} + \frac{C_5}{2}\langle|\xi|\rangle^{\nu},
\end{aligned} \tag{4.20}$$

which implies

$$\begin{aligned}
&2k\int_0^t \left\| (\log\langle D_v\rangle)^{1/2}(M_k f)(\tau) \right\|_{L^2}^2 d\tau + C_8 \int_0^t \|k M_k f\|_{L^2}^2 d\tau \\
&\quad \leq C_5 \int_0^t \|M_k f\|_{H^{\nu/2}}^2 d\tau + C_{10}\cdot k^{2+4/\nu}\cdot\int_0^t \|M_{k-1} f\|_{L^2}^2 d\tau.
\end{aligned} \tag{4.21}$$

Taking (4.21) into (4.19), and applying the assumption (E_k), we have

$$\begin{aligned}
\|M_k f(t,v)\|_{L^2}^2 &\leq \|f_0(v)\|_{L^2}^2 + C_9\left[C_0^{k+1}(k!)^s\right]^2 + C_{10}\cdot k^{2+4/\nu}\int_0^t \|M_{k-1}f\|_{L^2}^2 d\tau \\
&\leq \|f_0(v)\|_{L^2}^2 + C_9\left[C_0^{k+1}(k!)^s\right]^2 + C_{10}\cdot k^{2+4/\nu}\left\{C_0^k\cdot[(k-1)!]^s\right\}^2 \\
&\leq C_{11}\left[C_0^{k+1}(k!)^s\right]^2,
\end{aligned} \tag{4.22}$$

which implies that

$$\sup_{t\in(0,T]} \left\| M_k f(t,v) \right\|_{L^2} = \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{kt}} \le C_{11} \cdot C_0^{k+1} (k!)^s. \tag{4.23}$$

In other words, it follows from (E_k) that

$$(E_{k+1})\text{: for any } i \in [0,k], \quad \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{it}} \le C_{11} \cdot C_0^{i+1} (i!)^s. \tag{4.24}$$

Taking the same procedures as above, we can also gain (E_{k+2}) from (E_{k+1}), which is described as below:

$$(E_{k+2})\text{: for any } i \in [0,k+1], \quad \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{it}} \le C_{11}^2 \cdot C_0^{i+1} (i!)^s, \tag{4.25}$$

that is,

$$\begin{aligned}
\sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{0t}} \le C_0^1 (0!)^s &\Longrightarrow \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{1t}} \le C_{11}^1 \cdot C_0^2 (1!)^s \\
&\Longrightarrow \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{2t}} \le C_{11}^2 \cdot C_0^3 (2!)^s \\
&\quad \vdots \\
&\Longrightarrow \sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{kt}} \le C_{11}^k \cdot C_0^{k+1} (k!)^s.
\end{aligned} \tag{4.26}$$

Let $C_{12} = C_0 \cdot C_{11}$, we thus conclude that for any $k \in \mathbb{N}$,

$$\sup_{t\in(0,T]} \left\| f(t,\cdot) \right\|_{H^{kt}(U)} \le C_{12}^{k+1} (k!)^s. \tag{4.27}$$

For any fixed number $0 < t \le T \le 1$, suppose that

$$\begin{aligned}
s_0 &= \left(\left[\frac{1}{t} \right] + 1 \right) s, \\
s_1 &= \sum_{i=0}^{[1/t]} (1 - it),
\end{aligned} \tag{4.28}$$

where $\lceil 1/t\rceil$ denotes the smallest integer bigger than $1/t$. With a convention that $k! = 1$ if $0 \geq k \in \mathbb{Z}$, we have

$$\begin{aligned}
2^{ks_1} &= 2^k \cdot 2^{k(1-t)} \cdot 2^{k(1-2t)} \cdots 2^{k(1-\{1/t\}t)} \\
&\geq \frac{k!}{(kt)![(1-t)k]!} \cdot \frac{[(1-t)k]!}{(kt)![(1-2t)k]!} \cdots \frac{[(1-\lceil 1/t\rceil t)k]!}{(kt)![(1-(\lceil 1/t\rceil+1)t)k]!} \\
&\geq \frac{k!}{[(kt)!]^{(\lceil 1/t\rceil+1)}} \\
&\geq \frac{k!}{[(kt)!]^{s_0/s}}.
\end{aligned} \tag{4.29}$$

