Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 406540, 17 pages
doi:10.1155/2012/406540

Research Article

Some Estimates of Rough Bilinear Fractional Integral

Yun Fan[1, 2] and Guilian Gao[1]

[1] *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[2] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, China*

Correspondence should be addressed to Guilian Gao, gaoguilian305@163.com

Received 23 July 2012; Revised 11 September 2012; Accepted 25 September 2012

Academic Editor: Ti-Jun Xiao

We study the boundedness of rough bilinear fractional integral $B_{\Omega,\alpha}$ on Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ and modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and obtain some sufficient and necessary conditions on the parameters. Furthermore, we consider the boundedness of $B_{\Omega,\alpha}$ on generalized central Morrey space $\dot{B}^{p,\varphi}(\mathbb{R}^n)$. These extend some known results.

1. Introduction

In recent years, multilinear analysis becomes a very active research topic in studying harmonic analysis. As one of the most important operators, the multilinear fractional integral has also attracted much attention. In this note, we will consider the multilinear fractional integral with rough kernel. For fixed distinct and nonzero real numbers $\theta_1, \dots, \theta_m$, and $0 < \alpha < n$, the m-linear fractional with rough kernel is defined by

$$I_{\Omega,\alpha}\left(\vec{f}\right) = \int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(x - \theta_i y) \frac{\Omega(y)}{|y|^{n-\alpha}} dy, \tag{1.1}$$

where $\Omega \in L^s(S^{n-1})$ $(s \geq 1)$ is homogeneous of degree zero on $\mathbb{R}^n$, and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$.

When $\Omega \equiv 1$, The L^p boundedness of operator $I_{1,\alpha}$ has been well studied in [1, 2]. Recently, Hendar and Idha discussed the boundedness property of $I_{1,\alpha}$ on generalized Morrey space in [3].

Here, without loss of generality, we will study the case $m = 2$. More specifically, we will study the rough bilinear fractional integral:

$$B_{\Omega,\alpha}(f,g)(x) = \int_{\mathbb{R}^n} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^{n-\alpha}}dy, \quad 0 < \alpha < n. \tag{1.2}$$

The study of the operators $B_{\Omega,\alpha}$ and its related operators with rough kernel Ω recently attracted many attentions. In 2002, Ding and Chin first discussed its $L^p(\mathbb{R}^n)$ boundedness. The following theorem is their main result:

Theorem A (see [4]). *Let $0 < \alpha < n$, $1 \le s' < n/\alpha$ and $1 \le p_1, p_2 \le \infty$. If*

$$\frac{1}{p_1} + \frac{1}{p_2} \ge \frac{\alpha}{n}, \qquad \frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}, \tag{1.3}$$

there exists a positive constant C such that for any $f \in L^{p_1}(\mathbb{R}^n)$, $g \in L^{p_2}(\mathbb{R}^n)$,

(1) when $s' < \min\{p_1, p_2\}$,

$$\|B_{\Omega,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}; \tag{1.4}$$

(2) when $s' = \min\{p_1, p_2\}$,

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\infty}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{1.5}$$

Later, when $q > n/(n-\alpha)$, Chen and Fan in [5] relaxed the conditions of Ω in Theorem A using Hölder inequality. Their main result is as follows.

Theorem B. *Let $q > n/(n-\alpha)$, $0 < \alpha < n$, $p_1, p_2 > 1$ and*

$$\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2} - \frac{\alpha}{n}. \tag{1.6}$$

If $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f,g)\|_{L^q(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{1.7}$$

We note that when $q \le n/(n-\alpha)$, Hölder inequality is not sufficient in Theorem B. So how to relax the index of q is left. In fact, in [6, 7] the authors have obtained the necessary and sufficient conditions on the parameters for the m-linear fractional integral operator $I_{\Omega,\alpha}$ with rough kernel from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \cdots L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ by using the pointwise rearrangement estimate of the m-linear convolution.

Theorem C. *Let $0 < \alpha < n$, Ω and be homogeneous of degree zero on $\mathbb{R}^n$, $\Omega \in L^{n/(n-\alpha)}(S^{n-1})$, let p be the harmonic mean of $p_1, p_2, \ldots, p_m > 1$, and $n/(n-\alpha) \le p < n/\alpha$. Then the condition $1/q = 1/p - \alpha/n$ is necessary and sufficient for the boundedness of $I_{\Omega,\alpha}$ from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n) \times \cdots L^{p_m}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

This paper is organized as follows: in the second part of this work we prove some boundedness properties of $B_{\Omega,\alpha}$ on Morrey space and extend Theorem C to Morrey spaces; in the third part, we obtain the sufficient and necessary conditions on the parameters for the boundedness of $B_{\Omega,\alpha}$ on modified Morrey space; in the last part, we find the sufficient condition on the pair (φ, ν) which ensures the boundedness of the operators $B_{\Omega,\alpha}$ on the generalized center Morrey space. Since Morrey space, modified Morrey space and central Morrey space all can be seen as generalized L^p space.

2. The Boundedness of $B_{\Omega,\alpha}$ on Morrey Space

The classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ were originally introduced by Morrey in [8] to study the local behavior of solutions to second-order elliptic partial differential equations. The reader can find more details in [9].

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denotes the open ball centered at x of radius t, and $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$. When $1 \le p < \infty$ and $\lambda \ge 0$, Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ is defined by

$$L^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty \right\}, \tag{2.1}$$

where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} |f(x)|^p dx \right)^{1/p}. \tag{2.2}$$

If $1 \le p < \infty$, then $L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $L^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. When $\lambda > n$, $L^{p,\lambda}(\mathbb{R}^n) = \{0\}$. So we only consider the case $0 < \lambda < n$.

