Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 414201, 16 pages
doi:10.1155/2012/414201

Research Article

Toeplitz Operators with Quasihomogeneous Symbols on the Bergman Space of the Unit Ball

Bo Zhang[1] and Yufeng Lu[2]

[1] *College of Science, Dalian Ocean University, Dalian 116023, China*
[2] *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*

Correspondence should be addressed to Bo Zhang, zhangdlut@yahoo.cn

Received 24 May 2012; Revised 6 August 2012; Accepted 30 August 2012

Academic Editor: Nikolai M. Vasilevski

We consider when the product of two Toeplitz operators with some quasihomogeneous symbols on the Bergman space of the unit ball equals a Toeplitz operator with quasihomogeneous symbols. We also characterize finite-rank semicommutators or commutators of two Toeplitz operators with quasihomogeneous symbols.

1. Introduction

Let $dA(z)$ denote the Lebesgue volume measure on the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ normalized so that the measure of $\mathbb{B}_n$ equals 1. The Bergman space $L_a^2(\mathbb{B}_n)$ is the Hilbert space consisting of holomorphic functions on $\mathbb{B}_n$ that are also in $L^2(\mathbb{B}_n, dA)$. Hence, for each $z \in \mathbb{B}_n$, there exists an unique function $K_z \in L_a^2(\mathbb{B}_n)$ with the following property:

$$f(z) = \langle f, K_z \rangle \tag{1.1}$$

for all $f \in L_a^2(\mathbb{B}_n)$. As is well known that the reproducing kernel K_z is given by

$$K_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+1}}, \quad w \in \mathbb{B}_n. \tag{1.2}$$

Let P be the orthogonal projection from $L^2(\mathbb{B}_n, dA)$ onto $L^2_a(\mathbb{B}_n)$. Given a function $\varphi \in L^\infty(\mathbb{B}_n, dA)$, the Toeplitz operator T_φ: $L^2_a(\mathbb{B}_n) \to L^2_a(\mathbb{B}_n)$ is defined by the formula

$$T_\varphi(f)(z) = P(\varphi f)(z) = \int_{\mathbb{B}_n} \varphi(w) f(w) \overline{K_z(w)} dA(w), \tag{1.3}$$

for $f \in L^2_a(\mathbb{B}_n)$. Since the Bergman projection P has norm 1, it is clear that Toeplitz operators defined in this way are bounded linear operators on $L^2_a(\mathbb{B}_n)$ and $\|T_\varphi\| \le \|\varphi\|_\infty$.

We now consider a more general class of Toeplitz operators. For $F \in L^1(\mathbb{B}_n, dA)$, in analogy to (1.3) we define an operator T_F on $L^2_a(\mathbb{B}_n)$ by

$$T_F f(z) = \int_{\mathbb{B}_n} F(w) f(w) \overline{K_z(w)} dA(w), \qquad z \in \mathbb{B}_n. \tag{1.4}$$

Since the Bergman projection P can be extended to $L^1(\mathbb{B}_n, dA)$, the operator T_F is well defined on H^∞, the space of bounded analytic functions on $\mathbb{B}_n$. Hence, T_F is always densely defined on $\mathbb{B}_n$. Since P is not bounded on $L^1(\mathbb{B}_n, dA)$, it is well known that T_F can be unbounded in general. In [1], Zhou and Dong gave the following definitions, which are based on the definitions on the unit disk in [2].

Definition 1.1. Let $F \in L^1(\mathbb{B}_n, dA)$.

(a) We say that F is a T-function if (1.4) defines a bounded operator on $L^2_a(\mathbb{B}_n)$.

(b) If F is a T-function, we write T_F for the continuous extension of the operator (it is defined on the dense subset H^∞ of $L^2(\mathbb{B}_n)$) defined by (1.4). We say that T_F is a Toeplitz operator if and only if T_F is defined in this way.

(c) If there exists an $r \in (0,1)$, such that F is (essentially) bounded on the annulus $\{z : r < |z| < 1\}$, then we say F is "nearly bounded."

