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Research Article

On the Regularity of Solutions to an Adjoint Elliptic Equation with Partially VMO Coefficients

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We establish, in dimension two, a regularity result for nonnegative solutions to an adjoint elliptic equation, generalizing a previous result of Escauriaza (1994). We consider elliptic equations with coefficients $a_{ij}(x_1, x_2)$ which are measurable with respect to one variable and VMO with respect to the other.

1. Introduction

Let us consider a planar elliptic operator of nondivergence form:

$$\mathcal{M} = \sum_{ij} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1.1}$$

where $a_{ij} = a_{ji}$ for $i, j = 1, 2$ are measurable and the symmetric matrix

$$A(x) = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix} \tag{1.2}$$

is uniformly elliptic, that is,

$$\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi \rangle \leq \sqrt{K}|\xi|^2, \tag{1.3}$$

for all $\xi \in \mathbb{R}^2$ and a.e. $x = (x_1, x_2) \in \Omega$, a bounded open subset of $\mathbb{R}^2$. Here the ratio $\sqrt{K}/1/\sqrt{K} = K$ is the ellipticity constant.

The study of weak solutions v to the *adjoint equation* (adjoint solutions, for short)

$$\mathcal{M}^*[v] = \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x) v(x) \right) = 0 \tag{1.4}$$

often occurs in the literature (see Section 3, and for a very recent paper, see [1]).

We say that the function $v \in L^1_{\mathrm{loc}}(\Omega)$ is a weak solution to (1.4) if

$$\int_\Omega v \mathcal{M}[\varphi] = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.5}$$

In this paper we make the assumption that the coefficients $a_{ij}(x_1, x_2)$ are VMO with respect to one of the two variables (see Section 2). This kind of assumption has been recently considered mainly for divergence ($L[u] = \mathrm{div}(A(x)\nabla u) = 0$) or nondivergence ($M[w] = \mathrm{Tr}(A(x)D^2 w) = 0$) elliptic equations.

On the other hand, in [2] Escauriaza gave a regularity result for nonnegative solutions to adjoint equation with VMO coefficients.

Here, in case $n = 2$, we give a generalized form of Theorem 1.2 in which he proves that, in particular,

$$v \in \bigcap_{q>1} G_q, \tag{1.6}$$

where G_q is the Gehring class, as defined in Section 2.

2. Definitions and Notations

In order to describe the results of the present paper, it is necessary to introduce some definitions. We start recalling basic definitions of the G_q classes, introduced by Gehring [3], in connection with local integrability properties of the gradient of quasiconformal mappings.

Let us assume that v is a weight, that is, a nonnegative locally integrable function on $\mathbb{R}^2$ and consider cubes $Q \subset \mathbb{R}^2$ with sides parallel to the coordinate axes. We will set

$$v_Q = ⨍_Q v(x)\,dx = \frac{1}{|Q|} \int_Q v(x)\,dx \tag{2.1}$$

to denote the mean value of v over Q, where $|Q|$ denotes the 2-dimensional Lebesgue measure of a subset Q of $\mathbb{R}^2$.

Definition 2.1. A weight v satisfies the G_q-condition if there exists a constant $G \geq 1$ such that, for all cubes $Q \subset \mathbb{R}^2$ as above, one has

$$\frac{\left(\fint_Q v^q(x)dx\right)^{1/q}}{\fint_Q v(x)dx} \leq G, \tag{2.2}$$

and one refers to (2.2) as a "reverse" Hölder inequality.

In the following, we will consider elliptic differential equations with coefficients $a_{ij}(x)$ of the matrix A measurable with respect to one variable and vanishing mean oscillation (VMO) with respect to the other (we say *partially*-VMO, for short). We recall that the space VMO, introduced by Sarason [4], is a subspace of the functions in the John-Nirenberg space BMO. More precisely, VMO is defined as the closure in BMO of the subspace of uniformly continuous functions.

Definition 2.2. A locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ is in VMO if

$$\lim_{r \to 0} \fint_{B_r} |f(y) - f_{B_r}| dy = 0, \tag{2.3}$$

where $B_r = B(x, r)$ denotes a ball centered at $x \in \mathbb{R}^2$, with radius r. One will also assume that f is defined at ∞ in the following average sense:

$$f(\infty) = \lim_{r \to \infty} \frac{1}{\pi r^2} \int_{B_r(0)} f(y) dy, \tag{2.4}$$

(see [5]).

3. Examples

In the present section we collect a certain number of examples where solutions v to the adjoint equation

$$\mathcal{M}^*[v] = \sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) v(x)) = 0 \tag{3.1}$$

occur. The first example deals with adjoint solutions which are partial derivatives h_{x_1} and h_{x_2} of a very weak solution $h \in W^{1,2}_{\mathrm{loc}}$ to a particular diagonal divergence type equation, and an interesting relation comes out between regularity results.

