Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 283730, 13 pages
doi:10.1155/2012/283730

Research Article

Weighted Herz Spaces and Regularity Results

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Received 13 April 2012; Accepted 20 July 2012

Academic Editor: Hans G. Feichtinger

It is proved that, for the nondivergence form elliptic equations $\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = f$, if f belongs to the weighted Herz spaces $K_p^q(\varphi, w)$, then $u_{x_i x_j} \in K_p^q(\varphi, w)$, where u is the $W^{2,p}$-solution of the equations. In order to obtain this, the authors first establish the weighted boundedness for the commutators of some singular integral operators on $K_p^q(\varphi, w)$.

1. Introduction

For a sequence $\varphi = \{\varphi(k)\}_{-\infty}^{\infty}$, $\varphi(k) > 0$, we suppose that φ satisfies doubling condition of order (s, t) and write $\varphi \in D(s, t)$ if there exists $C \geq 1$ such that

$$C^{-1} 2^{s(k-j)} \leq \frac{\varphi(k)}{\varphi(j)} \leq C 2^{t(k-j)} \quad \text{for } k > j. \tag{1.1}$$

Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $E_k = B_k \setminus B_{k-1}$ for $k \in \mathbb{Z}$, and $\chi_k = \chi_{E_k}$ be the characteristic function of the set E_k for $k \in \mathbb{Z}$. Suppose that w is a weight function on $\mathbb{R}^n$. For $1 < p < \infty$, $0 < q < \infty$, the weighted Herz space is defined by

$$K_p^q(\varphi, w)(\mathbb{R}^n) = \left\{ f : f \text{ is a measurable function on } \mathbb{R}^n,\ \|f\|_{K_p^q(\varphi, w)} < \infty \right\}, \tag{1.2}$$

where

$$\|f\|_{K_p^q(\varphi, w)} = \left(\sum_{k=-\infty}^{\infty} \varphi(k)^q \|f \chi_k\|_{L^p(w)}^q \right)^{1/q}, \qquad \|f\|_{L^p(w)} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{1/p}. \tag{1.3}$$

Beurling in [1] introduced the Beurling algebras, and Herz in [2] generalized these spaces; many studies have been done for Herz spaces (see, e.g, [3, 4]). Weighted Herz spaces are also considered in [5, 6]. Lu and Tao in [7] studied nondivergence form elliptic equations on Morrey-Herz spaces, which are more general spaces. Ragusa in [8, 9] obtained some regularity results to the divergence form elliptic and parabolic equations on homogeneous Herz spaces.

The paper is organized as follows. In Section 2, we give some basic notions. In this section, we recall also continuity results regarding the Calderón-Zygmund singular integral operators that will appear in the representation formula of the $u_{x_ix_j}$ estimates. In Section 3, we prove the boundedness of the commutators of some singular integral operators on weighted Herz spaces. In Section 4, we study the interior estimates on weighted Herz spaces for the solutions of some nondivergence elliptic equations $\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j} = f$, and we prove that if $f \in K_p^q(\varphi, w)$, then $u_{x_ix_j} \in K_p^q(\varphi, w)$, where u is the $W^{2,p}$-solution of the equations.

Throughout this paper, unless otherwise indicated, C will be used to denote a positive constant that is not necessarily the same at each occurrence.

2. Preliminaries

We begin this section with some properties of A_p weights classes which play important role in the proofs of our main results. For more about A_p classes, we can refer to [10, 11].

Definition 2.1 (A_p weights $(1 \le p < \infty)$). Let $w(x) \ge 0$ and $w(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. One says that $w \in A_p$ for $1 < p < \infty$ if there exists a constant C such that for every ball $B \subset \mathbb{R}^n$,

$$\sup_B \left\{ \frac{1}{|B|} \int_B w(x)dx \right\} \left\{ \frac{1}{|B|} \int_B w(x)^{1-p'}dx \right\}^{p-1} \le C \tag{2.1}$$

holds, here and below, $1/p + 1/p' = 1$. One says that $w \in A_1$ if there exists a positive constant C such that

$$\frac{1}{|B|} \int_B w(x)dx \le C \operatorname*{essinf}_{x\in B} w(x). \tag{2.2}$$

The smallest constant appearing in (2.1) or (2.2) is called the A_p constant of w, denoted by C_w.

