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Research Article

Homogeneity Property of Besov and Triebel-Lizorkin Spaces

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We consider the classical Besov and Triebel-Lizorkin spaces defined via differences and prove a homogeneity property for functions with bounded support in the frame of these spaces. As the proof is based on compact embeddings between the studied function spaces, we present also some results on the entropy numbers of these embeddings. Moreover, we derive some applications in terms of pointwise multipliers.

1. Introduction

The present note deals with classical Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ defined via differences, briefly denoted as B- and F-spaces in the sequel. We study the properties of the dilation operator, which is defined for every $\lambda > 0$ as

$$T_\lambda : f \longrightarrow f(\lambda\cdot). \tag{1.1}$$

The norms of these operators on Besov and Triebel-Lizorkin spaces were studied already in [1] and [2, Sections 2.3.1 and 2.3.2] with complements given in [3–5].

We prove the so-called *homogeneity property*, showing that, for $s > 0$ and $0 < p,\ q \le \infty$,

$$\left\| f(\lambda\cdot) \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \sim \lambda^{s-(n/p)} \left\| f \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\|, \tag{1.2}$$

for all $0 < \lambda \leq 1$ and all

$$f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n) \quad \text{with } \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \leq \lambda\}. \tag{1.3}$$

The same property holds true for the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$. This extends and completes [6], where corresponding results for the spaces $B^s_{p,q}(\mathbb{R}^n)$, defined via Fourier-analytic tools, were established, which coincide with our spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ if $s > \max(0, n(1/p - 1))$. Concerning the corresponding F-spaces $F^s_{p,q}(\mathbb{R}^n)$, the same homogeneity property had already been established in [7, Corollary 5.16, page 66].

Our results yield immediate applications in terms of pointwise multipliers. Furthermore, we remark that the homogeneity property is closely related with questions concerning refined localization, nonsmooth atoms, local polynomial approximation, and scaling properties. This is out of our scope for the time being. But we use this property in the forthcoming paper [8] in connection with nonsmooth atomic decompositions in function spaces.

Our proof of (1.2) is based on compactness of embeddings between the function spaces under investigation. Therefore, we use this opportunity to present some closely related results on entropy numbers of such embeddings.

This paper is organized as follows. We start with the necessary definitions and the results about entropy numbers in Section 2. Then, we focus on equivalent quasinorms for the elements of certain subspaces of $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$, respectively, from which the homogeneity property will follow almost immediately in Section 3. The last section states some applications in terms of pointwise multipliers.

2. Preliminaries

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, $n \in \mathbb{N}$, $\mathbb{C}$ the complex plane. The set of multi-indices $\beta = (\beta_1, \ldots, \beta_n)$, $\beta_i \in \mathbb{N}_0$, $i = 1, \ldots, n$, is denoted by $\mathbb{N}_0^n$, with $|\beta| = \beta_1 + \cdots + \beta_n$, as usual. We use the symbol "$\lesssim$" in

$$a_k \lesssim b_k \quad \text{or} \quad \varphi(x) \lesssim \psi(x) \tag{2.1}$$

always to mean that there is a positive number c_1 such that

$$a_k \leq c_1 b_k \quad \text{or} \quad \varphi(x) \leq c_1 \psi(x) \tag{2.2}$$

for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are nonnegative sequences and φ, ψ are nonnegative functions. We use the equivalence "$\sim$" in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x) \tag{2.3}$$

for

$$a_k \lesssim b_k, \quad b_k \lesssim a_k \qquad \text{or} \qquad \varphi(x) \lesssim \psi(x), \quad \psi(x) \lesssim \varphi(x). \tag{2.4}$$

If $a \in \mathbb{R}$, then $a_+ := \max(a, 0)$ and $[a]$ denotes the integer part of a.

Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. $L_p(\mathbb{R}^n)$, with $0 < p \le \infty$, stands for the usual quasi-Banach space with respect to the Lebesgue measure, quasinormed by

$$\|f \mid L_p(\mathbb{R}^n)\| := \left(\int_{\mathbb{R}^n} |f(x)|^p \mathrm{d}x\right)^{1/p} \tag{2.5}$$

with the appropriate modification if $p = \infty$. Moreover, let Ω denote a domain in $\mathbb{R}^n$. Then, $L_p(\Omega)$ is the collection of all complex-valued Lebesgue measurable functions in Ω such that

$$\|f \mid L_p(\Omega)\| := \left(\int_{\Omega} |f(x)|^p \mathrm{d}x\right)^{1/p} \tag{2.6}$$

(with the usual modification if $p = \infty$) is finite.

Furthermore, B_R stands for an open ball with radius $R > 0$ around the origin,

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}. \tag{2.7}$$

Let $Q_{j,m}$ with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ denote a cube in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centered at $2^{-j}m$, and with side length 2^{-j+1}. For a cube Q in $\mathbb{R}^n$ and $r > 0$, we denote by rQ the cube in $\mathbb{R}^n$ concentric with Q and with side length r times the side length of Q. Furthermore, $\chi_{j,m}$ stands for the characteristic function of $Q_{j,m}$.