Since Morrey space can be seen as the generalized L^p space, we will be interested in the boundedness of $B_{\Omega,\alpha}$ on Morry space $L^{p,\lambda}(\mathbb{R}^n)$. In order to prove our results, we need the following bilinear maximal function:

$$M(f,g)(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|<r} |f(x-y)||g(x+y)| dy. \tag{2.3}$$

Lemma 2.1. *Let $p > 1$, $0 < \lambda < n$ and $1/p = 1/p_1 + 1/p_2$. If*

$$\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1,\ \lambda_2 < n, \tag{2.4}$$

then there exists a positive constant C such that

$$\|M(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda}(\mathbb{R}^n)}. \tag{2.5}$$

Proof. In [10], Fefferman and Stein have proved that for every p, $1 < p < \infty$, there is a constant $C_p > 0$ such that for any measurable functions f on $\mathbb{R}^n$ and $\varphi \geq 0$, the following inequality holds,

$$\int_{\mathbb{R}^n} (Mf(x))^p \varphi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p M\varphi(x)dx, \tag{2.6}$$

where M is the Hardy-LittleWood maximal function. Set $\varphi(x)$ be the characteristic function $\chi(x)$, when $1 \leq \delta < p$, by the above inequality, we can get

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \chi(x)dx \leq C_p \int_{\mathbb{R}^n} |f(x)|^p M\chi(x)dx, \tag{2.7}$$

where $M_\delta f(x) = (Mf^\delta)^{1/\delta}(x)$.

Taking $f \in L^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$, $\chi(x)$ is the characteristic function of a ball $B(x_0, r)$ in $\mathbb{R}^n$, by simple calculating,

$$\int_{B(x_0,r)} (M_\delta f(x))^p dx \leq C\|f\|^p_{L^{p,\lambda}(\mathbb{R}^n)} r^\lambda, \tag{2.8}$$

that is, $\|M_\delta f\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}$. For More details, see [11] about the boundedness of Hardy-Littlewood maximal function on Morrey space.

So when $p > 1$, $1/p = 1/p_1 + 1/p_2$, $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, we have

$$\begin{aligned} \|M(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} &\leq \left\|M_{p_1/p}(f) M_{p_2/p}(g)\right\|_{L^{p,\lambda}(\mathbb{R}^n)} \\ &\leq \left\|M_{p_1/p}(f)\right\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \left\|M_{p_2/p}(g)\right\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)} \\ &\leq C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \end{aligned} \tag{2.9}$$

□

Theorem 2.2. *Suppose $0 < \alpha < n$, and let $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$ and $s' < p$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}, \quad \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{2.10}$$

then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.11}$$

Proof. Let $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, $\sigma = (n - \alpha s' + \lambda)/2$, for $s' < p$ and $0 < \lambda < n - \alpha p$, we can get $\lambda < \sigma < n - \alpha s'$, $(n-\lambda)/p > \alpha > (n-\sigma)/s'$. First, $|B_{\Omega,\alpha}(f,g)(x)|$ is decomposed by

$$\begin{aligned} \left|B_{\Omega,\alpha}(f,g)(x)\right| &= \left(\int_{|y|\le\varepsilon} + \int_{|y|\ge\varepsilon}\right) f(x-y)g(x+y)\frac{\Omega(y)}{|y|^{n-\alpha}} \\ &=: I_1(x) + I_2(x). \end{aligned} \tag{2.12}$$

Estimate of $I_1(x)$ is

$$\begin{aligned} I_1(x) &= \sum_{m=1}^{\infty}\int_{|y|\sim\varepsilon 2^{-m}} |f(x-y)g(x+y)|\frac{|\Omega(y)|}{|y|^{n-\alpha}}dy \\ &\le \sum_{m=1}^{\infty}(\varepsilon 2^{-m})^{\alpha-n}\int_{|y|\sim\varepsilon 2^{-m}} |f(x-y)g(x+y)||\Omega(y)|dy \\ &\le \sum_{m=1}^{\infty}(\varepsilon 2^{-m})^{\alpha} M\left(f^{s'},g^{s'}\right)^{1/s'}(x) \\ &\le C\varepsilon^{\alpha} M\left(f^{s'},g^{s'}\right)^{1/s'}(x) \\ &=: C\varepsilon^{\alpha} M_{s'}(f,g)(x), \end{aligned} \tag{2.13}$$

and estimate of $I_2(x)$ is

$$\begin{aligned} I_2(x) &\le \left(\int_{|y|\ge\varepsilon} \frac{f^{s'}(x-y)g^{s'}(x+y)}{|y|^{\sigma}}dy\right)^{1/s'}\left(\int_{|y|\ge\varepsilon} |y|^{(\sigma/s'+\alpha-n)s}|\Omega(y)|^{s}dy\right)^{1/s} \\ &\le C\varepsilon^{(\sigma/s'+\alpha-n)+n/s}\left(\int_{|y|\ge\varepsilon} \frac{f^{s'}(x-y)g^{s'}(x+y)}{|y|^{\sigma}}dy\right)^{1/s'} \\ &=: C\varepsilon^{(\sigma/s'+\alpha-n)+n/s}F_{\sigma}(f,g)(x). \end{aligned} \tag{2.14}$$

For $F_\sigma(f,g)(x)$, we have the following estimates:

$$\begin{aligned} F_\sigma(f,g)(x) &\le \left(\sum_{k=0}^{\infty}\int_{|y|\sim\varepsilon 2^{k}} \frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^{\sigma}}dy\right)^{1/s'} \\ &\le \sum_{k=0}^{\infty}\left(\int_{|y|\sim\varepsilon 2^{k}} \frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^{\sigma}}dy\right)^{1/s'} \\ &\le \sum_{k=0}^{\infty}\left(\varepsilon 2^{k}\right)^{-\sigma/s'}\left(\int_{|y|\sim\varepsilon 2^{k}} \left|f^{s'}(x-y)g^{s'}(x+y)\right|dy\right)^{1/s'} \end{aligned}$$

$$\le \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p}\left(\int_{|y|\sim\varepsilon 2^k}\left|f^{p_1}(x-y)\right|dy\right)^{1/p_1}\left(\int_{|y|\sim\varepsilon 2^k}\left|g^{p_2}(x-y)\right|dy\right)^{1/p_2}$$
$$\le \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p+\lambda_1/p_1+\lambda_2/p_2}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}$$
$$\le C(\varepsilon)^{(n-\sigma)/s'-(n-\lambda)/p}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.15}$$