Lemma 2.2. *Let $1 \le p < \infty$ and $w \in A_p$. Then the following statements are true:*

(1) *(strong doubling) there exists a constant C such that*

$$\frac{w(B_k)}{w(B_j)} \le C2^{np(k-j)} \quad \text{for } k > j, \tag{2.3}$$

(2) *(centered reverse doubling) for some $\delta > 0$, $w \in RD(\delta)$, that is,*

$$\frac{w(B_k)}{w(B_j)} \ge C2^{\delta(k-j)} \quad \text{for } k > j, \tag{2.4}$$

(3) *for* $1 < p < \infty$, *one has* $w \in A_{\overline{p}}$ *for some* $\overline{p} < p$,

(4) *there exist two constants* C *and* $\delta > 0$ *such that for any measurable set* $B \subset E$,

$$\frac{w(B)}{w(E)} \leq C\left(\frac{|B|}{|E|}\right)^{\delta}. \tag{2.5}$$

If w *satisfies* (2.5), *one says* $w \in A_\infty$. *Obviously,* $A_\infty = \bigcup_{1\leq p<\infty} A_p$,

(5) *for all* $(1/p) + (1/p') = 1$, *one has* $w^{1-p'} \in A_{p'}$.

Remark 2.3. Note that $w(E) = \int_E w(x)dx$ and $w^p(E)^{1/p} = (\int_E w^p(x)dx)^{1/p}$.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be an open set. One says that any $f \in L^1_{\text{loc}}(\Omega)$ is in the bounded mean oscillation spaces BMO(Ω) if

$$\sup_{\gamma>0, x\in B_\gamma(x)\subset\Omega} \frac{1}{|B_\gamma(x)|}\int_{B_\gamma(x)} \left|f(y) - f_{B_\gamma(x)}\right| dy \equiv \|f\|_* < \infty, \tag{2.6}$$

where $f_{B_\gamma(x)}$ is the average over $B_\gamma(x)$ of f. Moreover, for any $f \in \text{BMO}(\Omega)$ and $r > 0$, one sets

$$\sup_{\gamma\leq r, x\in B_\gamma(x)\subset\Omega} \frac{1}{|B_\gamma(x)|}\int_{B_\gamma(x)} \left|f(y) - f_{B_\gamma(x)}\right| dy \equiv \eta(r). \tag{2.7}$$

One says that any $f \in \text{BMO}(\Omega)$ is in the vanishing mean oscillation spaces VMO(Ω) if $\eta(r) \to 0$ as $r \to 0$ and refer to $\eta(r)$ as the modulus of f.

Remark 2.5. $f \in \text{BMO}(\mathbb{R}^n)$ or $\text{VMO}(\mathbb{R}^n)$ if B ranges in the class of balls of $\mathbb{R}^n$.

Lemma 2.6 (see [12, Theorem 5]). *Let* $w \in A_\infty$. *Then the norm of* $BMO(w)$ *is equivalent to the norm of* $BMO(\mathbb{R}^n)$, *where*

$$\begin{gathered} BMO(w) = \left\{a : \|a\|_{*,w} = \sup \frac{1}{w(B)}\int_B |a(x) - a_{B,w}| w(x)dx\right\}, \\ a_{B,w} = \frac{1}{w(B)}\int_B a(z)w(z)dz. \end{gathered} \tag{2.8}$$

Definition 2.7. Let $K : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. One says that $K(x)$ is a constant Calderón-Zygmund kernel (constant C-Z kernel) if

(i) $K \in C^\infty(\mathbb{R}^n \setminus \{0\})$,

(ii) K is homogeneous of degree $-n$,

(iii) $\int_{S^{n-1}} K(x)d\sigma = 0$, $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Definition 2.8. Let Ω be an open set of $\mathbb{R}^n$ and $K : \Omega \times \{\mathbb{R}^n \setminus \{0\}\} \to \mathbb{R}$. One says that $K(x, y)$ is a variable Calderón-Zygmund kernel (variable C-Z kernel) on Ω if

(i) $K(x,\cdot)$ is a constant C-Z kernel for a.e. $x \in \Omega$,

(ii) $\max_{|j|\le 2n}\|(\partial^j/\partial z^j)K(x,z)\|_{L^\infty(\Omega\times S^{n-1})} \equiv M < \infty$.