2.1. Function Spaces Defined via Differences

If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$, and $r \in \mathbb{N}$, then

$$\left(\Delta_h^1 f\right)(x) = f(x+h) - f(x), \qquad \left(\Delta_h^{r+1} f\right)(x) = \Delta_h^1\left(\Delta_h^r f\right)(x) \tag{2.8}$$

are the usual iterated differences. Given a function $f \in L_p(\mathbb{R}^n)$, the *r-th modulus of smoothness* is defined by

$$\begin{aligned} \omega_r(f,t)_p &= \sup_{|h|\le t} \left\|\Delta_h^r f \mid L_p(\mathbb{R}^n)\right\|, \qquad t > 0,\ 0 < p \le \infty, \\ d_{t,p}^r f(x) &= \left(t^{-n}\int_{|h|\le t} \left|\left(\Delta_h^r f\right)(x)\right|^p \mathrm{d}h\right)^{1/p}, \quad t > 0,\ 0 < p < \infty, \end{aligned} \tag{2.9}$$

denotes its *ball means*.

Definition 2.1. (i) Let $0 < p, q \leq \infty,\ s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then, the Besov space $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ contains all $f \in L_p(\mathbb{R}^n)$ such that

$$\left\| f \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\|_r = \left\| f \mid L_p(\mathbb{R}^n) \right\| + \left(\int_0^1 t^{-sq} \omega_r(f,t)_p^q \,\frac{\mathrm{d}t}{t} \right)^{1/q} \tag{2.10}$$

(with the usual modification if $q = \infty$) is finite.

(ii) Let $0 < p < \infty,\ 0 < q \leq \infty,\ s > 0$, and $r \in \mathbb{N}$ such that $r > s$. Then, $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\left\| f \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\|_r = \left\| f \mid L_p(\mathbb{R}^n) \right\| + \left\| \left(\int_0^1 t^{-sq} d^r_{t,p} f(\cdot)^q \frac{\mathrm{d}t}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \tag{2.11}$$

(with the usual modification if $q = \infty$) is finite.

Remark 2.2. These are the *classical Besov* and *Triebel-Lizorkin spaces,* in particular, when $1 \leq p, q \leq \infty$ ($p < \infty$ *for the F-spaces)* and $s > 0$. We will sometimes write $\mathbf{A}^s_{p,q}(\mathbb{R}^n)$ when both scales of spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ and $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ are concerned simultaneously.

Concerning the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, the study for all admitted s, p, and q goes back to [9], we also refer to [10, Chapter 5, Definition 4.3] and [11, Chapter 2, Section 10]. There are as well many older references in the literature devoted to the cases $p, q \geq 1$.

The approach by differences for the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ has been described in detail in [12] for those spaces which can also be considered as subspaces of $\mathcal{S}'(\mathbb{R}^n)$. Otherwise, one finds in [13, Section 9.2.2, pp. 386–390] the necessary explanations and references to the relevant literature.

The spaces in Definition 2.1 are independent of r, meaning that different values of $r > s$ result in norms which are equivalent. This justifies our omission of r in the sequel. Moreover, the integrals $\int_0^1$ can be replaced by $\int_0^\infty$ resulting again in equivalent quasinorms, (cf. [14, Section 2]).

The spaces are quasi-Banach spaces (Banach spaces if $p, q \geq 1$). Note that we deal with subspaces of $L_p(\mathbb{R}^n)$, in particular, for $s > 0$ and $0 < q \leq \infty$, we have the embeddings

$$\mathbf{A}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n), \tag{2.12}$$

where $0 < p \leq \infty$ ($p < \infty$ for F-spaces). Furthermore, the B-spaces are closely linked with the Triebel-Lizorkin spaces via

$$\mathbf{B}^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow \mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathbf{B}^s_{p,\max(p,q)}(\mathbb{R}^n), \tag{2.13}$$

(cf. [15, Proposition 1.19 (i)]). The classical scale of Besov spaces contains many well-known function spaces. For example, if $p = q = \infty$, one recovers the Hölder-Zygmund spaces $\mathcal{C}^s(\mathbb{R}^n)$, that is,

$$\mathbf{B}^s_{\infty,\infty}(\mathbb{R}^n) = \mathcal{C}^s(\mathbb{R}^n), \qquad s > 0. \tag{2.14}$$

Recent results by Hedberg and Netrusov [16] on atomic decompositions, and by Triebel [13, Section 9.2] on the reproducing formula provide an equivalent characterization of Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ using *subatomic decompositions*, which introduces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ as those $f \in L_p(\mathbb{R}^n)$ which can be represented as

$$f(x) = \sum_{\beta\in\mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda^{\beta}_{j,m} k^{\beta}_{j,m}(x), \quad x \in \mathbb{R}^n, \tag{2.15}$$

with coefficients $\lambda = \{\lambda^{\beta}_{j,m} \in \mathbb{C} : \beta \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n\}$ belonging to some appropriate sequence space $b^{s,\varrho}_{p,q}$ defined as

$$b^{s,\varrho}_{p,q} := \left\{ \lambda : \left\| \lambda \mid b^{s,\varrho}_{p,q} \right\| < \infty \right\}, \tag{2.16}$$

where

$$\left\| \lambda \mid b^{s,\varrho}_{p,q} \right\| = \sup_{\beta\in\mathbb{N}_0^n} 2^{\varrho|\beta|} \left(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \left(\sum_{m\in\mathbb{Z}^n} \left| \lambda^{\beta}_{j,m} \right|^p \right)^{q/p} \right)^{1/q}, \tag{2.17}$$

$s > 0,\ 0 < p,\ q \le \infty$ (with the usual modification if $p = \infty$ and/or $q = \infty$), $\varrho \ge 0$, and $k^{\beta}_{j,m}(x)$ are certain standardized building blocks (which are universal). This subatomic characterization will turn out to be quite useful when studying entropy numbers.