On the Bergman space of the unit ball, Grudsky et al. [3] gave necessary and sufficient conditions for boundedness of Toeplitz operators with radial symbols. These conditions give a characterization of the radial functions in $L^1(\mathbb{B}_n, dA)$ which correspond to bounded operators and furthermore show that the T-functions form a proper subset of $L^1(\mathbb{B}_n, dA)$ which contains all bounded and "nearly bounded" functions.

We denote the semicommutator and commutator of two Toeplitz operators T_f and T_g by

$$(T_f, T_g] = T_{fg} - T_f T_g, \qquad [T_f, T_g] = T_f T_g - T_g T_f. \tag{1.5}$$

In the setting of the classical Hardy space, Brown and Halmos [4] gave a complete characterization for the product of two Toeplitz operators to be a Toeplitz operator. On the Bergman space of the unit disk, Ahern and Čučković [5] and Ahern [6] obtained a similar characterization for Toeplitz operators with bounded harmonic symbols. For general symbols, the situation is much more complicated. Louhichi et al. [2] gave necessary and sufficient conditions for the product of two Toeplitz operators with quasihomogeneous symbols to be a Toeplitz operator and Louhichi and Zakariasy made a further discussion in

[7]. But it remains open to determine when the product of two Toeplitz operators is a Toeplitz operator on the Bergman space.

The problem of determining when the semicommutator $(T_f, T_g]$ or commutator $[T_f, T_g]$ on the Bergman space has finite-rank seems to be far from solution. The analogous problem on the Hardy space has been completely solved (see [8, 9]). Guo et al. [10] completely characterized the finite-rank semicommutator or commutator of two Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk and Luecking [11] showed that finite-rank Toeplitz operators on the Bergman space of the unit disk must be zero. Recently, Čučković and Louhichi [12] studied finite-rank semicommutators and commutators of Toeplitz operators with quasihomogeneous symbols and obtained different results from the case of harmonic Toeplitz operators.

Motivated by recent work of Čučković and Louhichi, Zhang et al., and Zhou and Dong (see [1, 12, 13]), we discuss the finite-rank commutator (semicommutator) of Toeplitz operators with more general symbols on the unit ball in this paper. Let p and s be two multi-indexes. A function $f \in L^1(\mathbb{B}_n, dA)$ is called a quasihomogeneous function of quasihomogeneous degree (p, s) if f is of the form $\xi^p \bar{\xi}^s \varphi(r)$ for all ξ in the unit sphere $\mathbb{S}_n$ and some function $\varphi(r)$ defined on the interval $[0, 1)$.

Let f and g be two nonconstant quasihomogeneous functions (with certain restrictions on their quasihomogeneous degree). In this paper, we investigate the following problems:

(1) Under what conditions does $T_f T_g = T_F$ hold for some quasihomogeneous function F?

(2) Under what conditions does the semicommutator $(T_f, T_g]$ have finite rank?

(3) Under what conditions does the commutator $[T_f, T_g]$ have finite rank?

2. The Mellin Transform and Mellin Convolution

Main tool in this paper will be the Mellin transform. Recall that the Mellin transform $\widehat{\varphi}$ of a function $\varphi \in L^1([0, 1], rdr)$ is defined by the equation:

$$\widehat{\varphi}(z) = \int_0^1 \varphi(s) s^{z-1} ds. \tag{2.1}$$

It is easy to check that

$$\widehat{\varphi}(z_0 + p) = \widehat{z^p \varphi}(z_0), \tag{2.2}$$

where $p \geq 0$ and $z_0 \in \mathbb{R}$.

For convenience, we denote $\widehat{\varphi}(z)$ by $\varphi^{\wedge}(z)$ when the form of φ is complicated. It is clear that $\widehat{\varphi}$ is well defined on $\{z : \mathrm{Re}(z) > 2\}$ and analytic on $\{z : \mathrm{Re}(z) > 2\}$. It is well known that the Mellin transform $\widehat{\varphi}$ is uniquely determined by its values on $\{n_k\}_{k\geq 0}$, where $n_k \in \mathbb{N}$ and $\sum_{k\geq 0}(1/n_k) = \infty$. The following classical theorem is proved in [14, page 102].