Let K be a constant or a variable C-Z kernel on Ω. One defines the corresponding C-Z operator by

$$Tf(x) = \text{P.V.}\int_{\mathbb{R}^n} K(x-y)f(y)dy \quad \text{or } Tf(x) = \text{P.V.}\int_{\Omega} K(x,x-y)f(y)dy. \tag{2.9}$$

Lemma 2.9 (see [5, Theorem 3]). *Let $1<p<\infty$, $0<q<\infty$, $\delta>0$. One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1-(1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1-(t/n)))$,

(iii) $w \in RD(\delta)$.

If K is a constant or a variable C-Z kernel on $\mathbb{R}^n$ and T is the corresponding C-Z operator, then there exists a constant C such that for all $f \in K_p^q(\varphi,w)(\mathbb{R}^n)$,

$$\|Tf\|_{K_p^q(\varphi,w)(\mathbb{R}^n)} \le C\|f\|_{K_p^q(\varphi,w)(\mathbb{R}^n)}. \tag{2.10}$$

From this lemma, by a proof similar to that of Theorem 2.11 in [13], we obtain the following corollary.

Corollary 2.10. *Let $1<p<\infty$, $0<q<\infty$, $\delta>0$, and Ω be an open set of $\mathbb{R}^n$. One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1-(1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1-(t/n)))$,

(iii) $w \in RD(\delta)$.

If K is a constant or a variable C-Z kernel on Ω, and T is the corresponding C-Z operator, then there exists a constant C such that for all $f \in K_p^q(\varphi,w)(\Omega)$,

$$\|Tf\|_{K_p^q(\varphi,w)(\Omega)} \le C\|f\|_{K_p^q(\varphi,w)(\Omega)}. \tag{2.11}$$

3. Weighted Boundedness of Commutators

The aim of this section is to set up the weighted boundedness for the commutators formed by T and $\text{BMO}(\mathbb{R}^n)$ functions, where $[a,T]f(x) = T(af)(x) - a(x)T(f)(x)$. This kind of operators is useful in lots of different fields, see, for example, [13] as well as [14], then we consider important in themselves the related below results.

Lemma 3.1 (see [10, Theorem 7.1.6]). *Let $a \in BMO(\mathbb{R}^n)$. Then for any ball $B \subset \mathbb{R}^n$, there exist constants C_1, C_2 such that for all $\alpha > 0$,*

$$|\{x \in B : |a(x) - a_B| > \alpha\}| \le C_1|B|e^{-C_2\alpha/\|a\|_*}. \tag{3.1}$$

The inequality (3.1) *is also called John-Nirenberg inequality.*

Theorem 3.2. *Let $1 < p < \infty,\ 0 < q < \infty,\ \delta > 0$, and $a \in BMO(\mathbb{R}^n)$. One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1-(1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1-(t/n)))$,

(iii) $w \in RD(\delta)$.

If a linear operator T satisfies

$$|T(f)(x)| \le C\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^n}dy, \quad x \notin \text{ supp } f, \tag{3.2}$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $[a,T]$ is bounded on $L^p(w)$, then $[a,T]$ is also bounded on $K^q_p(\varphi,w)$.

Proof. Let $f \in K^q_p(\varphi,w)(\mathbb{R}^n)$ and $a \in \text{BMO}(\mathbb{R}^n)$, we write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x). \tag{3.3}$$

Then, we have

$$\begin{aligned}
\left\|[a,T]f\right\|_{K^q_p(\varphi,w)} &\le C\left(\sum_{k=-\infty}^{\infty}\varphi(k)^q\left(\sum_{j=-\infty}^{k-2}\left\|([a,T]f_j)\chi_k\right\|_{L^p(w)}\right)^q\right)^{1/q} \\
&\quad + C\left(\sum_{k=-\infty}^{\infty}\varphi(k)^q\left(\sum_{j=k-1}^{k+1}\left\|([a,T]f_j)\chi_k\right\|_{L^p(w)}\right)^q\right)^{1/q} \\
&\quad + C\left(\sum_{k=-\infty}^{\infty}\varphi(k)^q\left(\sum_{j=k+2}^{\infty}\left\|([a,T]f_j)\chi_k\right\|_{L^p(w)}\right)^q\right)^{1/q} \\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned} \tag{3.4}$$