In terms of pointwise multipliers in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, the following is known.

Proposition 2.3. *Let $0 < p, q \le \infty, s > 0, k \in \mathbb{N}$ with $k > s$, and let $h \in C^k(\mathbb{R}^n)$. Then,*

$$f \longrightarrow hf \tag{2.18}$$

is a linear and bounded operator from $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ into itself.

The proof relies on atomic decompositions of the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, (cf. [17, Proposition 2.5]). We will generalize this result in Section 4 as an application of our homogeneity property.

2.2. Function Spaces on Domains

Let Ω be a domain in $\mathbb{R}^n$. We define spaces $\mathbf{A}^s_{p,q}(\Omega)$ by restriction of the corresponding spaces on $\mathbb{R}^n$, that is, $\mathbf{A}^s_{p,q}(\Omega)$ is the collection of all $f \in L_p(\Omega)$ such that there is a $g \in \mathbf{A}^s_{p,q}(\mathbb{R}^n)$ with $g|_\Omega = f$. Furthermore,

$$\left\| f \mid \mathbf{A}^s_{p,q}(\Omega) \right\| = \inf \left\| g \mid \mathbf{A}^s_{p,q}(\mathbb{R}^n) \right\|, \tag{2.19}$$

where the infimum is taken over all $g \in \mathbf{A}^s_{p,q}(\mathbb{R}^n)$ such that the restriction $g|_\Omega$ to Ω coincides in $L_p(\Omega)$ with f.

In particular, the subatomic characterization for the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ from Remark 2.2 carries over. For further details on this subject, we refer to [18, Section 2.1].

Embeddings results between the spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$ hold also for the spaces $\mathbf{B}^s_{p,q}(\Omega)$, since they are defined by restriction of the corresponding spaces on $\mathbb{R}^n$. Furthermore, these results can be improved, if we assume $\Omega \subset \mathbb{R}^n$ to be bounded.

Proposition 2.4. *Let $0 < s_2 < s_1 < \infty$, $0 < p_1, p_2, q_1, q_2 \le \infty$, and $\Omega \subset \mathbb{R}^n$ be bounded. If*

$$\delta_+ = s_1 - s_2 - d\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0, \tag{2.20}$$

one has the embedding

$$\mathbf{B}^{s_1}_{p_1,q_1}(\Omega) \hookrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\Omega). \tag{2.21}$$

Proof. If $p_1 \le p_2$, the embedding follows from [19, Theorem 1.15], since the spaces on Ω are defined by restriction of their counterparts on $\mathbb{R}^n$. Therefore, it remains to show that, for $p_1 > p_2$, we have the embedding

$$\mathbf{B}^{s_2}_{p_1,q_2}(\Omega) \hookrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\Omega). \tag{2.22}$$

Let $\psi \in D(\mathbb{R}^n)$ with support in the compact set Ω_1 and

$$\psi(x) = 1 \quad \text{if } x \in \overline{\Omega} \subset \Omega_1. \tag{2.23}$$

Then, for $f \in \mathbf{B}^{s_2}_{p_1,q_2}(\Omega)$, there exists $g \in \mathbf{B}^{s_2}_{p_1,q_2}(\mathbb{R}^n)$ with

$$g\big|_\Omega = f, \qquad \left\| f \mid \mathbf{B}^{s_2}_{p_1,q_2}(\Omega) \right\| \sim \left\| g \mid \mathbf{B}^{s_2}_{p_1,q_2}(\mathbb{R}^n) \right\|. \tag{2.24}$$

We calculate

$$\begin{aligned} \left\| f \mid \mathbf{B}^{s_2}_{p_2,q_2}(\Omega) \right\| &\le \left\| \psi g \mid \mathbf{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n) \right\| \\ &\le \left\| \psi g \mid \mathbf{B}^{s_2}_{p_1,q_2}(\mathbb{R}^n) \right\| \\ &\le c_\psi \left\| g \mid \mathbf{B}^{s_2}_{p_1,q_2}(\mathbb{R}^n) \right\| \sim \left\| f \mid \mathbf{B}^{s_2}_{p_1,q_2}(\Omega) \right\|. \end{aligned} \tag{2.25}$$

The last inequality in (2.25) follows from Proposition 2.3. In the 2nd step, we used (2.10) together with the fact that

$$\left\| \Delta^r_h(\psi g) \mid L_{p_2}(\mathbb{R}^n) \right\| \le c_{\Omega_1} \left\| \Delta^r_h(\psi g) \mid L_{p_1}(\mathbb{R}^n) \right\|, \qquad p_1 > p_2, \tag{2.26}$$

which follows from Hölder's inequality since $\operatorname{supp} \psi g \subset \Omega_1$ is compact. □

2.3. Entropy Numbers

In order to prove the homogeneity results later on, we have to rely on the compactness of embeddings between B-spaces, $\mathbf{B}^s_{p,q}(\Omega)$, and F-spaces, $\mathbf{F}^s_{p,q}(\Omega)$, respectively. This will be established with the help of entropy numbers. We briefly introduce the concept and collect some properties afterwards.