Combining the above estimates, we have

$$\left|B_{\Omega,\alpha}(f,g)(x)\right| \le C\varepsilon^{\alpha}M_{s'}(f,g)(x)+C\varepsilon^{(\lambda-n)/p+\alpha}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.16}$$

Let $\varepsilon^{\alpha}M_{s'}(f,g)(x)=\varepsilon^{((\lambda-n)/p)+\alpha}\|f\|_{p_1,\lambda_1}\|g\|_{p_2,\lambda_2}$, then

$$\left|B_{\Omega,\alpha}(f,g)(x)\right| \le C\left(M_{s'}(f,g)(x)\right)^{p/q}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}^{1-p/q}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}^{1-p/q}. \tag{$\star$}$$

By computation, we get

$$\left(\frac{1}{r^{\lambda}}\int_{B(x,r)}\left(M_{s'}(f,g)(x)\right)^{(p/q)\times q}dx\right)^{1/q}$$
$$=\left(\frac{1}{r^{\lambda}}\int_{B(x,r)}\left(M\left(f^{s'},g^{s'}\right)(x)\right)^{p/s'}dx\right)^{1/p\times p/q}$$
$$\le\left(\frac{1}{r^{\lambda_1}}\int_{B(x,r)}f(x)^{p_1}dx\right)^{1/p_1\times p/q}\left(\frac{1}{r^{\lambda_2}}\int_{B(x,r)}g(x)^{p_2}dx\right)^{1/p_2\times p/q}$$
$$\le\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}^{p/q}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}^{p/q}. \tag{2.17}$$

Taking the supremum of r, we have

$$\left\|\left(M_{s'}(f,g)\right)^{p/q}\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \le \|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}^{p/q}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}^{p/q}. \tag{2.18}$$

Hence

$$\left\|B_{\Omega,\alpha}(f,g)\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.19}$$

□

Theorem 2.3. *Suppose $0<\alpha<n$, and let $\Omega\in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1<p<n/\alpha$, $0<\lambda<n-\alpha p$, $s'<p$ and $\lambda/p=\lambda_1/p_1+\lambda_2/p_2$, $0<\lambda_1,\lambda_2<n$, then the condition $1/q=1/p-\alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of $B_{\Omega,\alpha}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n)\times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

Proof. Sufficiency part of Theorem 2.3 is proved in Theorem 2.2.

Necessity. Let $1 < p < n/\alpha$ and $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$. Denote $f_t(x) =: f(tx)$ and $g_t(x) =: g(tx)$. Then we have

$$\|f_t\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} = t^{-n/p_1+\lambda_1/p_1}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}, \qquad \|g_t\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)} = t^{-n/p_2+\lambda_2/p_2}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)},$$
$$B_{\Omega,\alpha}(f_t, g_t)(x) = t^{-\alpha}B_{\Omega,\alpha}(f, g)(tx), \qquad \|B_{\Omega,\alpha}(f_t, g_t)\|_{L^{q,\lambda}(\mathbb{R}^n)} = t^{-\alpha-n/q+\lambda/q}\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\lambda}(\mathbb{R}^n)}. \tag{2.20}$$

Since $B_{\Omega,\alpha}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times L^{p_2,\lambda_2}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, it is true that

$$\begin{aligned}\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\lambda}(\mathbb{R}^n)} &= t^{\alpha+n/q-\lambda/q}\|B_{\Omega,\alpha}(f_t, g_t)\|_{L^{q,\lambda}(\mathbb{R}^n)}\\ &\le Ct^{\alpha+n/q-\lambda/q}\|f_t\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}\\ &\le Ct^{\alpha+n/q-\lambda/q-n/p+\lambda/p}\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)},\end{aligned} \tag{2.21}$$

where C depends only on p, q, λ, and n.

If $1/q < 1/p - \alpha/(n-\lambda)$, then in the case $t \to 0$, for all $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\lambda}(\mathbb{R}^n)} = 0$.

If $1/q > 1/p - \alpha/(n-\lambda)$, then in the case $t \to \infty$, for all $f \in L^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in L^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\lambda}(\mathbb{R}^n)} = 0$.

Therefore, we get $1/q = 1/p - \alpha/(n-\lambda)$. □

Corollary 2.4. *Let $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ be homogeneous of degree zero on $\mathbb{R}^n$, p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, and $s' < p$. If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}; \quad \frac{\mu}{q} = \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{2.22}$$

then there exists a positive constant C such that

$$\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\mu}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.23}$$

Proof. By Hölder inequality, it is easy to know when $t = (n-\lambda)q/(n-\mu)$, we have $L^{t,\lambda}(\mathbb{R}^n) \subseteq L^{q,\mu}(\mathbb{R}^n)$, through the given condition, $1/t = 1/p - \alpha/(n-\lambda)$. Applying Theorem 2.2, we get

$$\|B_{\Omega,\alpha}(f, g)\|_{L^{q,\mu}(\mathbb{R}^n)} \le \|B_{\Omega,\alpha}(f, g)\|_{L^{t,\lambda}(\mathbb{R}^n)} \le C\|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.24}$$

From the inequality ($\star$) and Theorem 2.2, we obtain an Olsen inequality involving a multiplication operator. □