For II, by the $L^p(w)$ boundedness of $[a,T]$, we have

$$\begin{aligned}
\mathrm{II} &\le C\left(\sum_{k=-\infty}^{\infty}\varphi(k)^q\left(\sum_{j=k-1}^{k+1}\|a\|_*^q\left\|f_j\chi_k\right\|_{L^p(w)}\right)^q\right)^{1/q} \\
&\le C\|a\|_*\|f\|_{K^q_p(\varphi,w)}.
\end{aligned} \tag{3.5}$$

For I, note that when $x \in E_k,\ y \in E_j$, and $j \le k-2$, $|x-y| \sim |x|$. So from the condition (3.2), we have

$$\begin{aligned}
\left|[a,T]f_j\right| &\le C\int_{\mathbb{R}^n} \frac{\left|a(x)-a(y)\right|}{\left|x-y\right|^n}\left|f_j(y)\right|dy \\
&\le C2^{-nk}|a(x)-a_{B_k,w}|\int_{\mathbb{R}^n}\left|f_j(y)\right|dy + C2^{-nk}\left|a_{B_k,w}-a_{B_j,w}\right|\int_{\mathbb{R}^n}\left|f_j(y)\right|dy \\
&\quad + C2^{-nk}\int_{\mathbb{R}^n}\left|a(y)-a_{B_j,w}\|f_j(y)\right|dy.
\end{aligned} \tag{3.6}$$

Thus,

$$\begin{aligned}
\left\|([a,T]f_j)\chi_k\right\|_{L^p(w)} &\le C2^{-nk}\left\|(a(x)-a_{B_k,w})\chi_k\right\|_{L^p(w)}\int_{\mathbb{R}^n}\left|f_j(y)\right|dy \\
&\quad + C2^{-nk}\left|a_{B_k,w}-a_{B_j,w}\right|w(B_k)^{1/p}\int_{\mathbb{R}^n}\left|f_j(y)\right|dy \\
&\quad + C2^{-nk}w(B_k)^{1/p}\int_{\mathbb{R}^n}\left|a(y)-a_{B_j,w}\|f_j(y)\right|dy \\
&= J_1+J_2+J_3.
\end{aligned} \tag{3.7}$$

According to Lemma 2.2, $w \in A_{\overline{r}}$ for some $\overline{r} < r$. By Hölder's inequality and Lemma 2.6,

$$\begin{aligned}
J_1 &\le C2^{-nk}\|a\|_* w(B_k)^{1/p}\left\|f_j\right\|_{L^p(w)} w^{-p'/p}(B_j)^{1/p'} \\
&= C2^{-nk}\|a\|_*\left\|f_j\right\|_{L^p(w)} w^{-p'/p}(B_j)^{1/p'} w(B_j)^{1/p}\left(\frac{w(B_k)}{w(B_j)}\right)^{1/p} \\
&\le C2^{-nk}\|a\|_*\left\|f_j\right\|_{L^p(w)}\left|B_j\right|2^{n\overline{r}(k-j)/p} \\
&\le C2^{k(-n+(\overline{r}n/p))}2^{nj(1-(\overline{r}/p))}\|a\|_*\left\|f_j\right\|_{L^p(w)}.
\end{aligned} \tag{3.8}$$

It is easy to see that $|a_{B_k,w}-a_{B_j,w}| \le C(k-j)\|a\|_*$. Therefore, similarly to J_1, we have

$$\begin{aligned}
J_2 &\le C(k-j)2^{-nk}\|a\|_* w(B_k)^{1/p}\left\|f_j\right\|_{L^p(w)} w^{-p'/p}(B_j)^{1/p'} \\
&\le Ck2^{k(-n+(\overline{r}n/p))}j2^{nj(1-(\overline{r}/p))}\|a\|_*\left\|f_j\right\|_{L^p(w)}.
\end{aligned} \tag{3.9}$$

Now, we establish the estimate for term J_3,

$$J_3 \le C2^{-nk}w(B_k)^{1/p}\left\|f_j\right\|_{L^p(w)}\left(\int_{B_j}\left|a(y)-a_{B_j,w}\right|^{p'} w^{1-p'}(y)dy\right)^{1/p'}. \tag{3.10}$$