Example 3.1. For $1/\sqrt{K} \le \beta \le \sqrt{K}$, we consider the following elliptic operators in $\mathbb{R}^2$:

$$\mathcal{M} = \frac{\partial^2}{\partial x_1^2} + \beta \frac{\partial^2}{\partial x_2^2},$$
$$\mathcal{L} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2}\left(\beta \frac{\partial}{\partial x_2}\right). \tag{3.2}$$

Fortuitous relations occur between adjoint solutions to $\mathcal{M}$ and solutions h to $\mathcal{L}[h] = 0$, as the following Lemma reveals (see [6]).

Lemma 3.2 (see [6]). *Let $h \in W^{1,2}_{\mathrm{loc}}(B_r)$, where B_r denotes the open ball in $\mathbb{R}^2$ centered at 0 with radius r, such that*

$$\mathcal{L}[h] = 0. \tag{3.3}$$

Set $w = \partial h/\partial x_1$, $v = \partial h/\partial x_2$ Then

$$\mathcal{M}^*[w] = 0, \qquad \mathcal{M}^*[v] = 0. \tag{3.4}$$

Proof. We proceed similarly as in [6]. If $\phi \in C_0^\infty(B_r)$, we have

$$\begin{aligned} \int_{B_r} w \cdot \mathcal{M}\phi \, dx &= \int_{B_r} \frac{\partial h}{\partial x_1}\left(\frac{\partial^2 \phi}{\partial x_1^2} + \beta \frac{\partial^2 \phi}{\partial x_2^2}\right) dx_1 dx_2 \\ &= \int_{B_r} \left(\frac{\partial h}{\partial x_1} \cdot \frac{\partial^2 \phi}{\partial x_1^2} + \beta \frac{\partial h}{\partial x_2} \cdot \frac{\partial^2 \phi}{\partial x_1 \partial x_2}\right) dx_1 dx_2 = 0. \end{aligned} \tag{3.5}$$

Thus $\mathcal{M}^*[w] = 0$. In analogous way one checks that $\mathcal{M}^*[v] = 0$. □

Corollary 3.3. *Let $h \in W^{1,2}_{\mathrm{loc}}$ such that $\mathcal{L}[h] = 0$. If $\partial h/\partial x_1 \geq 0$ and $\partial h/\partial x_2 \geq 0$, then for any ball $B_r \subset B_{2r} \subset \mathbb{R}^2$, one has*

$$\left(⨍_{B_r} |\nabla h|^p dx\right)^{1/p} \le c(K,p) ⨍_{B_r} |\nabla h| dx, \tag{3.6}$$

where $2 \le p < 2K/(K-1)$.

Proof. See [7, Theorem 3.1]. □

Compare this with the following well-known result of Astala [8] (see also Leonetti-Nesi [9]).

Theorem 3.4 (see [8, 9]). *Let $h \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a local solution to the equation*

$$\operatorname{div}(A(x)\nabla h(x)) = 0, \quad x \in \Omega, \tag{3.7}$$

where A is a real symmetric matrix satisfying the ellipticity bounds,

$$\frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi \rangle \leq \sqrt{K}|\xi|^2, \quad \forall \xi \in \mathbb{R}^2, \text{ for a.e. } x \in \Omega. \tag{3.8}$$

Then, for any ball $B_{2r} \subset \Omega$ one has

$$\left(⨍_{B_r} |\nabla h|^s dx \right)^{1/s} \leq c(K,s) ⨍_{B_{2r}} |\nabla h| dx, \tag{3.9}$$

where $2 \leq s < 2\sqrt{K}/(\sqrt{K}-1)$.

Notice that, while the exponent in the left-hand side of the reverse inequality (3.9) may be greater than the exponent in the reverse inequality (3.6), this one is stronger in another sense, because it involves the same support B_r at both sides.

Example 3.5. In [10] (see also [11]) the Jacobian $v = \det DU$, where $U : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is a locally univalent A-harmonic mapping; that is, its components are $W^{1,2}_{\text{loc}}$ solution to (3.7), is shown to be solution to an adjoint equation for the elliptic operator

$$\mathcal{M}[v] = \frac{\partial^2 v}{\partial x_1^2} + c\frac{\partial^2 v}{\partial x_2^2}, \tag{3.10}$$

where $1/K \leq c \leq K$.

Example 3.6. Very recently [12, 13], the reduced Beltrami differential equation

$$\frac{\partial f}{\partial \bar{z}} = \lambda(z)\mathcal{I}m\left(\frac{\partial f}{\partial z}\right), \quad |\lambda(z)| \leq k < 1, \ k = \frac{K-1}{K+1} \tag{3.11}$$

has been introduced and studied because it naturally arises in different contexts in the theory of quasiconformal mappings. It turns out that the partial derivatives of the components u, v of $f(z) = u(z) + iv(z)$ solution to (3.11) satisfy the equation

$$u_{x_2} = \frac{b(z)-1}{b(z)+1} v_{x_1} \tag{3.12}$$

with $b(z) = \mathcal{I}m\lambda(z)$. As a consequence of (3.12) in [13], it is proved that $u_{x_2} \neq 0$ a.e, and it is an adjoint solution for a suitable elliptic operator $\mathcal{M} = \sum_{ij} b_{ij}(z)(\partial^2/\partial x_i \partial x_j)$.