Theorem 2.1. *Assume that f is a bounded analytic function on $\{z : \mathrm{Re}(z) > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \ldots$, where*

(i) $\inf\{|z_n|\} > 0$ *and*

(ii) $\sum_{n\geq 1} \mathrm{Re}(1/z_n) = \infty$.

Then f vanishes identically on $\{z : \mathrm{Re}(z) > 0\}$.

Remark 2.2. We will often use this theorem to show that if $\varphi \in L^1([0,1], rdr)$ and if there exists a sequence $\{n_k\}_{k\geq 0} \subset \mathbb{N}$ such that

$$\widehat{\varphi}(n_k) = 0, \qquad \sum_{k\geq 0} \frac{1}{n_k} = \infty, \tag{2.3}$$

then $\widehat{\varphi}(z) = 0$ for all $z \in \{z : \mathrm{Re}(z) > 2\}$ and so $\varphi = 0$.

If f and g are defined on the interval $[0,1)$, then their Mellin convolution is defined by

$$(f *_M g)(r) = \int_r^1 f\left(\frac{r}{t}\right) g(t) \frac{dt}{t}. \tag{2.4}$$

The Mellin convolution theorem states that

$$\widehat{(f *_M g)}(r) = \widehat{f}(r)\widehat{g}(r), \tag{2.5}$$

and that, if f and g are in $L^1([0,1], rdr)$, then so is $f *_M g$.

3. Products of Toeplitz Operators with Quasihomogeneous Symbols

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_i is a nonnegative integer, we will write

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n, \\ \alpha! &= \alpha_1! \cdots \alpha_n!, \\ z^\alpha &= z_1^{\alpha_1} \cdots z_n^{\alpha_n} \end{aligned} \tag{3.1}$$

for $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$.

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, the notation $\alpha \succeq \beta$ means that

$$\alpha_i \geq \beta_i, \quad i = 1, \ldots, n \tag{3.2}$$

and $\alpha \perp \beta$ means that

$$\alpha_1\beta_1 + \cdots + \alpha_n\beta_n = 0. \tag{3.3}$$

We also define

$$\alpha - \beta = (\alpha_1 - \beta_1, \ldots, \alpha_n - \beta_n) \tag{3.4}$$

and obtain

$$|\alpha - \beta| = |\alpha| - |\beta|, \quad \text{if } \alpha \succeq \beta. \tag{3.5}$$

It is known that φ is radial if and only if $\varphi(Uz) = \varphi(z)$ for any unitary transform U of $\mathbb{C}^n$. So we have $\varphi(r\xi) = \varphi(r\eta)$ for $\xi, \eta \in \mathbb{S}_n$ and $r \in [0,1)$. That is, $\varphi(z)$ depends only on $|z|$. In this case, we denote φ by $\varphi = \varphi(r)$ for convenience. The definition of quasihomogeneous function on the unit disk has been given in many papers (see [2] or [7]), and a similar definition on the unit ball will be given in the following.

Definition 3.1. Let $p, s \succeq 0$. A function $f \in L^1(\mathbb{B}_n, dA)$ is called a quasihomogeneous function of quasihomogeneous degree (p,s) if f is of the form $\xi^p \overline{\xi}^s \varphi$ where φ is a radial function, that is,

$$f(r\xi) = \xi^p \overline{\xi}^s \varphi(r) \tag{3.6}$$

for any ξ in the unit sphere $\mathbb{S}_n$ and $r \in [0,1)$.

The following lemma is from [1] and we will use it often.

Lemma 3.2. *Let p, s be two multi-indexes and let φ be a bounded radial function on $\mathbb{B}_n$. Then for any multi-index α,*

$$T_{\xi^p\varphi}(z^\alpha) = 2(n + |\alpha| + |p|)\widehat{\varphi}(2n + 2|\alpha| + |p|)z^{\alpha+p};$$

$$T_{\overline{\xi}^s\varphi}(z^\alpha) = \begin{cases} 0 & \text{if } \alpha \not\succeq s, \\ \dfrac{2\alpha!(n + |\alpha| - |s|)!}{(\alpha - s)!(n - 1 + |\alpha|)!}\widehat{\varphi}(2n + 2|\alpha| - |s|)z^{\alpha - s} & \text{if } \alpha \succeq s; \end{cases}$$

$$T_{\xi^p\overline{\xi}^s\varphi}(z^\alpha) = \begin{cases} 0 & \text{if } \alpha + p \not\succeq s, \\ \dfrac{2(\alpha + p)!(n + |\alpha| + |p| - |s|)!}{(\alpha + p - s)!(n - 1 + |\alpha| + |p|)!}\widehat{\varphi}(2n + 2|\alpha| + |p| - |s|)z^{\alpha + p - s} & \text{if } \alpha + p \succeq s. \end{cases} \tag{3.7}$$