For the simplicity of analysis, we denote H as

$$\left(\int_{B_j}\left|a(y)-a_{B_j,w}\right|^{p'}w^{1-p'}(y)dy\right)^{1/p'}. \tag{3.11}$$

By an elementary estimate, we have

$$\begin{aligned}H &\leq C\left(\int_{B_j}\left[\left|a(y)-a_{B_j,w^{1-p'}}\right|+\left|a_{B_j,w^{1-p'}}-a_{B_j,w}\right|\right]^{p'}w^{1-p'}(y)dy\right)^{1/p'}\\ &\leq C\|a\|_{\mathrm{BMO}(w^{1-p'})}w^{1-p'}(B_j)^{1/p'}+\left|a_{B_j,w^{1-p'}}-a_{B_j,w}\right|w^{1-p'}(B_j)^{1/p'}.\end{aligned} \tag{3.12}$$

Note that

$$\begin{aligned}\left|a_{B_j,w^{1-p'}}-a_{B_j,w}\right| &\leq \left|a_{B_j,w^{1-p'}}-a_{B_j}\right|+\left|a_{B_j}-a_{B_j,w}\right|\\ &= J_{31}+J_{32}.\end{aligned} \tag{3.13}$$

Combining (2.5) with (3.1),

$$\begin{aligned}J_{32} &= \frac{1}{w(B_j)}\int_{B_j}\left|a(y)-a_{B_j}\right|w(y)dy\\ &= \frac{1}{w(B_j)}\int_0^\infty w\left(\left\{x\in B_j:\left|a(y)-a_{B_j}\right|>\alpha\right\}\right)d\alpha\\ &\leq C\int_0^\infty e^{-C_2\alpha\delta/\|a\|_*}d\alpha\leq C.\end{aligned} \tag{3.14}$$

In the same manner, we can see that

$$J_{31}\leq C. \tag{3.15}$$

By Lemma 2.6, we get

$$\begin{aligned}J_3 &\leq C2^{-nk}\|a\|_*w(B_k)^{1/p}\left\|f_j\right\|_{L^p(w)}w^{-p'/p}(B_j)^{1/p'}\\ &\leq C2^{k(-n+(\overline{r}n/p))}2^{nj(1-(\overline{r}/p))}\|a\|_*\left\|f_j\right\|_{L^p(w)}.\end{aligned} \tag{3.16}$$

Using hypotheses $\varphi \in D(s,t)$ and the estimates of J_1, J_2, and J_3, we obtain the following inequality:

$$\begin{aligned} \mathrm{I} \leq{}& C\|a\|_*\left(\sum_k 2^{k(-n+(\overline{r}n/p)+t)q}\cdot\left(\sum_{j\leq k-2}2^{nj(1-(\overline{r}/p)-(t/n))}\varphi(j)\left\|f\chi_j\right\|_{L^p(w)}\right)^q\right)^{1/q}\\ &+C\|a\|_*\left(\sum_k\left(k2^{k(-n+(\overline{r}n/p)+t)}\right)^q\cdot\left(\sum_{j\leq k-2}j2^{nj(1-(\overline{r}/p)-(t/n))}\varphi(j)\left\|f\chi_j\right\|_{L^p(w)}\right)^q\right)^{1/q}\\ ={}& \mathrm{I}_1+\mathrm{I}_2. \end{aligned} \tag{3.17}$$

When $q \leq 1$, we have

$$\begin{aligned} \mathrm{I}_1 &\leq C\|a\|_*\left(\sum_j 2^{nj(1-(\overline{r}/p)-(t/n))q}\varphi(j)^q\left\|f\chi_j\right\|^q_{L^p(w)}\cdot\sum_{k=j+2}^{\infty}2^{k(-n+(\overline{r}n/p)+t)q}\right)^{1/q}\\ &\leq C\|a\|_*\|f\|_{K^q_p(\varphi,w)}, \end{aligned} \tag{3.18}$$

because $-n+nr/p+t\leq 0$, that is, $-n+n\overline{r}/p+t<0$.