Namely, it has been proved [13] that u is a solution to an elliptic equation of divergence form

$$\operatorname{div} A(z)\nabla u = 0, \tag{3.13}$$

where $A(z)$ is of the type

$$A(z) = \begin{pmatrix} 1 & a_{12}(z) \\ 0 & a_{22}(z) \end{pmatrix}. \tag{3.14}$$

As a consequence, the function $v = u_{x_2}$ is a solution to the adjoint equation

$$\mathcal{M}^*[v] = 0, \tag{3.15}$$

where

$$\mathcal{M} = \frac{\partial^2}{\partial x_1^2} + a_{12}\frac{\partial^2}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2}{\partial x_2^2}. \tag{3.16}$$

Note that the matrix $A(z)$ is not symmetric; however, the operator $\mathcal{M}$ can also be represented by the symmetric and uniformly elliptic matrix

$$B(x) = \begin{pmatrix} 1 & \dfrac{a_{12}(z)}{2} \\ \dfrac{a_{12}(z)}{2} & a_{22}(z) \end{pmatrix}. \tag{3.17}$$

Notice also that $v = u_{x_2} > 0$ a.e. (see [13]), and moreover, by general properties of nonnegative adjoint solutions, v satisfies a reverse Hölder inequality [7, 14, 15]

$$\left(\fint_{B_r} v(z)^2 dz \right)^{1/2} \leq c(K) \fint_{B_r} v(z) dz, \tag{3.18}$$

in every ball $B_r \subset \Omega$ such that $B_{2r} \subset \Omega$. Hence v is identically zero or $v > 0$ a.e. [13].

Example 3.7. The properties of the adjoint solutions are also very useful for studing the G-convergence of non divergence operators, as shown, for example, in a paper of D'Onofrio and Greco [16]. In that paper the authors consider elliptic operators $\mathcal{M}$ of non divergence type, defined by

$$\mathcal{M}[u] = \operatorname{Tr}\left(AD^2 u\right), \quad \text{for } u \in W^{2,2}(\Omega),\ \Omega \subset \mathbb{R}^2, \tag{3.19}$$

where $A = (a_{ij}) \in \mathbb{M}$, the set of all symmetric 2×2 real matrices and satisfy the ellipticity condition (1.3).

The adjoint to the operator $\mathcal{M}$ is given by $\mathcal{M}^*[v] = (a_{11}v)_{x_1x_2} + 2(a_{12}v)_{x_1x_2} + (a_{22}v)_{x_2x_2}$ and reveals useful behaviour with respect to G-convergence of sequence of operators of the form (3.19).

Proposition 3.8 (see[16]). *Let $\mathcal{M}_k$, $k = 1, 2, \ldots, \mathcal{M}$ be operators whose coefficient matrices A_k, $A \in \mathbb{M}$ and satisfying* (1.3). *Assume that $v_k \in L^2(\Omega)$ are solutions to the adjoint equations $\mathcal{M}_k^*[v_k] = 0$ and verify that*

$$\begin{aligned} v_k &\rightharpoonup v \quad \text{in } L^2(\Omega), \\ v_k A_k &\rightharpoonup vA \quad \text{in } L^2(\Omega; \mathbb{M}), \end{aligned} \tag{3.20}$$

where $v(x) > 0$ a.e. in Ω. Then, one has $\mathcal{M}_k \xrightarrow{G} \mathcal{M}$.

In order to prove Proposition 3.8, the following lemma is crucial.

Lemma 3.9 (see [16]). *Let $\mathcal{M}_k$, $k = 1, 2, \ldots, \mathcal{M}$ be operators with coefficient matrices A_k, $A \in \mathbb{M}$ and satisfy* (1.3), *$v_k \in L^2(\Omega)$ satisfying $\mathcal{M}_k^*[v_k] = 0$, and let $v_k \in W^{2,2}_{\text{loc}}(\Omega)$ be given. If*

$$\begin{aligned} u_k &\rightharpoonup u \quad \text{in } W^{2,2}_{\text{loc}}(\Omega), \\ v_k A_k &\rightharpoonup vA \quad \text{in } L^2_{\text{loc}}(\Omega; \mathbb{M}), \end{aligned} \tag{3.21}$$

then

$$\operatorname{Tr}\left(v_k A_k D^2 u_k\right) \longrightarrow \operatorname{Tr}\left(vAD^2 u\right) \tag{3.22}$$

in the sense of distributions.