Corollary 2.5. *Suppose* $0 < \alpha < n$, *and let* $\Omega \in L^s(S^{n-1})$ *be homogeneous of degree zero on* $\mathbb{R}^n$, *let* p *be the harmonic mean of* p_1 *and* p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $s' < p$, *and* $W \in L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)$. *If*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}; \quad \frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n. \tag{2.25}$$

One has

$$\|W \cdot B_{\Omega,\alpha}(f,g)\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|W\|_{L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)} \|f\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \|g\|_{L^{p_2,\lambda_2}(\mathbb{R}^n)}. \tag{2.26}$$

3. The Boundedness of $B_{\Omega,\alpha}$ on Modified Morrey Space

After studying Morrey spaces in detail, people are led to considering the local and global counterpart. There are many famous work by V. I. Burenkov, H. V. Guliyev and V. S. Guliyev, and so forth and (see [12–20]). Recently, Guliyev et al. have considered the following modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ in [21].

Definition 3.1. Let $1 \le p < \infty$, $0 \le \lambda \le n$ and $[t]_1 = \min\{1,t\}$. $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, with the finite norms

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(\frac{1}{[t]_1^{\lambda}} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}. \tag{3.1}$$

Note that

$$\begin{gathered} \widetilde{L}^{p,0}(\mathbb{R}^n) = L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n), \\ \widetilde{L}^{p,\lambda}(\mathbb{R}^n) \subset_{\succ} L^{p,\lambda}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n), \qquad \max\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\} \le \|f\|_{\widetilde{L}^{p,\lambda}}, \end{gathered} \tag{3.2}$$

and if $\lambda < 0$ or $\lambda > n$, then $\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) = \{0\}$.

In [21], the authors discussed the boundedness of maximal function in modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and obtained the following generalized Hardy-Littlewood-Sobolev inequalities in modified Morrey spaces.

Theorem D. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n - \alpha$. *If* $1 < p < (n-\lambda)/\alpha$, *then condition* $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$ *is necessary and sufficient for the boundedness of the operator* I_α *from* $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

We also can extend Theorem D to the multilinear case.

Lemma 3.2. *Let $p > 1$, $0 < \lambda < n$ and $1/p = 1/p_1 + 1/p_2$. If*

$$\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}, \quad 0 < \lambda_1, \lambda_2 < n, \tag{3.3}$$

then there exists a positive constant C such that

$$\|M(f,g)\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{\widetilde{L}^{p_1,\lambda}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda}(\mathbb{R}^n)}. \tag{3.4}$$

Proof. When $1 \le \delta < p$, the following inequality:

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \chi(x)dx \le C_p \int_{\mathbb{R}^n} |f(x)|^p M\chi(x)dx \tag{3.5}$$

holds, where M is the Hardy-littlewood maximal function and $M_\delta f(x) = (Mf^\delta)^{1/\delta}(x)$.

Taking $f \in \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, $0 < \lambda < n$. Using the method in [21], we get $\|M_\delta f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} \le C\|f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)}$.

Hence, with the same arguments in Lemma 2.1, we complete the proof of Lemma 3.2. □

Theorem 3.3. *Suppose $0 < \alpha < n$, $\Omega \in L^s(S^{n-1})$ and let be homogeneous of degree zero on $\mathbb{R}^n$, let p be the harmonic mean of p_1 and p_2, $1 < p < n/\alpha$, $0 < \lambda < n - \alpha p$, $s' < p$ and $\lambda/p = \lambda_1/p_1 + \lambda_2/p_2$, $0 < \lambda_1, \lambda_2 < n$. Then the condition $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of $B_{\Omega,\alpha}$ from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.*

Proof. (1) *Sufficiency*. Let $f \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g \in \widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, $\sigma = (n - \alpha s' + \lambda)/2$, since $s' < p$ and $0 < \lambda < n - \alpha p$, we can get $\lambda < \sigma < n - \alpha s'$, $(n-\lambda)/p > \alpha > (n-\sigma)/s'$ and $\lambda < n - ((n-\sigma)/s')p < n - \alpha p$.

Do the same decomposition of $B_{\Omega,\alpha}(f,g)(x)$ in the proof of Theorem 2.2, then we only need to estimate $F_\sigma(f,g)(x)$. We can easily obtain

$$\begin{aligned} F_\sigma(f,g)(x) &\le \left(\sum_{k=0}^{\infty}\int_{|y|\sim\varepsilon 2^k}\frac{\left|f^{s'}(x-y)g^{s'}(x+y)\right|}{|y|^\sigma}dy\right)^{1/s'} \\ &\le \sum_{k=0}^{\infty}\left(\varepsilon 2^k\right)^{(n-\sigma)/s'-n/p}\left(\int_{|y|\sim\varepsilon 2^k}\left|f^{p_1}(x-y)\right|dy\right)^{1/p_1} \\ &\quad\times\left(\int_{|y|\sim\varepsilon 2^k}\left|g^{p_2}(x-y)\right|dy\right)^{1/p_2} \\ &\le (\varepsilon)^{(n-\sigma)/s'-n/p}\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}. \end{aligned} \tag{3.6}$$

For $0 < \varepsilon < 1/2$, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p} &= \sum_{k=0}^{[\log_2(1/2\varepsilon)]}\varepsilon^{\lambda/p}\left(2^k\right)^{(n-\sigma)/s'-n/p+\lambda/p} \\ &\quad + \sum_{k=[\log_2(1/2\varepsilon)]+1}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p} \\ &\le C\left(\varepsilon^{\lambda/p}+\varepsilon^{(n-\sigma)/s'-n/p}\right) \le C\varepsilon^{\lambda/p}.\end{aligned} \tag{3.7}$$