When $q>1$, we take $\varepsilon>0$ such that $-n+n\overline{r}/p+t+n\varepsilon<0$. Then

$$\begin{aligned} \mathrm{I}_1 \leq{}& C\|a\|_*\left[\sum_k 2^{k(-n+(\overline{r}n/p)+t)q}\cdot\left(\sum_{j=-\infty}^{k-2}2^{nj(1-(\overline{r}/p)-(t/n)-\varepsilon)q}\varphi(j)^q\left\|f\chi_j\right\|^q_{L^p(w)}\cdot\left(\sum_{j=-\infty}^{k-2}2^{n\varepsilon q' j}\right)^{q/q'}\right)\right]^{1/q}\\ \leq{}& C\|a\|_*\left[\sum_k 2^{k(-n+(\overline{r}n/p)+t+n\varepsilon)q}\cdot\left(\sum_{j=-\infty}^{k-2}2^{nj(1-(\overline{r}/p)-(t/n)-\varepsilon)q}\varphi(j)^q\left\|f\chi_j\right\|^q_{L^p(w)}\right)\right]^{1/q}\\ \leq{}& C\|a\|_*\|f\|_{K^q_p(\varphi,w)}. \end{aligned} \tag{3.19}$$

Similar to I_1, we have

$$\mathrm{I}_2 \leq C\|a\|_*\|f\|_{K^q_p(\varphi,w)}. \tag{3.20}$$

Finally we estimate III. The proof of this part is analogue to I, so we just give out an outline. Note that $j \geq k+2$ and $x \in E_k,\ y \in E_j$, $|x-y| \sim |y|$. So from the condition (3.2), we have

$$\begin{aligned}\left\|\left([a,T]f_j\right)\chi_k\right\|_{L^p(w)} &\leq C2^{-nj}\|a(x)-a_{B_k,w}\|_{L^p(w)}\int_{\mathbb{R}^n}|f_j(y)|dy \\ &\quad + C2^{-nj}\left|a_{B_k,w}-a_{B_j,w}\right|w(B_k)^{1/p}\int_{\mathbb{R}^n}|f_j(y)|dy \\ &\quad + C2^{-nj}w(B_k)^{1/p}\int_{\mathbb{R}^n}\left|a(y)-a_{B_j,w}\right||f_j(y)|dy \\ &= J_1'+J_2'+J_3'.\end{aligned} \tag{3.21}$$

Using hypotheses (iii) for w in place of strong doubling,

$$\begin{aligned}J_1' &\leq C2^{-jn}\|a\|_*\|f_j\|_{L^p(w)}w^{-p'/p}(B_j)^{1/p'}w(B_j)^{1/p}\left(\frac{w(B_k)}{w(B_j)}\right)^{1/p} \\ &\leq C2^{k\delta/p}2^{-j\delta/p}\|a\|_*\|f_j\|_{L^p(w)}.\end{aligned} \tag{3.22}$$

Similarly,

$$J_2' \leq Ck2^{k\delta/p}j2^{-j\delta/p}\|a\|_*\|f_j\|_{L^p(w)}, \qquad J_3' \leq C2^{k\delta/p}2^{-j\delta/p}\|a\|_*\|f_j\|_{L^p(w)}. \tag{3.23}$$

Using hypotheses (i) for w, that is, $\varphi \in D(s,t)$, we obtain the following inequality:

$$\begin{aligned}\mathrm{III} &\leq C\|a\|_*\left(\sum_k 2^{k(s+\delta/p)q}\cdot\left(\sum_{j\geq k+2}2^{j(-s-\delta/p)}\varphi(j)\|f\chi_j\|_{L^p(w)}\right)^q\right)^{1/q} \\ &\quad + C\|a\|_*\left(\sum_k\left(k2^{(s+\delta/p)}\right)^q\cdot\left(\sum_{j\geq k+2}j2^{j(-s-\delta/p)}\varphi(j)\|f\chi_j\|_{L^p(w)}\right)^q\right)^{1/q} \\ &= \mathrm{III}_1+\mathrm{III}_2.\end{aligned} \tag{3.24}$$

According to $s+\delta/p>0$, when $q \leq 1$,

$$\begin{aligned}\mathrm{III}_1 &\leq C\|a\|_*\left(\sum_j 2^{j(-s-\delta/p)q}\varphi(j)^q\|f\chi_j\|^q_{L^p(w)}\cdot\sum_{k=-\infty}^{j-2}2^{k(s+\delta/p)q}\right)^{1/q} \\ &\leq C\|a\|_*\|f\|_{K_p^q(\varphi,w)}.\end{aligned} \tag{3.25}$$