Proposition 3.3. *Let $p^1, \ldots, p^m, s^1, \ldots, s^m$ be multi-indexes and let $\varphi_1, \ldots, \varphi_m$ be bounded radial functions on $\mathbb{B}_n$. If the product $\prod_{i=0}^{m-1} T_{\xi^{p^{m-i}}\overline{\xi}^{s^{m-i}}\varphi_{m-i}}$ is of finite rank, then there exists α_0 such that*

$$\prod_{i=0}^{m-1} T_{\xi^{p^{m-i}}\overline{\xi}^{s^{m-i}}\varphi_{m-i}}(z^\alpha) = 0 \quad \text{for } \alpha \succeq \alpha_0. \tag{3.8}$$

Proof. Denote by S the product of Toeplitz operators $\prod_{i=0}^{m-1} T_{\xi^{p^{m-i}}\bar{\xi}^{s^{m-i}}\varphi_{m-i}}$ and let rank $S = N$. For multi-indexes $\alpha \geq \sum_{i=1}^{m}(p^i + s^i)$, we have

$$\begin{aligned} S(z^\alpha) &= \prod_{i=0}^{m-2} T_{\xi^{p^{m-i}}\bar{\xi}^{s^{m-i}}\varphi_{m-i}} \left(\frac{2(\alpha+p^1)!(n+|\alpha|+|p^1|-|s^1|)!}{(\alpha+p^1-s^1)!(n-1+|\alpha|+|p^1|)!} \widehat{\varphi_1}\left(2n+2|\alpha|+\left|p^1\right|-\left|s^1\right|\right) z^{\alpha+p^1-s^1} \right) \\ &= 2^m a(\alpha) b(|\alpha|) \prod_{i=1}^{m} \widehat{\varphi_i} \left(2n+2\left(|\alpha| + \sum_{j=0}^{i-1}\left(\left|p^j\right|-\left|s^j\right|\right)\right) + \left|p^i\right| - \left|s^i\right| \right) z^{\alpha+\sum_{i=1}^{m}(p^i-s^i)}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} a(\alpha) &= \frac{\prod_{i=1}^{m}\left(\left(\alpha+\sum_{j=0}^{i-1}(p^j-s^j)+p^i\right)!\right)}{\prod_{i=1}^{m}\left(\left(\alpha+\sum_{j=1}^{i}(p^j-s^j)\right)!\right)}, \\ b(|\alpha|) &= \frac{\prod_{i=1}^{m}\left(\left(n+|\alpha|+\sum_{j=1}^{i}(|p^j|-|s^j|)\right)!\right)}{\prod_{i=1}^{m}\left(\left(n-1+|\alpha|+\sum_{j=0}^{i-1}(|p^j|-|s^j|)+|p^i|\right)!\right)}. \end{aligned} \tag{3.10}$$

It follows that $S(z^\alpha) = C_{(\alpha,p,s)} z^{\alpha+\sum_{i=1}^{m}(p^i-s^i)}$, where $\alpha \geq \sum_{i=1}^{m}(p^i+s^i)$ and $C_{(\alpha,p,s)}$ is a constant dependent on α, p and s. Thus the set $\{S(z^\alpha) : \alpha \geq \sum_{i=1}^{m}(p^i+s^i)\} = \vee_{\alpha\geq\sum_{i=1}^{m}(p^i+s^i)}\{z^{\alpha+\sum_{i=1}^{m}(p^i-s^i)}\}$ contains at most N elements. Let $n_0 = (N,\ldots,N)$, then there exists $\alpha_0 \geq n_0 + \sum_{i=1}^{m}(p^i+s^i)$ such that

$$S(z^\alpha) = 0 \quad \text{for any } \alpha \geq \alpha_0. \tag{3.11}$$

□

Proposition 3.4. *$p^1,\ldots,p^m, s^1,\ldots,s^m$, $\varphi_1,\ldots,\varphi_m$ are defined as in Proposition 3.3. The product $\prod_{i=0}^{m-1} T_{\xi^{p^{m-i}}\bar{\xi}^{s^{m-i}}\varphi_{m-i}}$ is of finite rank if and only if $\varphi_i = 0$ for some $i \in \{1,\ldots,m\}$.*