While $\varepsilon \ge 1/2$, we obtain

$$\sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p}\left[\varepsilon 2^k\right]_1^{\lambda/p} = \sum_{k=0}^{\infty}\left(2^k\right)^{(n-\sigma)/s'-n/p} \le C. \tag{3.8}$$

Thus, we obtain

$$\begin{aligned}F_\sigma(f,g)(x) &\le C(\varepsilon)^{((n-\sigma)/s')-(n/p)}[2\varepsilon]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}, \\ \left|B_{\Omega,\alpha}(f,g)(x)\right| &\le C\left(\varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-(n/p)}[\varepsilon]_1^{\lambda/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right) \\ &\le C\min\Big\{\varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-n/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}, \\ &\qquad \varepsilon^{\alpha}M_{s'}(f,g)(x)+\varepsilon^{\alpha-(n-\lambda)/p}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\Big\}.\end{aligned} \tag{3.9}$$

Set

$$\begin{aligned}\varepsilon &= \left(M_{s'}(f,g)(x)^{-1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right)^{p/(n-\lambda)}, \\ \varepsilon &= \left(M_{s'}(f,g)(x)^{-1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\right)^{p/n},\end{aligned} \tag{3.10}$$

we have

$$\begin{aligned}&\left|B_{\Omega,\alpha}(f,g)(x)\right| \\ &\quad \le C\min\left\{\left(\frac{M_{s'}(f,g)(x)}{\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}}\right)^{1-p\alpha/(n-\lambda)},\left(\frac{M_{s'}(f,g)(x)}{\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}}\right)^{1-p\alpha/n}\right\} \\ &\qquad \times \|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)} \\ &\quad \le C\left(M_{s'}(f,g)(x)\right)^{p/q}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}^{1-p/q}\|g\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}^{1-p/q}.\end{aligned} \tag{3.11}$$

Hence, by the boundedness of $M(f,g)(x)$ in Lemma 3.2, we prove that $B_{\Omega,\alpha}$ is bounded from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.

(2) *Necessity*. Let $1<p<n/\alpha$ and $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$. Denote $f_t(x)=:f(tx)$, $g_t(x)=:g(tx)$, and $[t]_{1,+}=\max\{1,t\}$. Then from [21], we have

$$\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}=t^{-n/p_1}[t]_{1,+}^{\lambda_1/p_1}\|f\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)},\qquad \|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}=t^{-n/p_2}[t]_{1,+}^{\lambda_2/p_2}\|f\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)},$$

$$B_{\Omega,\alpha}(f_t,g_t)(x)=t^{-\alpha}B_{\Omega,\alpha}(f,g)(tx),\qquad \|B_{\Omega,\alpha}(f_t,g_t)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}=t^{-\alpha-n/q}[t]_{1,+}^{\lambda/q}\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}. \tag{3.12}$$

By the boundedness of $B_{\Omega,\alpha}$, we have

$$\begin{aligned}\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}&=t^{\alpha+n/q}[t]_{1,+}^{-\lambda/q}\|B_{\Omega,\alpha}(f_t,g_t)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}\\&\le Ct^{\alpha+n/q}[t]_{1,+}^{-\lambda/q}\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}\\&\le Ct^{\alpha+n/q-(n/p)}[t]_{1,+}^{\lambda/p-\lambda/q}\|f_t\|_{\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)}\|g_t\|_{\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)}.\end{aligned} \tag{3.13}$$

If $1/q>1/p-\alpha/(n-\lambda)$, then in the case $t\to 0$, for all $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{\widetilde{L}^{q,\lambda}(\mathbb{R}^n)}=0$.

If $1/q<1/p-\alpha/(n-\lambda)$, then in the case $t\to\infty$, for all $f\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)$, $g\in\widetilde{L}^{p_2,\lambda_2}(\mathbb{R}^n)$, we have $\|B_{\Omega,\alpha}(f,g)\|_{L^{q,\lambda}(\mathbb{R}^n)}=0$.

Therefor $\alpha/n\le 1/p-1/q\le\alpha/(n-\lambda)$. □

4. The Boundedness of $B_{\Omega,\alpha}$ on Generalized Center Morrey Space

Definition 4.1. Let $\varphi(r)$ be a positive measurable function on $\mathbb{R}_+$ and $1\le p<\infty$. We denote by $\dot{B}^{p,\varphi}(\mathbb{R}^n)$ the generalized central Morrey space, the space of all functions $f\in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{\dot{B}^{p,\varphi}(\mathbb{R}^n)}=\sup_{r>0}\varphi(r)^{-1}|B(0,r)|^{-1/p}\|f\|_{L^p(B(0,r))}, \tag{4.1}$$

where $B(0,r)$ denotes a ball centered at 0 with side length r and $|B(0,r)|$ is the Lebesgue measure of the ball $B(0,r)$.

According to this definition, we recover the spaces $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ under the choice $\varphi(r)=r^{n\lambda}$. About the $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ space, the readers can refer to [22], In fact, we can easily check that $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ is a Banach space, $\dot{B}^{p,\lambda}(\mathbb{R}^n)$ reduce to $\{0\}$ when $\lambda<-1/p$, $\dot{B}^{p,(-1/p)}(\mathbb{R}^n)=L^p(\mathbb{R}^n)$ and $\dot{B}^{p,0}(\mathbb{R}^n)=\dot{B}^p(\mathbb{R}^n)$.

There are many papers that discussed the conditions on φ to obtain the boundedness of fractional integral on the generalized Morrey spaces, see [23, 24]. In [25] the following condition was imposed on the pair (φ_1, φ_2):

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t<s<\infty} \varphi_1(s) s^{n/p}}{t^{n/q+1}} \leq C\varphi_2(r) \tag{4.2}$$

for the fractional integral I_α, where $1/q = 1/p - \alpha/n$ and C (> 0) does not depend on r.