When $q > 1$, we take $\varepsilon > 0$ such that $s + \delta/p - \varepsilon > 0$. Then

$$\begin{aligned}
\mathrm{III}_1 &\le C\|a\|_* \left[\sum_k 2^{k(s+\delta/p)q} \cdot \left(\sum_{j\ge k+2} 2^{j(-s-\delta/p+\varepsilon)q} \varphi(j)^q \|f\chi_j\|^q_{L^p(w)} \cdot \left(\sum_{j\ge k+2} 2^{-jq'\varepsilon} \right)^{q/q'} \right) \right]^{1/q} \\
&\le C\|a\|_* \left[\sum_k 2^{k(s+\delta/p-\varepsilon)q} \cdot \left(\sum_{j\ge k+2} 2^{j(-s-\delta/p+\varepsilon)q} \varphi(j)^q \|f\chi_j\|^q_{L^p(w)} \right) \right]^{1/q} \\
&\le C\|a\|_* \|f\|_{K^q_p(\varphi,w)}.
\end{aligned} \tag{3.26}$$

Similar to III_1, we have

$$\mathrm{III}_2 \le C\|a\|_* \|f\|_{K^q_p(\varphi,w)}. \tag{3.27}$$

This finishes the proof of Theorem 3.2. □

The condition (3.2) in Theorem 3.2 can be satisfied by many operators such as Bochner-Riesz operators at the critical index, Ricci-Stein's oscillatory singular integrals, Fefferman's multiplier, and the *C-Z* operators. From this theorem and Theorem 2.7 and 2.10 in [13], we easily deduce the following corollary.

Corollary 3.3. *Let* $1 < p < \infty$, $0 < q < \infty$, $\delta > 0$, *and* $a \in BMO(\mathbb{R}^n)$. *One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1-(1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1-(t/n)))$,

(iii) $w \in RD(\delta)$.

If K is a constant or a variable C-Z kernel on $\mathbb{R}^n$ *and T is the corresponding C-Z operator, then there exists a constant such that for all* $f \in K^q_p(\varphi,w)(\mathbb{R}^n)$,

$$\|[a,T]f\|_{K^q_p(\varphi,w)(\mathbb{R}^n)} \le C\|a\|_* \|f\|_{K^q_p(\varphi,w)(\mathbb{R}^n)}. \tag{3.28}$$

From this and the extension theorem of BMO(Ω)-functions in [15], by a procedure similar to Theorem 2.11 in [13] and Theorem 2.2 in [16], we can obtain the following corollary.

Corollary 3.4. *Let* $1 < p < \infty$, $0 < q < \infty$, *and* $\delta > 0$. *Suppose that* Ω *is an open set of* $\mathbb{R}^n$ *and* $a \in VMO(\Omega)$. *One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1-(1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1-(t/n)))$,

(iii) $w \in RD(\delta)$.

If K is a variable C-Z kernel on Ω and T is the corresponding C-Z operator, then for any $\varepsilon > 0$, there exists a positive number $\rho_0 = \rho_0(\varepsilon, \eta)$ such that for any ball B_R with the radius $R \in (0, \rho_0)$, $B_R \subseteq \Omega$ and all $f \in K_p^q(\varphi, w)(B_R)$,

$$\|[a,T]f\|_{K_p^q(\varphi,w)(B_R)} \leq C\varepsilon \|f\|_{K_p^q(\varphi,w)(B_R)}, \tag{3.29}$$

where $C = C(n, p, q, a, \varphi, M)$ is independent of ε, f, and R.

4. Interior Estimate of Elliptic Equation

In this section, we will establish the interior regularity of the strong solutions to elliptic equations in weighted Herz spaces by applying the estimates about singular integral operators and linear commutators obtained in the above section.