This, together with (4.27), implies that

$$\sup_{t\in(0,T]} \|f(t,\cdot)\|_{H^{kt}(U)} \leq C_{12}^{k+1}(k!)^s \leq 2^{kss_1}C_{12}^{k+1}[(kt!)]^{s_0} \leq C_{13}^{k+1}[(kt!)]^{s_0}, \tag{4.30}$$

where $k \in \mathbb{N}$, and C_{16} is a constant only depending on t. Furthermore, for any fixed number $t_0 > 0$, put

$$\begin{aligned}
s_0' &= \left(\left\lceil \frac{1}{t_0}\right\rceil + 1\right)s, \\
s_1' &= \sum_{i=0}^{\lceil 1/t_0\rceil}(1 - it_0).
\end{aligned} \tag{4.31}$$

Then for any $k \in \mathbb{N}$, we can choose $C_{13}' = 2^{ss_1'}C_{12}$ and have the fact that

$$\sup_{t\in[t_0,T]} \|f(t,\cdot)\|_{H^{kt}(U)} \leq \left(C_{13}'\right)^{k+1}[(k!)]^{s_0'}. \tag{4.32}$$

This completes the proof of Theorem 1.3.

5. Proof of Propositions 3.1 and 3.2

Proof of Proposition 3.1. We first notice that

$$\begin{aligned}
[M_k(D_v),\Phi^*]f(v) &= M_k(D_v)(\Phi^* f)(v) - \Phi^*(v)M_k(D_v)f(v) \\
&= \left(\mathcal{F}^{-1}M_k(\xi) * \Phi^* f\right)(v) - \Phi^*(v)\left(\mathcal{F}^{-1}M_k(\xi) * f\right)(v) \\
&= \int_{\mathbb{R}^6} e^{i(v-y)\xi}M_k(\xi)d\xi f(y)(\Phi^*(y) - \Phi^*(v))dy.
\end{aligned} \tag{5.1}$$

Using the Taylor formula of order $k+6$, we get

$$\Phi^*(y) - \Phi^*(v) = \sum_{j=1}^{k+5} \frac{(y-v)^j}{j!} \partial_v^j \Phi^*(v) + \frac{(y-v)^{k+6}}{(k+6)!} \partial_v^{k+6} \Phi^*(c) \tag{5.2}$$

for some $c \in (y, v)$. Hence,

$$[M_k(D_v), \Phi^*] f(v) = \sum_{j=1}^{k+5} \Gamma_j f(v) + \Gamma_{k+6} f, \tag{5.3}$$

where

$$\begin{aligned}
\Gamma_j f(v) &= \int_{\mathbb{R}^6} e^{i(v-y)\xi} M_k(\xi) d\xi f(y) \frac{(y-v)^j}{j!} \partial_v^j \Phi^*(v) dy \\
&= \frac{(-i)^j}{j!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^j M_k(\xi) d\xi f(y) \partial_v^j \Phi^*(v) dy \\
&= \frac{(-i)^j}{j!} \left(\mathcal{F}^{-1} \partial_\xi^j M_k * f \right)(v) \cdot \partial_v^j \Phi^*(v), \\
\Gamma_{k+6} f &= \int_{\mathbb{R}^6} e^{i(v-y)\xi} M_k(\xi) d\xi f(y) \frac{(y-v)^{k+6}}{(k+6)!} \partial_v^{k+6} \Phi^*(c) dy \\
&= \frac{(-i)^{k+6}}{(k+6)!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^{k+6} M_k(\xi) d\xi f(y) \partial_v^{k+6} \Phi^*(c) dy.
\end{aligned} \tag{5.4}$$