Theorem E (see [26]). *The inequality*

$$\operatorname*{ess\,sup}_{t>0} \omega(t) Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t) g(t) \tag{4.3}$$

holds for all nonnegative and nonincreasing g on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{\omega(t)}{t} \int_0^t \frac{dr}{\operatorname{ess\,sup}_{t>0} v(s)} < \infty, \tag{4.4}$$

and $c \approx A$, where the H is the Hardy operator

$$Hg(t) := \frac{1}{t} \int_0^t g(r) dr, \quad 0 < t < \infty. \tag{4.5}$$

In this section we are going to discuss the boundedness of $B_{\Omega,\alpha}$ on generalized central Morrey space.

Lemma 4.2. *Suppose $0 < \alpha < n$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$, and $s \geq p'$, then for $1 < p < n/\alpha$, the inequality*

$$\begin{aligned} &\|B_{\Omega,\alpha}(f,g)\|_{L^q(B(0,r))} \\ &\quad \leq C r^{n/q} \left(\int_{2r}^\infty \|f\|_{L^{p_1}(B(0,t))}^{p_1/p} \frac{dt}{t^{n/q+1}} \right)^{p/p_1} \left(\int_{2r}^\infty \|g\|_{L^{p_2}(B(0,t))}^{p_2/p} \frac{dt}{t^{n/q+1}} \right)^{p/p_2} \end{aligned} \tag{4.6}$$

holds for any ball $B(0,r)$ and for all $f \in L_{p_1}^{\mathrm{loc}}(\mathbb{R}^n)$ and $g \in L_{p_2}^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. Let $1 < p < n/\alpha$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/p - \alpha/n$ and $s \geq p'$. For any $r > 0$, set $B = B(0,r)$, we write

$$\begin{aligned} f(x) &= f(x)\chi_{3B}(x) + f(x)\chi_{(3B)^c}(x) := f_1(x) + f_2(x), \\ g(x) &= g(x)\chi_{3B}(x) + g(x)\chi_{(3B)^c}(x) := g_1(x) + g_2(x). \end{aligned} \tag{4.7}$$

Hence

$$\begin{aligned} \|B_{\Omega,\alpha}(f,g)\|_{L^q(B)} \leq{}& \|B_{\Omega,\alpha}(f_1,g_1)\|_{L^q(B)} + \|B_{\Omega,\alpha}(f_1,g_2)\|_{L^q(B)} \\ &+ \|B_{\Omega,\alpha}(f_2,g_1)\|_{L^q(B)} + \|B_{\Omega,\alpha}(f_2,g_1)\|_{L^q(B)}. \end{aligned} \tag{4.8}$$

Since $B_{\Omega,\alpha}$ is bounded from $L^{p_1}\times L^{p_2}$ to L^q, we have

$$\begin{aligned}\|B_{\Omega,\alpha}(f_1,g_1)\|_{L^q(B)} &\le \|B_{\Omega,\alpha}(f_1,g_1)\|_{L^q(\mathbb{R}^n)} \le C\|f_1\|_{L^{p_1}(\mathbb{R}^n)}\|g_1\|_{L^{p_2}(\mathbb{R}^n)} \\ &\le C\|f\|_{L^{p_1}(3B)}\|g\|_{L^{p_2}(3B)},\end{aligned} \tag{4.9}$$

where the constant $C>0$ is independent of f and g.

To estimate $B_{\Omega,\alpha}(f_1,g_2)$, it follows that

$$\begin{aligned}
|B_{\Omega,\alpha}(f_1,g_2)| &= \left|\int_{\mathbb{R}^n}\frac{f_1(x-y)g_2(x+y)\Omega(y)}{|y|^{n-\alpha}}dy\right| \\
&\le \left(\int_{\mathbb{R}^n}\left|f_1^{p_1/p}(x-y)\Omega(y)\right|dy\right)^{p/p_1}\left(\int_{\mathbb{R}^n}\frac{\left|g_2^{p_2/p}(x-y)\Omega(y)\right|}{|y|^{(n-\alpha)p_2/p}}dy\right)^{p/p_2} \\
&\le \left(\int_{4B}|f^{p_1}(y)|dy\right)^{1/p_1}\left(\int_{4B}\left|\Omega^{p'}(x-y)\right|dy\right)^{p/p_1p'} \\
&\quad\times\left(\int_{(2B)^c}\frac{|g^{p_2/p}(y)\Omega(x-y)|}{|y|^{(n-\alpha)p_2/p}}dy\right)^{p/p_2} \\
&\le Cr^{pn/p_1p'}\left(r^{n/q}\|f\|_{L^{p_1}(4B)}^{p_1/p}\int_{4r}^{\infty}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\
&\quad\times\left(\int_{(2B)^c}\left|g^{p_2/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{(n-\alpha)p_2/p+1}}dy\right)^{p/p_2} \\
&\le Cr^{pn/p_1p'+np/qp_1}\left(\int_{4r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\
&\quad\times\left(\int_{2r}^{\infty}\int_{2r\le|y|<t}\left|g^{p_2/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{(n-\alpha)p_2/p+1}}\right)^{p/p_2} \\
&\le Cr^{(p/p_1)(n-\alpha)}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\
&\quad\times\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{(n-\alpha)p_2/p+1-(n/p')}}\right)^{p/p_2} \\
&\le C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1} \\
&\quad\times\left(\int_{2r}^{\infty}|t|^{(p/p_1)(n-\alpha)}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{(n-\alpha)p_2/p+1-(n/p')}}\right)^{p/p_2}
\end{aligned}$$

$$\leq C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}$$
$$\times\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}. \tag{4.10}$$

So

$$\|B_{\Omega,\alpha}(f_1,g_2)\|_{L^q(B(0,r))}\leq Cr^{n/q}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_2}. \tag{4.11}$$