Suppose that $n \geq 3$ and Ω is an open set of $\mathbb{R}^n$. We are concerned with the nondivergence form elliptic equations

$$\mathcal{L}u(x) = -\sum_{i,j=1}^{n} a_{i,j}(x) u_{x_i x_j} = f(x), \quad \text{a.e. in } \Omega, \tag{4.1}$$

whose coefficients a_{ij} are assumed such that

$$\begin{gathered} a_{ij}(x) = a_{ji}(x), \quad \text{a.e. } x \in \Omega,\ i, j = 1, 2, \ldots, n, \\ a_{ij} \in L^\infty(\Omega) \cap \mathrm{VMO}(\Omega), \\ \mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \leq \mu|\xi|^2, \qquad \exists \mu > 0, \quad \text{a.e.} \quad x \in \Omega,\ \xi \in \mathbb{R}^n. \end{gathered} \tag{4.2}$$

Let

$$\begin{gathered} \Gamma(x,t) = \frac{1}{(n-2)\omega_n \left(\det a_{ij}(x)\right)^{1/2}} \left(\sum_{i,j=1}^{n} A_{ij}(x) t_i t_j\right)^{(2-n)/2}, \\ \Gamma_i(x,t) = \frac{\partial}{\partial t_i}\Gamma(x,t), \qquad \Gamma_{ij}(x,t) = \frac{\partial^2}{\partial t_i \partial t_j}\Gamma(x,t), \end{gathered} \tag{4.3}$$

for a.e. $x \in B$ and $\forall t \in \mathbb{R}^n \setminus \{0\}$, where the A_{ij} are the entries of the inverse of the matrix $(a_{ij})_{i,j=1,2,\ldots,n}$.

From [13], we deduce the interior representation, that is, if $u \in W_0^{2,p}$,

$$u_{x_i x_j}(x) = \text{P.V.} \int_B \Gamma_{ij}(x, x-y) \left[\sum_{h,l=1}^{n} (a_{hl}(x) - a_{hl}(y)) u_{x_h x_l}(y) + \mathcal{L}u(y) \right] dy$$
$$+ \mathcal{L}u(x) \int_{|t|=1} \Gamma_i(x,t) t_j d\sigma_t, \quad \text{a.e. for } x \in B \subset \Omega, \tag{4.4}$$

where B is a ball in Ω. We also set

$$M \equiv \max_{i,j=1,\dots,n} \max_{|\beta| \le 2n} \left\| \frac{\partial^\beta}{\partial t^\beta} \Gamma_{ij}(x,t) \right\|_{L^\infty(\Omega \times S^{n-1})} < \infty. \tag{4.5}$$

Theorem 4.1. *Let $1 < p < \infty$, $0 < q < \infty$, and $\delta > 0$. Suppose that Ω is an open set of $\mathbb{R}^n$ and a_{ij} satisfies (4.2) for $i, j = 1, 2, \dots, n$. One assumes that*

(i) $\varphi \in D(s,t)$, *where* $-(\delta/p) < s \le t < n(1 - (1/p))$,

(ii) $w \in A_r$, *where* $r = \min(p, p(1 - (t/n)))$,

(iii) $w \in RD(\delta)$.

Then there exists a constant C such that for all balls $B \subset \Omega$ and $u \in W_0^{2,p}$, One has $u_{x_i x_j} \in K_p^q(\varphi, w)(B)$ and

$$\left\| u_{x_i x_j} \right\|_{K_p^q(\varphi,w)(B)} \le C \|\mathcal{L}u\|_{K_p^q(\varphi,w)(B)}. \tag{4.6}$$

Proof. It is well known that $\Gamma_{ij}(x,t)$ are C-Z kernels in the t variable. Thus, using the technology of [13, 16] and the Corollaries 2.10 and 3.4, we deduce that, for any $\varepsilon > 0$,

$$\left\| u_{x_i x_j} \right\|_{K_p^q(\varphi,w)(B)} \le C\varepsilon \left\| u_{x_i x_j} \right\|_{K_p^q(\varphi,w)(B)} + C\|\mathcal{L}u\|_{K_p^q(\varphi,w)(B)}. \tag{4.7}$$

Choosing ε to be small enough (e.g., $C\varepsilon < 1$), we obtain

$$\left\| u_{x_i x_j} \right\|_{K_p^q(\varphi,w)(B)} \le \left(\frac{C}{(1 - C\varepsilon)} \right) \|\mathcal{L}u\|_{K_p^q(\varphi,w)(B)}. \tag{4.8}$$

This finishes the proof of Theorem 4.1. □

Acknowledgment

This research is supported by the NSF of China (no. 11161044).

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