Moreover, if we consider the Hessian matrix of any $w \in W^{2,2}(\Omega)$,

$$D^2 w = \begin{pmatrix} w_{x_1x_1} & w_{x_1x_2} \\ w_{x_1x_2} & w_{x_2x_2} \end{pmatrix}. \tag{3.23}$$

In [17] it is proved that w is a solution to

$$\mathcal{M}[w] = \operatorname{Tr}\left(BD^2 w\right) = 0, \tag{3.24}$$

where B is a suitable coefficient matrix, if and only if

$$\left(w^2_{x_1x_2} - w_{x_1x_1} w_{x_2x_2}\right)\left(K + \frac{1}{K}\right) \geq w^2_{x_1x_1} + 2w^2_{x_1x_2} + w^2_{x_2x_2}, \tag{3.25}$$

where $\sqrt{K}$, $K \leq 1$, is the elliptic constant. In the case where the Hessian matrix is diagonal, that is, $w_{x_1x_2} = 0$, it is easy to see that a solution of $\mathcal{M}^*[v] = 0$ is the positive function $v = \sqrt{-\det D^2 w} = \sqrt{|w_{x_1x_1} w_{x_2x_2}|}$.

4. The Coefficients Measurable with Respect to One Variable and VMO with Respect to the Other

It is well known that, for linear elliptic operators in nondivergence form with continuous coefficients, the $W^{2,p}$ estimates hold for all $p > 1$. It was shown that these estimates still hold in the same range when the coefficients are in VMO [18] or *partially* in VMO [19]. Our aim here is to generalize a regularity result of Escauriaza (Theorem 1.2, [2]) for the nonnegative adjoint solutions v to

$$\mathcal{M}^*[v] = 0 \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{4.1}$$

as defined in (1.4), with

$$\begin{gathered} A = (a_{ij}) = {}^tA, \\ \frac{|\xi|^2}{\sqrt{K}} \leq \langle A(x)\xi, \xi\rangle \leq \sqrt{K}|\xi|^2, \end{gathered} \tag{4.2}$$

for a.e. $x = (x_1, x_2) \in \Omega$ and for $\xi \in \mathbb{R}^2$.

Theorem 4.1. *If $v \in L^1(\Omega)$ is a nonnegative solution to* (4.1) *and the coefficient matrix $A(x)$ satisfies* (4.2)*, and moreover*

$$A(x_1, \cdot) \in \text{VMO}, \tag{4.3}$$

then

$$v \in \bigcap_{q>1} G_q. \tag{4.4}$$

Let us begin with the following L^p-global regularity (for all $p \geq 2$) result for the complex Beltrami equation

$$F_{\overline{z}} + \mu(z)F_z + \overline{\mu(z)}\ \overline{F_z} = H(z) \quad z \in \mathbb{R}^2, \tag{4.5}$$

under a partially-VMO assumption on the Beltrami coefficients μ, as defined in Section 2.

Proposition 4.2. *Let $B_r = B(0, r) \subset \mathbb{R}^2$, and let $\mu(z) = \mu(x_1, x_2)$ be measurable, such that $|\mu(z)| + |\overline{\mu(z)}| = 2|\mu(z)| \leq k < 1$ with $k = (K-1)/(K+1)$ and $\mu(z) = 0$ for $|z| \geq r > 0$. Moreover, assume that $\mu(x_1, \cdot) \in VMO(\mathbb{R}, \mathbb{R})$ for a.e. $x_1 \in \mathbb{R}$. Then for any $p \geq 2$ and for $H \in L^p(\mathbb{R}^2)(H(z) = 0$ for $|z| > r)$, there exists a unique solution F to the Beltrami equation* (4.5) *such that $F_z \in L^p$ and*

$$\|F_{\overline{z}}\|_{L^p(\mathbb{R}^2)} \leq c(p, k)\|H\|_{L^p(\mathbb{R}^2)}. \tag{4.6}$$

Remark 4.3. We note that in general, elliptic Beltrami operator

$$I - \mu T - \nu \overline{T}, \tag{4.7}$$

where T is the Beurling transform defined via the relation

$$T(F_{\overline{z}}) = F_z \tag{4.8}$$

under the assumption

$$\mu, \nu \in \mathrm{VMO}\left(\widehat{\mathbb{C}}\right), \tag{4.9}$$

is invertible in all $L^p(\mathbb{C})$ spaces, $p > 1$. The proof is much the same [5, 20], considering the complex Beltrami equation

$$F_{\overline{z}} - \mu(z) F_z - \nu(z) \overline{F_z} = h, \;\; h \in L^p(\mathbb{C}). \tag{4.10}$$

The meaning of the condition (4.9) is that μ and ν have vanishing mean oscillation in the usual sense, that is, belong to the closure of $C_0^\infty(\mathbb{C})$ in $\mathrm{BMO}(\mathbb{C})$ and that μ and ν are defined at infinity in the following average sense:

$$\begin{aligned} \mu(\infty) &= \lim_{r \to \infty} \fint_{B_r(0)} \mu(z) |dz|^2, \\ \nu(\infty) &= \lim_{r \to \infty} \fint_{B_r(0)} \nu(z) |dz|^2. \end{aligned} \tag{4.11}$$

The following example, due to T. Iwaniec, shows that without such condition the result fails.