Proof. Using Proposition 3.3 and Theorem 2.1, we can easily get the result. □

This result is analogous to Theorem 3.2 in [15], but we get it in a different way.

Similar to the proof of Proposition 3.3, we can get a result about finite-rank commutators (semicommutators).

Proposition 3.5. *Let p^1, p^2, s^1, and s^2 be multi-indexes and let φ_1, φ_2 be bounded radial functions on $\mathbb{B}_n$. If the commutator $[T_{\xi^{p^1}\bar{\xi}^{s^1}\varphi_1} - T_{\xi^{p^2}\bar{\xi}^{s^2}\varphi_2}]$ (or the semicommutator $(T_{\xi^{p^1}\bar{\xi}^{s^1}\varphi_1} - T_{\xi^{p^2}\bar{\xi}^{s^2}\varphi_2}]$) is of finite rank, then there exists α_0 such that $[T_{\xi^{p^1}\bar{\xi}^{s^1}\varphi_1} - T_{\xi^{p^2}\bar{\xi}^{s^2}\varphi_2}](z^\alpha) = 0$ (or $(T_{\xi^{p^1}\bar{\xi}^{s^1}\varphi_1} - T_{\xi^{p^2}\bar{\xi}^{s^2}\varphi_2}](z^\alpha) = 0$) for $\alpha \geq \alpha_0$.*

Now we are in a position to characterize when the product of two Toeplitz operators with some quasihomogeneous symbols equals a Toeplitz operator with quasihomogeneous symbols.

When $n = 1$, if $p \perp s$, then $p = 0$ or $s = 0$. But when $n \geq 2$, there exist nonzero multi-indexes p and s such that $p \perp s$. In this case, we have the following theorem.

Theorem 3.6. *Suppose p and s are two nonzero multi-indexes with $p \perp s$. Let φ_1 and φ_2 be bounded radial functions on $\mathbb{B}_n$. If there exists a bounded radial function φ such that $T_{\xi^p\varphi_1}T_{\overline{\xi}^s\varphi_2} = T_{\xi^p\overline{\xi}^s\varphi}$, then φ must be a solution of the equation*

$$r^{2|p|}*_M\cdots*_M r^{2|p|+2(|s|-1)}*_M\left(r^{|p|+|s|}\varphi\right) = r^2*_M\cdots*_M r^{2(|s|-1)}*_M\left(r^{|p|}\varphi_1\right)*_M\left(r^{|s|}\varphi_2\right). \tag{3.12}$$

Proof. Obviously, the equality $T_{\xi^p\varphi_1}T_{\overline{\xi}^s\varphi_2}z^\alpha = T_{\xi^p\overline{\xi}^s\varphi}z^\alpha$ holds for each monomial z^α with the multi-index α.

Since $p \perp s$, $\alpha + p \geq s$ is equivalent to $\alpha \geq s$. By Lemma 3.2, it is easy to check that

$$\widehat{\varphi}(2n+2|\alpha|+|p|-|s|) = 2c(\alpha)d(|\alpha|)\widehat{\varphi}_1(2n+2|\alpha|-2|s|+|p|)\widehat{\varphi}_2(2n+2|\alpha|-|s|), \tag{3.13}$$

where

$$c(\alpha) = \frac{\alpha!(\alpha+p-s)!}{(\alpha-s)!(\alpha+p)!}, \qquad d(|\alpha|) = \frac{(n+|\alpha|-|s|)!(n-1+|\alpha|+|p|)!}{(n-1+|\alpha|)!(n+|\alpha|+|p|-|s|-1)!}. \tag{3.14}$$