Let X and Y be quasi-Banach spaces, and let $T : X \to Y$ be a bounded linear operator. If additionally, T is continuous, we write $T \in L(X, Y)$. Let $U_X = \{x \in X : \|x \mid X\| \leq 1\}$ denote the unit ball in the quasi-Banach space X. An operator T is called compact if, for any given $\varepsilon > 0$ we can cover the image of the unit ball U_X with finitely many balls in Y of radius ε.

Definition 2.5. Let X, Y be quasi-Banach spaces, and let $T \in L(X, Y)$. Then, for all $k \in \mathbb{N}$, the kth dyadic entropy number $e_k(T)$ of T is defined by

$$e_k(T) = \inf\left\{ \varepsilon > 0 : T(U_X) \subset \bigcup_{j=1}^{2^{k-1}} \left(y_j + \varepsilon U_Y\right) \text{ for some } y_1, \dots, y_{2^{k-1}} \in Y \right\}, \tag{2.27}$$

where U_X and U_Y denote the unit balls in X and Y, respectively.

These numbers have various elementary properties which are summarized in the following lemma.

Lemma 2.6. *Let X, Y, and Z be quasi-Banach spaces, and let $S, T \in L(X, Y)$ and $R \in L(Y, Z)$.*

(i) *(Monotonicity) $\|T\| \geq e_1(T) \geq e_2(T) \geq \cdots \geq 0$. Moreover, $\|T\| = e_1(T)$, provided that Y is a Banach space.*

(ii) *(Additivity) If Y is a p-Banach space $(0 < p \leq 1)$, then, for all $j, k \in \mathbb{N}$,*

$$e^p_{j+k-1}(S + T) \leq e^p_j(S) + e^p_k(T). \tag{2.28}$$

(iii) *(Multiplicativity) For all $j, k \in \mathbb{N}$,*

$$e_{j+k-1}(RT) \leq e_j(R) e_k(T). \tag{2.29}$$

(iv) *(Compactness) T is compact if and only if*

$$\lim_{k \to \infty} e_k(T) = 0. \tag{2.30}$$

Remark 2.7. As for the general theory, we refer to [20–22]. Further information on the subject is also covered by the more recent books [2, 23].

Some problems about entropy numbers of compact embeddings for function spaces can be transferred to corresponding questions in related sequence spaces. Let $n > 0$ and $\{M_j\}_{j \in \mathbb{N}_0}$ be a sequence of natural numbers satisfying

$$M_j \sim 2^{jn}, \quad j \in \mathbb{N}_0. \tag{2.31}$$

Concerning entropy numbers for the respective sequence spaces $b_{p,q}^{s,\varrho}(M_j)$, which are defined as the sequence spaces $b_{p,q}^{s,\varrho}$ in (2.17) with the sum over $m \in \mathbb{Z}^n$ replaced by a sum over $m = 1, \ldots, M_j$, the following result was proved in [24, Proposition 3.4].

Proposition 2.8. *Let $d > 0$, $0 < \sigma_1, \sigma_2 < \infty$, and $0 < q_1, q_2 \leq \infty$. Furthermore, let $\varrho_1 > \varrho_2 \geq 0$,*

$$0 < p_1 \leq p_2 \leq \infty, \qquad \delta = \sigma_1 - \sigma_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0. \tag{2.32}$$

Then the identity map

$$\mathrm{id} : b_{p_1,q_1}^{\sigma_1,\varrho_1}(M_j) \longrightarrow b_{p_2,q_2}^{\sigma_2,\varrho_2}(M_j) \tag{2.33}$$

is compact, where M_j is restricted by (2.31).

The next theorem provides a sharp result for entropy numbers of the identity operator related to the sequence spaces $b_{p,q}^{s,\varrho}(M_j)$.

Theorem 2.9. *Let $n > 0$, $0 < s_1, s_2 < \infty$, and $0 < q_1, q_2 \leq \infty$. Furthermore, let $\varrho_1 > \varrho_2 \geq 0$,*

$$0 < p_1 \leq p_2 \leq \infty, \qquad \delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0. \tag{2.34}$$

For the entropy numbers e_k of the compact operator

$$\mathrm{id} : b_{p_1,q_1}^{s_1,\varrho_1}(M_j) \longrightarrow b_{p_2,q_2}^{s_2,\varrho_2}(M_j), \tag{2.35}$$

one has

$$e_k(\mathrm{id}) \sim k^{-\delta/n + 1/p_2 - 1/p_1}, \qquad k \in \mathbb{N}. \tag{2.36}$$

Remark 2.10. The proof of Theorem 2.9 follows from [25, Theorem 9.2]. Using the notation from this book, we have

$$b_{p_i,q_i}^{s_i,\varrho_i}(M_j) = \ell_\infty\left[2^{\varrho_i}\ell_{q_i}\left(2^{j(s_i - n/p_i)}\ell_{p_i}^{M_j}\right)\right], \quad i = 1, 2. \tag{2.37}$$

Recall the embedding assertions for Besov spaces $\mathbf{B}_{p,q}^{s}(\Omega)$ from Proposition 2.4. We will give an upper bound for the corresponding entropy numbers of these embeddings. For our purposes, it will be sufficient to assume $\Omega = B_R$.