Example 4.4. There exists a function $f \in \mathrm{VMO}(\mathbb{R}^n)$, $0 \leq f(x) \leq 1$ everywhere, such that

$$0 = \liminf_{|B| \to \infty} \fint_B f(x) dx < \limsup_{|B| \to \infty} \fint_B f(x) dx = 1, \tag{4.12}$$

where B stands for a ball centered at the origin.

Preliminaries

Let $L : \mathbb{R} \to [0, 1]$ be a Lipschitz function given by

$$L(t) = \frac{1}{2}(1 + |t| - |t - 1|). \tag{4.13}$$

The Lipschitz constant of L equals 1, and, therefore, for each $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$, we have

$$\|L \circ \varphi\|_{\mathrm{BMO}} \le 2\|\varphi\|_{\mathrm{BMO}}. \tag{4.14}$$

Next, denote by $C(n)$ the BMO-norm of the function $x \to \log|x|$. We will truncate this function to make building blocks to our construction.

The Building Blocks

For a nonnegative integer k, we set

$$\begin{gathered}\varphi_k(x) = L\left(7 - 2^{-k}\log|x|\right),\\ \psi_k(x) = L\left(5 - 2^{-k}\log|x|\right).\end{gathered} \tag{4.15}$$

We define the building block as $f_k = \varphi_k - \psi_k$. Note that each f_k is continuous and supported in the ball $|x| \le e^{7\cdot 2^k}$, whereas f_{k+1} vanishes on this ball. The BMO-norm of f_k can be estimated as

$$\|f_k\|_{\mathrm{BMO}} \le \|\varphi_k\|_{\mathrm{BMO}} + \|\psi_k\|_{\mathrm{BMO}} \le 2 \cdot 2^{-k}C(n) + 2 \cdot 2^{-k}C(n) = 2^{-k+2}C(n). \tag{4.16}$$

Thus the infinite series

$$f = \sum_{k=0}^{\infty} f_k \tag{4.17}$$

represents a VMO function.

Computation of L^1-Averages

Given any positive integer N, we consider concentric balls $B_r \subset B_R$ centered at the origin and with radii $r = e^{4\cdot 2^N} < e^{6\cdot 2^N} = R$. Elementary geometric observation reveals that

$$\begin{aligned}\fint_{B_R} f &\ge \frac{1}{|B_R|}\int_{B_R} f_N \ge \frac{1}{|B_R|}\int_{e^{5\cdot 2^N} \le |x| \le e^{6\cdot 2^N}} f_N\\ &= \frac{e^{6n2^N} - e^{5n2^N}}{e^{6n2^N}} = 1 - e^{-n2^N} \longrightarrow 1, \quad \text{as } N \longrightarrow \infty.\end{aligned} \tag{4.18}$$

On the other hand

$$\begin{aligned}\fint_{B_R} f &= \frac{1}{|B_R|}\sum_{k=1}^{N-1}\int_{|x|\le e^{4\cdot 2^N}} f_k(x)dx \le \frac{1}{|B_R|}\sum_{k=1}^{N-1}\int_{|x|\le e^{7\cdot 2^k}} dx \\ &\le \sum_{k=1}^{N-1}\frac{e^{7n2^k}}{e^{4n2^N}} \le (N-1)\frac{e^{7n2^{N-1}}}{e^{4n2^N}} = (N-1)e^{-n2^{N-1}} \longrightarrow 0, \quad \text{as } N \longrightarrow \infty,\end{aligned} \tag{4.19}$$

as desired.

Proof of Proposition 4.2. If we set $K = (1+k)/(1-k)$, then there exists a symmetric matrix

$$A(z) = a_{ij}(z) \tag{4.20}$$

such that

$$\begin{gathered}\frac{I}{\sqrt{K}} \le A(z) \le \sqrt{K}I, \\ \mu(z) = \frac{1}{2}\left[\frac{a_{11}(z) - a_{22}(z) + 2a_{12}(z)i}{a_{11}(z) + a_{22}(z)}\right].\end{gathered} \tag{4.21}$$

Moreover $a_{ij}(z) = 0$ for $|z| \ge r$.

We may assume the following familiar normalization:

$$\operatorname{Tr} A(z) = a_{11}(z) + a_{22}(z) = 1. \tag{4.22}$$

With the previous prescriptions we easily check that for $w \in W^{2,1}_{\mathrm{loc}}(B_r)$ if we define the complex gradient of w as

$$F(z) = w_z = \frac{1}{2}(w_{x_1} - iw_{x_2}), \tag{4.23}$$

we have

$$\operatorname{Tr}\Big(A(z)D^2w\Big) = 2\Big(F_{\overline{z}} + \mu F_z + \overline{\mu}\overline{F_z}\Big) \tag{4.24}$$

(see [5]). Hence (4.5) is equivalent to

$$\operatorname{Tr}\Big(A(z)D^2w\Big) = H \quad \text{in } \mathbb{R}^2 \tag{4.25}$$

with coefficient matrix $A(z) = A(x_1, x_2)$ allowed to be only measurable with respect to x_1 and VMO with respect to $x_2 \in \mathbb{R}$, (thanks to (4.22)).