From Lemma 2.1, Remark 2.2, (3.6), and (3.7), it follows that

$$\begin{aligned}
\sum_{j=1}^{k} \left\| \Gamma_j f(v) \right\|_{L^2} &= \sum_{j=1}^{k} \left\| \frac{(-i)^j}{j!} \left(\mathcal{F}^{-1} \partial_\xi^j M_k * f \right)(v) \cdot \partial_v^j \Phi^*(v) \right\|_{L^2} \\
&\leq \sum_{j=1}^{k} \frac{1}{j!} \left\| \partial_\xi^j M_k(\xi) \mathcal{F} f(\xi) \right\|_{L^2} \cdot \left\| \partial_v^j \Phi^*(v) \right\|_{L^\infty} \\
&\leq C \cdot \sum_{j=1}^{k} 16^j k \cdots (k-j+1) \{(k-j)!\}^s \cdot C_0^{k-j+1} \\
&\leq C \cdot C_0^{k+1} \{(k-1)!\}^s k^2 \\
&\leq C_1 \cdot C_0^{k+1} (k!)^s,
\end{aligned}$$

$$
\begin{aligned}
\sum_{j=k+1}^{k+5} \left\|\Gamma_j f(v)\right\|_{L^2} &\leq \sum_{j=k+1}^{k+5} \frac{1}{j!} \left\|\partial_\xi^j M_k(\xi) \mathcal{F} f(\xi)\right\|_{L^2} \cdot \left\|\partial_v^j \Phi^*(v)\right\|_{L^\infty} \\
&\leq C \cdot (k+5)! \sup_{t\in(0,T]} \left\|f(t,v)\right\|_{L^2} \\
&\leq C_2 \cdot C_0^{k+1} (k!)^s, \\
\left\|\Gamma_{k+6} f\right\|_{L^2} &= \left\| \frac{(-i)^{k+6}}{(k+6)!} \int_{\mathbb{R}^6} e^{i(v-y)\xi} \partial_\xi^{k+6} M_k(\xi) d\xi f(y) \partial_v^{k+6} \Phi^*(c) dy \right\|_{L^2} \\
&\leq \frac{1}{(k+6)!} \left\| \int_{\mathbb{R}^6} \left|\partial_\xi^{k+6} M_k(\xi)\right| d\xi \cdot \left|f(y)\right| \cdot \left|\partial_v^{k+6} \Phi^*(c)\right| dy \right\|_{L^2} \\
&\leq C \cdot C_0^{k+1} (k!)^s \int_{\mathbb{R}^3} \left(1+|\xi|^2\right)^{-3} d\xi \\
&\leq C_3 \cdot C_0^{k+1} (k!)^s.
\end{aligned}
\tag{5.5}
$$

Combining (5.5), we complete the proof of Proposition 3.1. □

Proof of Proposition 3.2. One has

$$
\begin{aligned}
\left\|\nabla_v [M_k(D_v), \Phi^*] f(t,v)\right\|_{L^2} &\leq C \cdot \left\|\langle|\xi|\rangle \mathcal{F}\left([M_k, \Phi^*] f\right)(t,\xi)\right\|_{L^2} \\
&\leq C \cdot \left\|[\langle D_v\rangle M_k, \Phi^*] f(t,v) - [\langle D_v\rangle, \Phi^*] M_k f(t,v)\right\|_{L^2}.
\end{aligned}
\tag{5.6}
$$

Similar to the proof of Proposition 3.1, we obtain

$$
\begin{aligned}
\left\|[\langle D_v\rangle M_k, \Phi^*] f(t,v)\right\|_{L^2} &\leq C\left\{(k+1)\left\|M_k f(t,v)\right\|_{L^2} + C_0^{k+1}(k!)^s\right\}, \\
\left\|[\langle D_v\rangle, \Phi^*] M_k f(t,v)\right\|_{L^2} &\leq C\left\{\left\|M_k f(t,v)\right\|_{L^2} + C_0^{k+1}(k!)^s\right\}.
\end{aligned}
\tag{5.7}
$$

Then (3.9) is obtained. The proof of (3.10) is similar so is omitted. This completes the proof of Proposition 3.2. □

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