Theorem 2.11. *Let*

$$\begin{gathered} 0 < s_2 < s_1 < \infty, \qquad 0 < p_1, p_2 \leq \infty, \qquad 0 < q_1, q_2 \leq \infty, \\ \delta_+ = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ > 0. \end{gathered} \tag{2.38}$$

Then, the embedding

$$\mathrm{id} : \mathbf{B}^{s_1}_{p_1,q_1}(\Omega) \longrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\Omega) \tag{2.39}$$

is compact, and, for the related entropy numbers, one computes

$$e_k(\mathrm{id}) \lesssim k^{-(s_1-s_2)/n}, \quad k \in \mathbb{N}. \tag{2.40}$$

Proof.

Step 1. Let $p_2 \geq p_1$, $\delta_+ = \delta$, and let $f \in \mathbf{B}^{s_1}_{p_1,q_1}(\Omega)$, then, by [26, Theorem 6.1], there is a (nonlinear) bounded extension operator

$$g = \mathrm{Ex} f \quad \text{such that } \mathrm{Re}_\Omega g = g\big|_\Omega = f, \tag{2.41}$$

$$\left\| g \mid \mathbf{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \right\| \leq c \left\| f \mid \mathbf{B}^{s_1}_{p_1,q_1}(\Omega) \right\|. \tag{2.42}$$

We may assume that g is zero outside a fixed neighbourhood Λ of Ω. Using the subatomic approach for $\mathbf{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)$, cf. Remark 2.2, we can find an optimal decomposition of g, that is,

$$g(x) = \sum_{\beta \in \mathbb{N}_0^n} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{j,m} k^{\beta}_{j,m}(x), \qquad \left\| g \mid \mathbf{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \right\| \sim \left\| \lambda \mid b^{s_1,\varrho_1}_{p_1,q_1} \right\| \tag{2.43}$$

with $\varrho_1 > 0$ large.

Let M_j for fixed $j \in \mathbb{N}_0$ be the number of cubes $Q_{j,m}$ such that

$$rQ_{j,m} \cap \Omega \neq \emptyset. \tag{2.44}$$

Since $\Omega \subset \mathbb{R}^n$ is bounded, we have

$$M_j \sim 2^{jn}, \quad j \in \mathbb{N}_0. \tag{2.45}$$

This coincides with (2.31). We introduce the (nonlinear) operator S,

$$S : \mathbf{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \longrightarrow b^{s_1,\varrho_1}_{p_1,q_1}(M_j) \tag{2.46}$$

by

$$Sg = \lambda, \qquad \lambda = \left\{ \lambda^{\beta}_{j,m} : \beta \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ rQ_{j,m} \cap \Omega \neq \emptyset \right\}, \tag{2.47}$$

where g is given by (2.43). Recall that the expansion is not unique, but this does not matter. It follows that S is a bounded map since

$$\|S\| = \sup_{g \neq 0} \frac{\left\| \lambda \mid b^{s_1,\varrho_1}_{p_1,q_1}(M_j) \right\|}{\left\| g \mid \mathbf{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n) \right\|} \leq c. \tag{2.48}$$

Next we construct the linear map T,

$$T : b^{s_2,\varrho_2}_{p_2,q_2}(M_j) \longrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n), \tag{2.49}$$

given by

$$T\lambda = \sum_{\beta \in \mathbb{N}^n_0} \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} \lambda^{\beta}_{j,m} k^{\beta}_{j,m}(x). \tag{2.50}$$

It follows that T is a linear (since the subatomic approach provides an expansion of functions via universal building blocks) and bounded map,

$$\|T\| = \sup_{\lambda \neq 0} \frac{\left\| T\lambda \mid \mathbf{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n) \right\|}{\left\| \lambda \mid b^{s_2,\varrho_2}_{p_2,q_2}(M_j) \right\|} \leq c. \tag{2.51}$$

We complement the three bounded maps Ex,S, T by the identity operator

$$\mathrm{id} : b^{s_1,\varrho_1}_{p_1,q_1}(M_j) \longrightarrow b^{s_2,\varrho_2}_{p_2,q_2}(M_j) \quad \text{with } \varrho_1 > \varrho_2, \tag{2.52}$$

which is compact by Proposition 2.8 and the restriction operator

$$\mathrm{Re}_{\Omega} : \mathbf{B}^{s_2}_{p_2,q_2}(\mathbb{R}^n) \longrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\Omega), \tag{2.53}$$

which is continuous. From the constructions, it follows that

$$\mathrm{id}\left(\mathbf{B}^{s_1}_{p_1,q_1}(\Omega) \longrightarrow \mathbf{B}^{s_2}_{p_2,q_2}(\Omega)\right) = \mathrm{Re}_{\Omega} \circ T \circ \mathrm{id} \circ S \circ \mathrm{Ex}. \tag{2.54}$$

Hence, taking finally Re_{Ω}, we obtain f by (2.41), where we started from. In particular, due to the fact that we used the subatomic approach, the final outcome is independent of ambiguities in the nonlinear constructions Ex and S. The unit ball in $\mathbf{B}^{s_1}_{p_1,q_1}(\Omega)$ is mapped by $S \circ \mathrm{Ex}$ into a bounded set in

$$b^{s_1,\varrho_1}_{p_1,q_1}(M_j). \tag{2.55}$$

Since the identity operator id from (2.52) is compact, this bounded set is mapped into a precompact set in

$$b^{s_2,Q_2}_{p_2,q_2}(M_j), \tag{2.56}$$

which can be covered by 2^k balls of radius $ce_k(\mathrm{id})$ with

$$e_k(\mathrm{id}) \le ck^{-\delta/n+1/p_2-1/p_1}, \qquad k \in \mathbb{N}. \tag{2.57}$$