Under these assumptions, in [19, Theorem 2.4], the existence of a unique solution $w \in W^{2,p}$ to (4.25) for $H \in L^p$ has been established ($p \geq 2$), together with the estimate

$$\left\| D^2 w \right\|_{L^p(B_r)} \leq c(p,k) \|H\|_{L^p(B_r)}. \tag{4.26}$$

Hence (4.6) follows. □

Let us now give the following sharp version of the Alexandrov-Bakelman-Pucci maximum principle for non divergence elliptic operators

$$\mathcal{M}[w] = \sum a_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} \tag{4.27}$$

with partially VMO coefficients.

Lemma 4.5. *Under the assumptions* (4.2), (4.3), (4.22) *on A, suppose $B_r = B(0,r) \subset \mathbb{R}^2$ and that $w \in W^{2,1}_{\mathrm{loc}}(B_r) \cap C^0(\overline{B}_r)$ satisfies, for $h \in L^p(B_r)$, $p > 1$,*

$$\begin{aligned} \mathcal{M}[w] &= h \quad \text{in } B_r, \\ w &= 0 \quad \text{on } \partial B_r. \end{aligned} \tag{4.28}$$

Then one has

$$\|w\|_{L^\infty(B_r)} \leq c(K,p) r^{2-2/p} \|h\|_{L^p(B_r)}. \tag{4.29}$$

Proof. In view of [19, Theorem 2.4], we know that the Dirichlet problem (4.28) always has a unique solution $w \in W^{2,p}(B_r) \cap W^{1,2}_0(B_r)$ for every $h \in L^p(B_r)$, $p > 1$.

Define $h(z) = 0$ for $z = (x_1, x_2) \in \mathbb{R}^2 \setminus B_r$ and

$$\mu(z) = \frac{1}{2}[a_{11}(z) - a_{22}(z) + 2a_{12}(z)i], \tag{4.30}$$

and set $\mu(z) = 0$ for $z \in \mathbb{R}^2 \setminus B_r$. According to Proposition 4.2, the equation

$$F_{\overline{z}} + \mu F_z + \overline{\mu}\, \overline{F_z} = \frac{h}{2} \tag{4.31}$$

has unique solution $F \in W^{1,p}_{\mathrm{loc}}(B_r)$ such that

$$\|F_{\overline{z}}\|_{L^p(\mathbb{R}^2)} \leq c(p,K) \|h\|_{L^p(\mathbb{R}^2)}. \tag{4.32}$$

Now, let us see that

$$F_{|B_r} \in W^{1,2}_{\mathrm{loc}}(B_r). \tag{4.33}$$

Define $f = w_z = (1/2)(w_{x_1} - iw_{x_2})$. Then

$$(f-F)_{\overline{z}} + \mu(f-F)_z + \overline{\mu}\,\overline{(f-F)_z} = 0 \quad \text{in } B_r. \tag{4.34}$$

This means that the mapping

$$g = f - F \tag{4.35}$$

is weakly K-quasiregular, and since $F \in W^{1,p}_{\text{loc}}$ and $f \in W^{1,2}_{\text{loc}}$, we deduce $g \in W^{1,p}_{\text{loc}}$ and actually g is K-quasiregular and in particular

$$g \in W^{1,2}_{\text{loc}}(B_r). \tag{4.36}$$

Then (4.33) follows.

Now, let us introduce the solution U to the problem

$$\begin{gathered} \Delta U = 4F_{\overline{z}}, \\ U(0) = 0. \end{gathered} \tag{4.37}$$

We have $\Delta U \in L^p(\mathbb{R}^2) \cap L^2_{\text{loc}}(B_r)$ and $\mathcal{M}[U] = h$ a.e. $z \in \mathbb{R}^2$. Moreover, classically

$$\|\Delta U\|_{L^p(\mathbb{R}^2)} \le c(K,p)\|h\|_{L^p}. \tag{4.38}$$

(Notice that $c(K,p)$ is optimal in Talenti [21].)