By the same estimating, we also can obtain

$$\|B_{\Omega,\alpha}(f_2,g_1)\|_{L^q(B(0,r))}\leq Cr^{n/q}\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{t^{(n/q)+1}}\right)^{p/p_2}. \tag{4.12}$$

To estimate $B_{\Omega,\alpha}(f_2,g_2)$, we get

$$\begin{aligned}
\left|B_{\Omega,\alpha}(f_2,g_2)\right| &= \left|\int_{\mathbb{R}^n}\frac{f_1(x-y)g_2(x+y)\Omega(y)}{|y|^{n-\alpha}}dy\right| \\
&\leq \left(\int_{\mathbb{R}^n}\frac{\left|f_2^{p_1/p}(x-y)\Omega(y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\quad\times\left(\int_{\mathbb{R}^n}\frac{\left|g_2^{p_2/p}(x-y)\Omega(y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\leq \left(\int_{(2B)^c}\frac{\left|f^{p_1/p}(y)\Omega(x-y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_1} \\
&\quad\times\left(\int_{(2B)^c}\frac{\left|g^{p_2/p}(y)\Omega(x-y)\right|}{|y|^{n-\alpha}}dy\right)^{p/p_2} \\
&\leq C\left(\int_{(2B)^c}\left|f^{p_1/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{n-\alpha+1}}dy\right)^{p/p_1} \\
&\quad\times\left(\int_{(2B)^c}\left|g^{p_2/p}(y)\Omega(x-y)\right|\int_{|y|}^{\infty}\frac{dt}{|t|^{n-\alpha+1}}dy\right)^{p/p_2}
\end{aligned}$$

$$\leq C\left(\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|f^{p_1/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{n-\alpha+1}}\right)^{p/p_1}$$
$$\times\left(\int_{2r}^{\infty}\int_{2r\leq|y|<t}\left|g^{p_2/p}(y)\Omega(x-y)\right|dy\frac{dt}{|t|^{n-\alpha+1}}\right)^{p/p_2}$$
$$\leq C\left(\int_{2r}^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}\left(\int_{2r}^{\infty}\|g_2\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}. \tag{4.13}$$

Combining the above estimates, we end the proof of Lemma 4.2. □

Theorem 4.3. *Suppose $0<\alpha<n$, $1/p=1/p_1+1/p_2$, $1<p<n/\alpha$, $1/q=1/p-\alpha/n$, and $s\geq p'$. If (φ_1,ν_1) satisfies the condition*

$$\int_r^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_1^{p_1/p}(s)s^{n/p}}{t^{n/q+1}}\leq C\nu_1^{p_1/p}(r), \tag{4.14}$$

and (φ_2,ν_2) satisfies the condition

$$\int_r^{\infty}\frac{\operatorname{ess\,inf}_{t<s<\infty}\varphi_2^{p_1/p}(s)s^{n/p}}{t^{n/q+1}}\leq C\nu_2^{p_1/p}(r), \tag{4.15}$$

where the constant $C>0$ does not depend on r. Let $\varphi=\nu_1\nu_2$, then $B_{\Omega,\alpha}$ is bounded from $\dot{B}^{p_1,\varphi_1}\times\dot{B}^{p_2,\varphi_2}$ to $\dot{B}^{q,\varphi}$.

Proof. By Theorem E and Lemma 4.2, we have

$$\|B_{\Omega,\alpha}(f,g)\|_{\dot{B}^{q,\varphi}(\mathbb{R}^n)}\leq C\sup_{r>0}\varphi(r)^{-1}\left(\int_r^{\infty}\|f\|_{L^{p_1}(B(0,t))}^{p_1/p}\frac{dt}{t^{n/q+1}}\right)^{p/p_1}$$
$$\times\left(\int_r^{\infty}\|g_2\|_{L^{p_2}(B(0,t))}^{p_2/p}\frac{dt}{|t|^{n/q+1}}\right)^{p/p_2}$$
$$=C\sup_{r>0}\left(\nu_1(r)^{-p_1/p}\int_0^{r^{-n/q}}\|f\|_{L^{p_1}(B(0,t^{-q/n}))}^{p_1/p}dt\right)^{p/p_1}$$
$$\times\left(\nu_2(r)^{-p_2/p}\int_0^{r^{-n/q}}\|g_2\|_{L^{p_2}(B(0,t^{-q/n}))}^{p_2/p}dt\right)^{p/p_2}$$
$$=C\sup_{r>0}\left(\nu_1\left(r^{-q/n}\right)^{-p_1/p}\int_0^{r}\|f\|_{L^{p_1}(B(0,t^{-q/n}))}^{p_1/p}dt\right)^{p/p_1}$$
$$\times\left(\nu_2(r^{-q/n})^{-p_2/p}\int_0^{r}\|g_2\|_{L^{p_2}(B(0,t^{-q/n}))}^{p_2/p}dt\right)^{p/p_2}$$

$$\begin{aligned}
&\leq C \sup_{r>0}\left(\varphi_1\left(r^{-q/n}\right)^{-p_1/p} r^{q/p}\|f\|_{L^{p_1}(B(0,r^{-q/n}))}^{p_1/p}\right)^{p/p_1} \\
&\quad \times \sup_{r>0}\left(\varphi_2\left(r^{-q/n}\right)^{-p_2/p} r^{q/p}\|g\|_{L^{p_2}(B(0,r^{-q/n}))}^{p_2/p}\right)^{p/p_2} \\
&\leq C\|f\|_{\dot{B}^{p_1,\varphi_1}(\mathbb{R}^n)}\|g\|_{\dot{B}^{p_2,\varphi_2}(\mathbb{R}^n)}.
\end{aligned} \tag{4.16}$$

□

Corollary 4.4. *Suppose* $0 < \alpha < n$, $1/p = 1/p_1 + 1/p_2$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $s \geq p'$, $\lambda_1 < -\alpha p/np_1$, $\lambda_2 < -\alpha p/np_2$, *and* $\lambda < \lambda_1 + \lambda_2 + \alpha/n$, *then* $B_{\Omega,\alpha}$ *is bounded from* $\dot{B}^{p_1,\lambda_1} \times \dot{B}^{p_2,\lambda_2}$ *to* $\dot{B}^{q,\lambda}$.