This follows from Theorem 2.9, where we used $p_2 \ge p_1$. Applying the two linear and bounded maps T and Re_Ω afterwards does not change this covering assertion—using Lemma 2.6 (iii) and ignoring constants for the time being. Hence, we arrive at a covering of the unit ball in $\mathbf{B}^{s_1}_{p_1,q_1}(\Omega)$ by 2^k balls of radius $ce_k(\mathrm{id})$ in $\mathbf{B}^{s_2}_{p_2,q_2}(\Omega)$. Inserting

$$\delta = s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \tag{2.58}$$

in the exponent, we finally obtain the desired estimate

$$e_k(\mathrm{id}) \le ck^{-(s_1-s_2)/n}, \qquad k \in \mathbb{N}. \tag{2.59}$$

Step 2. Let $p_1 > p_2$. Since, by Proposition 2.4,

$$\mathbf{B}^{s_2}_{p_1,q_2}(\Omega) \subset \mathbf{B}^{s_2}_{p_2,q_2}(\Omega), \tag{2.60}$$

we see that

$$\mathbf{B}^{s_1}_{p_1,q_1}(\Omega) \subset \mathbf{B}^{s_2}_{p_1,q_2}(\Omega) \subset \mathbf{B}^{s_2}_{p_2,q_2}(\Omega), \tag{2.61}$$

and, therefore, (2.40) is a consequence of Step 1 applied to $p_1 = p_2$. This completes the proof for the upper bound.

□

Remark 2.12. By (2.13) and the above definitions, we have

$$\mathbf{B}^{s}_{p,\min(p,q)}(\Omega) \hookrightarrow \mathbf{F}^{s}_{p,q}(\Omega) \hookrightarrow \mathbf{B}^{s}_{p,\max(p,q)}(\Omega). \tag{2.62}$$

In other words, any assertion about entropy numbers for B-spaces where the parameter q does not play any role applies also to the related F-spaces.

Therefore, using Lemma 2.6 (iv) and Theorem 2.11, we deduce compactness of the corresponding embeddings related to B- and F-spaces under investigation.

3. Homogeneity

Our first aim is to prove the following characterization.

Proposition 3.1. *Let $0 < p, q \le \infty$, $s > 0$, and let $R > 0$ be a real number. Then,*

$$\left\| f \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \sim \left(\int_0^\infty t^{-sq} \omega_r(f,t)_p^q \frac{dt}{t} \right)^{1/q} \tag{3.1}$$

for all $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B_R$.

Proof. We will need that $\mathbf{B}^s_{p,q}(B_R)$ embeds compactly into $L_p(B_R)$. This follows at once from the fact that $\mathbf{B}^s_{p,q}(B_R)$ is compactly embedded into $\mathbf{B}^{s-\varepsilon}_{p,q}(B_R)$, cf. Remark 2.12, and $\mathbf{B}^{s-\varepsilon}_{p,q}(B_R) \hookrightarrow L_p(B_R)$, which is trivial.We argue similarly to [6]. We have to prove that

$$\| f \mid L_p(\mathbb{R}^n) \| \lesssim \left(\int_0^\infty t^{-sq} \omega_r(f,t)_p^q \frac{dt}{t} \right)^{1/q} \tag{3.2}$$

for every $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B_R$. Let us assume that this is not true. Then, we find a sequence $(f_j)_{j=1}^\infty \subset \mathbf{B}^s_{p,q}(\mathbb{R}^n)$, such that

$$\| f_j \mid L_p(\mathbb{R}^n) \| = 1, \qquad \left(\int_0^\infty t^{-sq} \omega_r(f_j,t)_p^q \frac{dt}{t} \right)^{1/q} \le \frac{1}{j}, \tag{3.3}$$

that is, we obtain that $\| f_j \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \|$ is bounded. The trivial estimates

$$\| f_j \mid L_p(\mathbb{R}^n) \| = \| f_j \mid L_p(B_R) \|, \qquad \left\| f_j \mid \mathbf{B}^s_{p,q}(B_R) \right\| \le \left\| f_j \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \tag{3.4}$$

imply that this is true also for $\| f_j \mid \mathbf{B}^s_{p,q}(B_R) \|$. Due to the compactness of $\mathbf{B}^s_{p,q}(B_R) \hookrightarrow L_p(B_R)$, we may assume, that $f_j \to f$ in $L_p(B_R)$ with $\| f | L_p(B_R) \| = 1$. Using the subadditivity of $\omega(\cdot,t)_p$, we obtain that

$$\left(\int_0^\infty t^{-sq} \omega_r(f_j - f_{j'},t)_p^q \frac{dt}{t} \right)^{1/q} \le \frac{1}{j} + \frac{1}{j'}. \tag{3.5}$$

Together with the estimate $\| f_j - f_{j'} \mid L_p(\mathbb{R}^n) \| \to 0$, this implies that $(f_j)_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$, that is, $f_j \to g$ in $\mathbf{B}^s_{p,q}(\mathbb{R}^n)$. Obviously, $f = g$ follows.