Finally, let us introduce

$$u = U - w. \tag{4.39}$$

Then u is continuous in $\overline{B}_r$ by the Sobolev imbedding, and it is the solution to the Dirichlet problem

$$\begin{gathered} \mathcal{M}[u] = 0 \quad \text{a.e. } z \in B_r, \\ u_{/\partial B_r} = U, \end{gathered} \tag{4.40}$$

and $u \in W^{2,2}_{\text{loc}}(B_r) \cap C^0(\overline{B}_r)$. By the classical maximum principle [22],

$$\|U - w\|_{L^\infty(B_r)} = \|U\|_{L^\infty(\partial B_r)}. \tag{4.41}$$

Hence, we use Sobolev and the condition $U(0) = 0$ to conclude

$$\|w\|_{L^\infty(B_r)} \le 2\|U\|_{L^\infty(B_r)} \le cr^{2-2/p}\|\Delta U\|_{L^p(B_r)} \le cr^{2-2/p}\|h\|_{L^p(B_r)}. \tag{4.42}$$

$\square$

Proof of Theorem 4.1. Let us fix $q > 1$, set $p = q/(q-1)$, and fix a ball B_r such that $B_{2r} \subset \Omega$. As in [7, 15], we make use of the dual formulation of the L^q-norm

$$\left(\int_{B_r} v^q\right)^{1/q} = \sup\left\{\int_{B_r} vh : h \geq 0,\ h \in C_0^1(B_r), \|h\|_{L^p(\mathbb{R}^2)} \leq 1\right\}. \tag{4.43}$$

Fix $h \in C_0^1(B_r), \|h\|_{L^p(\mathbb{R}^2)} \leq 1, h \geq 0$, and solve the Dirichlet problem

$$\begin{gathered} \mathcal{M}[w] = \sum a_{ij}\frac{\partial^2 w}{\partial x_i \partial x_j} = h \quad \text{in } B_{2r}, \\ w = 0 \quad \text{on } \partial B_{2r}. \end{gathered} \tag{4.44}$$

Since $A(x_1, \cdot) \in \mathrm{VMO}$, the problem has a unique solution $w \in W^{2,p}(B_{2r})$ vanishing on ∂B_{2r}, satisfying the estimate

$$\left\|D^2 w\right\|_{L^p(B_{2r})} \leq c\|h\|_{L^p(B_{2r})}, \tag{4.45}$$

$c = c(K, p, \|A\|_{\mathrm{VMO}})$.

Fix a nonnegative function $\varphi_r \in C_0^1(B_{3r/2})$ such that $\varphi_r = 1$ on B_r and $|\partial^\alpha \varphi_r/\partial x^\alpha| \leq C(\alpha)/r^{|\alpha|}$.

Then, we have

$$\begin{aligned} \int_{B_r} vh &\leq \int_{B_{2r}} v\mathcal{M}[w]\varphi_r = -\int_{B_{2r}} vw\mathcal{M}[\varphi_r] - 2\int_{B_{2r}} v\langle A\nabla w, \nabla\varphi_r\rangle \\ &\leq \frac{c}{r^2}\|w\|_{L^\infty}(B_{2r})\int_{B_{3r/2}} v + \frac{c\sqrt{K}}{r}\int_{B_{3r/2}} v|\nabla w|. \end{aligned} \tag{4.46}$$

By (4.29) $\|w\|_{L^\infty(B_{2r})} \leq c(K,p)r^{2-2/p}\|h\|_{L^p(B_{2r})} \leq c\ r^{2/q}$; hence, (4.46) implies

$$\int_{B_r} vh \leq \frac{c}{r^2}r^{2/q}\int_{B_{3r/2}} v + \frac{c}{r}\left(\int_{B_{3r/2}} v\right)^{1/2}\left(\int_{B_{2r}} v|\nabla w|^2\right)^{1/2}. \tag{4.47}$$

Now, we estimate the last integral in the right-hand side. By (1.3), one has

$$\int_{B_{2r}} v|\nabla w|^2 \leq \sqrt{K}\int_{B_{2r}} v\langle A\nabla w, \nabla w\rangle = \sqrt{K}\int_{B_{2r}} v\left[\frac{M[w^2]}{2} - wh\right]. \tag{4.48}$$

Since $w^2 = 0$ and $\nabla(w^2) = 0$ on ∂B_{2r}, then we deduce

$$\int_{B_{2r}} v\mathcal{M}\left[w^2\right] = 0 \quad \text{as } \mathcal{M}^*[v] = 0. \tag{4.49}$$

Using again (4.29) yields

$$\int_{B_{2r}} v|\nabla w|^2 \leq 2\sqrt{K}\int_{B_{2r}} v|w|h \leq 2\sqrt{K}\|w\|_{L^\infty(B_{2r})}\int_{B_r} vh$$
$$\leq 2\sqrt{K}cr^{2/q}\int_{B_r} vh. \tag{4.50}$$

By (4.47) and (4.50), it follows that

$$\int_{B_r} vh \leq \frac{c}{r^{2(1-1/q)}}\int_{B_{3r/2}} v + \frac{c}{r^{(1-1/q)}}\left(\int_{B_{3r/2}} v\right)^{1/2}\left(\int_{B_r} vh\right)^{1/2}. \tag{4.51}$$

By elementary inequality $\sqrt{a}\sqrt{b} \leq a/2 + b/2$, we obtain

$$\int_{B_r} vh \leq \frac{c}{r^{2(1-1/q)}}\int_{B_{3r/2}} v + \frac{c}{r^{2(1-1/q)}}\int_{B_{3r/2}} v + \frac{1}{2}\int_{B_r} vh. \tag{4.52}$$