Remark 4.5. Although we worked on the bilinear case. Applying same ideas in the argument, we may obtain similar extension of $I_{\Omega,\alpha}(\vec{f})$.

Acknowledgments

The authors thank the referees for useful comments which improve the presentation of this paper. This research is supported by NSFZJ (Grant no. Y604563) and NSFC (Grant no. 11271330).

References

[1] L. Grafakos, “On multilinear fractional integrals,” *Studia Mathematica*, vol. 102, no. 1, pp. 49–56, 1992.
[2] C. E. Kenig and E. M. Stein, “Multilinear estimates and fractional integration,” *Mathematical Research Letters*, vol. 6, no. 1, pp. 1–15, 1999.
[3] G. Hendar and S. Idha, “Multilinear maximal functions andfractional integrals on generalized Morrey spaces,” http://personal.fmipa.itb.ac.id/hgunawan/files/2007/11/multilinear-maximal-functions-n-fractional-integrals-ver-3.pdf.
[4] Y. Ding and C.-L. Lin, “Rough bilinear fractional integrals,” *Mathematische Nachrichten*, vol. 246-247, pp. 47–52, 2002.
[5] J. Chen and D. Fan, “Rough bilinear fractional integrals with variable kernels,” *Frontiers of Mathematics in China*, vol. 5, no. 3, pp. 369–378, 2010.
[6] V. S. Guliev and Sh. A. Nazirova, “Rearrangement estimates for generalized multilinear fractional integrals,” *Siberian Mathematical Journal*, vol. 48, no. 3, pp. 463–470, 2007.
[7] V. S. Guliyev and Sh. A. Nazirova, “O’Neil inequality for multilinear convolutions and some applications,” *Integral Equations and Operator Theory*, vol. 60, no. 4, pp. 485–497, 2008.
[8] C. B. Morrey, Jr., “On the solutions of quasi-linear elliptic partial differential equations,” *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.
[9] J. Peetre, “On the theory of $M_{p,\lambda}$ spaces,” *Journal of Functional Analysis*, vol. 4, pp. 71–87, 1969.
[10] C. Fefferman and E. M. Stein, “Some maximal inequalities,” *American Journal of Mathematics*, vol. 93, pp. 107–115, 1971.
[11] F. Chiarenza and M. Frasca, “Morrey spaces and Hardy-Littlewood maximal function,” *Rendiconti di Matematica e delle sue Applicazioni*, vol. 7, no. 3-4, pp. 273–279, 1987.
[12] V. I. Burenkov and H. V. Guliyev, “Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces,” *Studia Mathematica*, vol. 163, no. 2, pp. 157–176, 2004.
[13] V. I. Burenkov, V. S. Guliyev, A. Serbetci, and T. V. Tararykova, “Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces,” *Eurasian Mathematical Journal*, vol. 1, no. 1, pp. 32–53, 2010.

[14] V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, “Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces,” *Journal of Computational and Applied Mathematics*, vol. 208, no. 1, pp. 280–301, 2007.

[15] V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, “On boundedness of the fractional maxiaml operator from complementary Morrey type spaces,” in *The Interaction of Analysis and Geometry*, vol. 424 of *Contemporary Mathematics*, pp. 17–32, American Mathematical Society, Providence, RI, USA, 2007.

[16] V. I. Burenkov and V. S. Guliyev, “Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces,” *Potential Analysis*, vol. 30, no. 3, pp. 211–249, 2009.

[17] V. S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$ [Ph.D. thesis]*, Steklov Mathematical Institute, Moscow, Russia, 1994.

[18] V. S. Guliyev, *Fuction Spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups, Some Applications*, Casioglu, Baku, Azerbaijan, 1999.

[19] V. S. Guliev and R. Ch. Mustafaev, “Integral operators of potential type in spaces of homogeneous type,” *Doklady Ross. Skad. Nauk. Matematika*, vol. 354, no. 6, pp. 730–732, 1997 (Russian).

[20] V. S. Guliev and R. Ch. Mustafaev, “Fractional integrals in spaces of functions defined on spaces of homogeneous type,” *Analysis Mathematica*, vol. 24, no. 3, pp. 181–200, 1998.

[21] V. S. Guliyev, J. J. Hasanov, and Y. Zeren, “Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces,” *Journal of Mathematical Inequalities*, vol. 5, no. 4, pp. 491–506, 2011.

[22] J. Alvarez, J. Lakey, and M. Guzmán-Partida, “Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures,” *Collectanea Mathematica*, vol. 51, no. 1, pp. 1–47, 2000.

[23] E. Nakai, “Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces,” *Mathematische Nachrichten*, vol. 166, pp. 95–103, 1994.

[24] V. S. Guliyev, “Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces,” *Journal of Inequalities and Applications*, vol. 2009, Article ID 503948, 20 pages, 2009.

[25] V. S. Guliyev, S. S. Aliyev, T. Karaman, and P. S. Shukurov, “Boundedness of sublinear operators and commutators on generalized Morrey spaces,” *Integral Equations and Operator Theory*, vol. 71, no. 3, pp. 327–355, 2011.

[26] M. Carro, L. Pick, J. Soria, and V. D. Stepanov, “On embeddings between classical Lorentz spaces,” *Mathematical Inequalities & Applications*, vol. 4, no. 3, pp. 397–428, 2001.