The subadditivity of $\omega(\cdot,t)_p$ used to the sum $(f - f_j) + f_j$ implies finally that

$$\left(\int_0^\infty t^{-sq} \omega_r(f,t)_p^q \frac{dt}{t} \right)^{1/q} = 0. \tag{3.6}$$

As $\omega_r(f,t)$ is a nondecreasing function of t, this implies that $\omega_r(f,t) = 0$ for all $0 < t < \infty$ and finally $\| \Delta_h^r f \mid L_p(\mathbb{R}^n) \| = 0$ for all $h \in \mathbb{R}^n$. By standard arguments, this is satisfied only if f is a polynomial of order at most r. Due to its bounded support, we conclude that $f = 0$, which is a contradiction with $\|f \mid L_p(\mathbb{R}^n)\| = 1$. $\square$

With the help of this proposition, the proof of homogeneity quickly follows.

Theorem 3.2. *Let $0 < \lambda \leq 1$ and $f \in \mathbf{B}^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B_\lambda$. Then,*

$$\left\| f(\lambda \cdot) \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \sim \lambda^{s-n/p} \left\| f \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \tag{3.7}$$

with constants of equivalence independent of λ and f.

Proof. We know from Proposition 3.1 that

$$\left\| f(\lambda \cdot) \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\| \sim \left(\int_0^\infty t^{-sq} \omega_r(f(\lambda \cdot), t)_p^q \frac{dt}{t} \right)^{1/q}, \tag{3.8}$$

as $\operatorname{supp} f(\lambda \cdot) \subset B_1$. Using $\Delta^r_h(f(\lambda \cdot))(x) = (\Delta^r_{\lambda h} f)(\lambda x)$, we get

$$\begin{aligned} \omega_r(f(\lambda \cdot), t)_p &= \sup_{|h| \leq t} \left\| \Delta^r_h (f(\lambda \cdot)) \right\|_p = \sup_{|h| \leq t} \left\| (\Delta^r_{\lambda h} f)(\lambda \cdot) \right\|_p = \lambda^{-n/p} \sup_{|h| \leq t} \left\| (\Delta^r_{\lambda h} f)(\cdot) \right\|_p \\ &= \lambda^{-n/p} \sup_{|\lambda h| \leq \lambda t} \left\| (\Delta^r_{\lambda h} f)(\cdot) \right\|_p = \lambda^{-n/p} \omega_r(f, \lambda t)_p, \end{aligned} \tag{3.9}$$

which finally implies

$$\begin{aligned} \left(\int_0^\infty t^{-sq} \omega_r(f(\lambda \cdot), t)_p^q \frac{dt}{t} \right)^{1/q} &= \lambda^{-n/p} \left(\int_0^\infty t^{-sq} \omega_r(f, \lambda t)_p^q \frac{dt}{t} \right)^{1/q} \\ &= \lambda^{s-n/p} \left(\int_0^\infty t^{-sq} \omega_r(f, t)_p^q \frac{dt}{t} \right)^{1/q} \sim \lambda^{s-n/p} \left\| f \mid \mathbf{B}^s_{p,q}(\mathbb{R}^n) \right\|. \end{aligned} \tag{3.10}$$

$\square$

The homogeneity property for Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ follows similarly.

Proposition 3.3. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s > 0$, and let $R > 0$ be a real number. Then,*

$$\left\| f \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\| \sim \left\| \left(\int_0^\infty t^{-sq} d^r_{t,p} f(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \tag{3.11}$$

for all $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B_R$.

Proof. We have to prove that

$$\left\| f \mid L_p(\mathbb{R}^n) \right\| \lesssim \left\| \left(\int_0^\infty t^{-sq} d^r_{t,p} f(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \tag{3.12}$$

for every $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$ with supp $f \subset B_R$. Let us assume again that this is not true. Then, we find a sequence $(f_j)_{j=1}^\infty \subset \mathbf{F}^s_{p,q}(\mathbb{R}^n)$ such that

$$\left\| f_j \mid L_p(\mathbb{R}^n) \right\| = 1, \qquad \left\| \left(\int_0^\infty t^{-sq} d^r_{t,p} f_j(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \le \frac{1}{j}, \tag{3.13}$$

which in turn implies that $\| f_j \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n)\|$ is bounded. Again, the same is true also for $\| f_j | \mathbf{F}^s_{p,q}(B_R)\|$. Due to the compactness of $\mathbf{F}^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n)$, we may assume that $f_j \to f$ in $L_p(B_R)$ with $\| f \mid L_p(B_R)\| = 1$. A straightforward calculation shows again that $(f_j)_{j=1}^\infty$ is a Cauchy sequence in $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ and, therefore, $f_j \to f$ also in $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$. Finally, we obtain

$$\left\| \left(\int_0^\infty t^{-sq} d^r_{t,p} f(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| = 0 \tag{3.14}$$

or, equivalently,

$$\int_0^\infty t^{-sq} d^r_{t,p} f(x)^q \frac{dt}{t} = 0 \tag{3.15}$$

for almost every $x \in \mathbb{R}^n$. Hence, $d^r_{t,p} f(x) = 0$ for almost all $x \in \mathbb{R}^n$ and almost all $t > 0$. By standard arguments, it follows that f must be almost everywhere equal to a polynomial of order smaller than r. Together with the bounded support of f, we obtain that f must be equal to zero almost everywhere. □

Theorem 3.4. *Let* $0 < \lambda \le 1$ *and* $f \in \mathbf{F}^s_{p,q}(\mathbb{R}^n)$ *with* supp$f \subset B_\lambda$. *Then,*

$$\left\| f(\lambda \cdot) \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\| \sim \lambda^{s-n/p} \left\| f \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\| \tag{3.16}$$

with constants of equivalence independent of λ *and* f.