Rearranging yields

$$\int_{B_r} vh \leq \frac{c}{r^{2(1-(1/q))}}\int_{B_{3r/2}} v. \tag{4.53}$$

Since h is arbitrary, by (4.43) and (4.53), we obtain

$$\left(\fint_{B_r} v^p\right)^{1/p} \leq c\fint_{B_{3r/2}} v, \tag{4.54}$$

with $c = c(K,p)$. An application of [14, Lemma 2.0] concludes the proof. □

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References

[1] V. Maz'ya and R. McOwen, "Asymptotics for solutions of elliptic equations in double divergence form," *Communications in Partial Differential Equations*, vol. 32, no. 1–3, pp. 191–207, 2007.
[2] L. Escauriaza, "Weak type-(1, 1) inequalities and regularity properties of adjoint and normalized adjoint solutions to linear nondivergence form operators with VMO coefficients," *Duke Mathematical Journal*, vol. 74, no. 1, pp. 177–201, 1994.
[3] F. W. Gehring, "The L^p-integrability of the partial derivatives of a quasiconformal mapping," *Acta Mathematica*, vol. 130, pp. 265–277, 1973.

[4] D. Sarason, "Functions of vanishing mean oscillation," *Transactions of the American Mathematical Society*, vol. 207, pp. 391–405, 1975.

[5] K. Astala, T. Iwaniec, and G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, vol. 48 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 2009.

[6] P. Bauman, "Equivalence of the Green's functions for diffusion operators in $\mathbb{R}^n$: a counterexample," *Proceedings of the American Mathematical Society*, vol. 91, no. 1, pp. 64–68, 1984.

[7] G. Moscariello and C. Sbordone, "Optimal L^p-properties of Green's functions for non-divergence elliptic equations in two dimensions," *Studia Mathematica*, vol. 169, no. 2, pp. 133–141, 2005.

[8] K. Astala, "Area distortion of quasiconformal mappings," *Acta Mathematica*, vol. 173, no. 1, pp. 37–60, 1994.

[9] F. Leonetti and V. Nesi, "Quasiconformal solutions to certain first order systems and the proof of a conjecture of G. W. Milton," *Journal de Mathématiques Pures et Appliquées*, vol. 76, no. 2, pp. 109–124, 1997.

[10] G. Alessandrini and V. Nesi, "Univalent σ-harmonic mappings," *Archive for Rational Mechanics and Analysis*, vol. 158, no. 2, pp. 155–171, 2001.

[11] G. Alessandrini and V. Nesi, "Beltrami operators, non-symmetric elliptic equations and quantitative Jacobian bounds," *Annales Academiæ Scientiarium Fennicæ. Mathematica*, vol. 34, no. 1, pp. 47–67, 2009.

[12] B. Bojarski, L. D'Onofrio, T. Iwaniec, and C. Sbordone, "G-closed classes of elliptic operators in the complex plane," *Ricerche di Matematica*, vol. 54, no. 2, pp. 403–432, 2005.

[13] K. Astala and J. Jääskeläinen, "Homeomorphic solutions to reduced beltrami equations," *Annales Academiæ Scientiarum Fennicæ Mathematica*, vol. 34, pp. 607–613, 2009.

[14] E. B. Fabes and D. W. Stroock, "The L^p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations," *Duke Mathematical Journal*, vol. 51, no. 4, pp. 997–1016, 1984.

[15] C. Sbordone, "The precise L^p-theory of elliptic equations in the plane," *Progress in Nonlinear Differential Equations and Their Applications*, vol. 63, pp. 415–421, 2005.

[16] L. D'Onofrio and L. Greco, "A counter-example in G-convergence of non-divergence elliptic operators," *Proceedings of the Royal Society of Edinburgh: Section A*, vol. 133, no. 6, pp. 1299–1310, 2003.

[17] C. Pucci, "Un problema variazionale per i coefficienti di equazioni differenziali di tipo ellittico," vol. 16, pp. 159–172, 1962.

[18] F. Chiarenza, M. Frasca, and P. Longo, "Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients," *Ricerche di Matematica*, vol. 40, no. 1, pp. 149–168, 1991.

[19] D. Kim and N. V. Krylov, "Elliptic differential equations with coefficients measurable with respect to one variable and VMO with respect to the others," *SIAM Journal on Mathematical Analysis*, vol. 39, no. 2, pp. 489–506, 2007.

[20] K. Astala, T. Iwaniec, and E. Saksman, "Beltrami operators in the plane," *Duke Mathematical Journal*, vol. 107, no. 1, pp. 27–56, 2001.

[21] G. Talenti, "Equazioni lineari ellittiche in due variabili," *Le Matematiche*, vol. 21, no. 2, pp. 339–376, 1966.

[22] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, 1998.