Since $p \perp s$, we have $\alpha!(\alpha+p-s)! = (\alpha+p)!(\alpha-s)!$. So (3.13) implies that

$$\widehat{\varphi}(2n+2|\alpha|+|p|-|s|) = 2d(|\alpha|)\widehat{\varphi}_1(2n+2|\alpha|-2|s|+|p|)\widehat{\varphi}_2(2n+2|\alpha|-|s|). \tag{3.15}$$

As $|s| \geq 1$, we have

$$\frac{\widehat{\varphi}(2n+2|\alpha|+|p|-|s|)}{\prod_{i=1}^{|s|}(2n+2|\alpha|+2|p|-2i)} = \frac{\widehat{\varphi}_1(2n+2|\alpha|-2|s|+|p|)\widehat{\varphi}_2(2n+2|\alpha|-|s|)}{\prod_{i=1}^{|s|-1}(2n+2|\alpha|-2i)}. \tag{3.16}$$

A direct calculation gives that $\widehat{r^{2|p|+2i}}(2n+2|\alpha|-2|s|) = 1/(2n+2|\alpha|-2|s|+2|p|+2i)$ for $0 \leq i \leq |s|-1$. Then we have

$$\begin{aligned}&\prod_{i=0}^{|s|-1}\widehat{r^{2|p|+2i}}(2n+2|\alpha|-2|s|)\widehat{r^{|p|+|s|}\varphi}(2n+2|\alpha|-2|s|)\\ &\quad= \prod_{i=1}^{|s|-1}\widehat{r^{2i}}(2n+2|\alpha|-2|s|)\widehat{r^{|p|}\varphi_1}(2n+2|\alpha|-2|s|)\widehat{r^{|s|}\varphi_2}(2n+2|\alpha|-2|s|),\end{aligned} \tag{3.17}$$

or equivalently,

$$\begin{aligned}&\left(r^{2|p|}*_M\cdots*_M r^{2|p|+2(|s|-1)}*_M\left(r^{|p|+|s|}\varphi\right)\right)^{\wedge}(2n+2|\alpha|-2|s|)\\&\quad=\left(r^2*_M\cdots*_M r^{2(|s|-1)}*_M\left(r^{|p|}\varphi_1\right)*_M\left(r^{|s|}\varphi_2\right)\right)^{\wedge}(2n+2|\alpha|-2|s|).\end{aligned}\tag{3.18}$$

Combining the above equality with Remark 2.2, we get the conclusion. □

In the following, we give some explicit examples in which Theorem 3.6 is applied.

Example 3.7. Suppose $p=(1,0)$, $s=(0,1)$, $\varphi_1=r$, $\varphi_2=r$, φ is a bounded radial function such that $T_{\xi^p r}T_{\overline{\xi}^s r}=T_{\xi^p\overline{\xi}^s\varphi}$. Using Theorem 3.6, we can get that $\varphi=1$.

Example 3.8. Suppose $s=0$, φ is a bounded radial function such that $T_{\xi^p\varphi_1}T_{\varphi_2}=T_{\xi^p\varphi}$, then φ must be a solution of the equation

$$r^{|p|}\varphi_1*_M\varphi_2=\chi_{[0,1]}*_M r^{|p|}\varphi,\tag{3.19}$$

where $\chi_{[0,1]}$ is the characteristic function of the set $[0,1]$. For example, suppose $p=(1,1)$, $\varphi_1=r$, $\varphi_2=r^4$, φ is a bounded radial function such that $T_{\xi^p r}T_{r^4}=T_{\xi^p\varphi}$, then it follows that $|p|=2$ and $\varphi=4r^2-3r$.

Louhichi et al. [2] showed that there exist two nontrivial quasihomogeneous Toeplitz operators on the Bergman space of the unit disk such that the product of those Toeplitz operators is also a nontrivial Toeplitz operator, for example $T_{e^{i\theta}r}T_{e^{-i\theta}r}=T_{1-\log(1/r^2)}$. On weighted Bergman space of the unit ball $\mathbb{B}_n$ $(n>2)$, Vasilevski [16, 17] showed that there exist parabolic quasihomogeneous (It is clear that the quasihomogeneous function is also a parabolic quasihomogeneous function) symbol Toeplitz operators such that the finite product of those Toeplitz operators is also a Toeplitz operator of this type. However, on the unit ball $\mathbb{B}_n$ $(n\geq 2)$, if p and s are two nonzero multi-indexes which are not orthogonal, we can get that there exist no nontrivial φ_1 and φ_2 such that $T_{\xi^p\varphi_1}T_{\overline{\xi}^s\varphi_2}=T_{\xi^p\overline{\xi}^s\varphi}$.