Proof. We know from Proposition 3.3 that

$$\left\| f(\lambda \cdot) \mid \mathbf{F}^s_{p,q}(\mathbb{R}^n) \right\| \sim \left\| \left(\int_0^\infty t^{-sq} d^r_{t,p} (f(\lambda \cdot))(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\|, \tag{3.17}$$

as supp $f(\lambda \cdot) \subset B_1$. Using $\Delta^r_h (f(\lambda \cdot))(x) = (\Delta^r_{\lambda h} f)(\lambda x)$, we get using the substitution $\widetilde{h} = \lambda h$,

$$\begin{aligned} d^r_{t,p}(f(\lambda \cdot))(x) &= \left(t^{-n} \int_{|h| \le t} \left| \Delta^r_h f(\lambda \cdot)(x) \right|^p dh \right)^{1/p} = \left(t^{-n} \int_{|h| \le t} \left| (\Delta^r_{\lambda h} f)(\lambda x) \right|^p dh \right)^{1/p} \\ &= \left((\lambda t)^{-n} \int_{|\widetilde{h}| \le \lambda t} \left| \left(\Delta^r_{\widetilde{h}} f \right)(\lambda x) \right|^p d\widetilde{h} \right)^{1/p} = d^r_{\lambda t,p}(f)(\lambda x), \end{aligned} \tag{3.18}$$

which finally implies

$$\begin{aligned}\left\| \left(\int_0^\infty t^{-sq} d_{t,p}^r (f(\lambda \cdot))(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| &= \left\| \left(\int_0^\infty t^{-sq} d_{\lambda t,p}^r f(\lambda \cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \\ &= \lambda^s \left\| \left(\int_0^\infty t^{-sq} d_{t,p}^r f(\lambda \cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \\ &= \lambda^{s-n/p} \left\| \left(\int_0^\infty t^{-sq} d_{t,p}^r f(\cdot)^q \frac{dt}{t} \right)^{1/q} \mid L_p(\mathbb{R}^n) \right\| \\ &\sim \lambda^{s-n/p} \left\| f \mid \mathbf{F}_{p,q}^s(\mathbb{R}^n) \right\|. \end{aligned} \tag{3.19}$$

□

4. Pointwise Multipliers

We briefly sketch an application of the above homogeneity results in terms of pointwise multipliers. A locally integrable function φ in $\mathbb{R}^n$ is called a *pointwise multiplier* in $\mathbf{A}_{p,q}^s(\mathbb{R}^n)$ if

$$f \longmapsto \varphi f \tag{4.1}$$

maps the considered space into itself. For further details on the subject, we refer to [27, pp. 201–206] and [28, Chapter 4]. Our aim is to generalize Proposition 2.3 as a direct consequence of Theorems 3.2 and 3.4. Again let B_λ be the balls introduced in (2.7).

Corollary 4.1. *Let $s > 0$, $0 < p, q \leq \infty$, and $0 < \lambda \leq 1$. Let φ be a function having classical derivatives in $B_{2\lambda}$ up to order $1 + [s]$ with*

$$\left| D^\gamma \varphi(x) \right| \leq a \lambda^{-|\gamma|}, \qquad \left| \gamma \right| \leq 1 + [s],\ x \in B_{2\lambda}, \tag{4.2}$$

for some constant $a > 0$. Then, φ is a pointwise multiplier in $\mathbf{B}_{p,q}^s(B_\lambda)$,

$$\left\| \varphi f \mid \mathbf{B}_{p,q}^s(B_\lambda) \right\| \leq c \left\| f \mid \mathbf{B}_{p,q}^s(B_\lambda) \right\|, \tag{4.3}$$

where c is independent of $f \in \mathbf{B}_{p,q}^s(B_\lambda)$ and of λ (but depends on a).

Proof. By Proposition 2.3, the function $\varphi(\lambda \cdot)$ is a pointwise multiplier in $\mathbf{B}_{p,q}^s(B_1)$. Then, (4.3) is a consequence of (3.7),

$$\begin{aligned}\left\| \varphi f \mid \mathbf{B}_{p,q}^s(B_\lambda) \right\| &\sim \lambda^{-(s-n/p)} \left\| \varphi f(\lambda \cdot) \mid \mathbf{B}_{p,q}^s(B_1) \right\| \\ &\lesssim \lambda^{-(s-n/p)} \left\| f(\lambda \cdot) \mid \mathbf{B}_{p,q}^s(B_1) \right\| \sim \left\| f \mid \mathbf{B}_{p,q}^s(B_\lambda) \right\|. \end{aligned} \tag{4.4}$$

□

Remark 4.2. In terms of Triebel-Lizorkin spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$, we obtain corresponding results (assuming $p < \infty$) with the additional restriction on the smoothness parameter s that

$$s > n\left(\frac{1}{\min(p,q)} - \frac{1}{p}\right). \tag{4.5}$$

This follows from the fact that the analogue of Proposition 2.3 for F-spaces is established using an atomic characterization of the spaces $\mathbf{F}^s_{p,q}(\mathbb{R}^n)$ which is only true if we impose (4.5), (cf. [13, Proposition 9.14]).

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