Theorem 3.9. *Let p, s be two nonzero multi-indexes which are not orthogonal. Given $n\geq 2$, and let φ_1 and φ_2 be two bounded radial functions on $\mathbb{B}_n$. If there exists a bounded radial function φ such that $T_{\xi^p\varphi_1}T_{\overline{\xi}^s\varphi_2}=T_{\xi^p\overline{\xi}^s\varphi}$, then $\varphi_1=0$ or $\varphi_2=0$.*

Proof. If $T_{\xi^p\varphi_1}T_{\overline{\xi}^s\varphi_2}=T_{\xi^p\overline{\xi}^s\varphi}$, as in Theorem 3.6, we can get

$$\widehat{\varphi}\left(2n+2|\alpha|+\left|p\right|-|s|\right)=2c(\alpha)d(|\alpha|)\widehat{\varphi}_1\left(2n+2|\alpha|-2|s|+\left|p\right|\right)\widehat{\varphi}_2(2n+2|\alpha|-|s|),\quad \alpha\succeq s,\tag{3.20}$$

where

$$c(\alpha)=\frac{\alpha!(\alpha+p-s)!}{(\alpha-s)!(\alpha+p)!},\qquad d(|\alpha|)=\frac{(n+|\alpha|-|s|)!(n-1+|\alpha|+\left|p\right|)!}{(n-1+|\alpha|)!(n+|\alpha|+\left|p\right|-|s|-1)!}.\tag{3.21}$$

We claim that there exists M_0 such that

$$\widehat{\varphi}_1(2n+2M-2|s|+|p|)\widehat{\varphi}_2(2n+2M-|s|)=0, \quad \text{for } M\geq M_0. \tag{3.22}$$

Since p is not orthogonal to s, without loss of generality, we can suppose $p_1s_1\neq 0$.

Case 1 ($n=2$). We denote $p=(p_1,p_2)$, $s=(s_1,s_2)$. For $M>s_1+s_2+1$, let $\alpha_M=(s_1,M-s_1)$ and $\beta_M=(s_1+1,M-s_1-1)$.

Let $F(M)=c(\beta_M)/c(\alpha_M)-1$. Then

$$F(M)=\frac{s_1p_1M^2+\big((p_1+1)(s_1+1)(p_2-2s_2-s_1)+(s_1+p_1+1)(2s_1+s_2-p_2)\big)M+N}{(M-s_1)(M-s_1+p_2-s_2)(s_1+p_1+1)}, \tag{3.23}$$

where $N=(s_1+s_2)(p_2-s_2)(s_1+1)(p_1+1)-(s_1+p_1+1)s_1(s_1-p_2+s_2)$.

Therefore, the equation $F(M)=0$ has two solutions at most. It means that there exists $M_0\geq|s|+1$ such that $F(M)\neq 0$ for any $M\geq M_0$. Thus for each $M\geq M_0$, we have $|\alpha_M|=|\beta_M|=M$ and $c(\alpha_M)\neq c(\beta_M)$.

Case 2 ($n\geq 3$). Given a multi-index $\gamma=(\gamma_1,\gamma_2,\gamma_3,\ldots,\gamma_n)\geq s$, let $\alpha_M=(s_1,M-s_1,\gamma_3,\ldots,\gamma_n)$ and $\beta_M=(s_1+1,M-s_1-1,\gamma_3,\ldots,\gamma_n)$ for $M>s_1+s_2+1$. As in Case 1, we can find an integer M_0 with $M_0\geq|s|+1$ such that $|\alpha_M|=|\beta_M|=M$ and $c(\alpha_M)\neq c(\beta_M)$ for any $M\geq M_0$.

Using (3.20), we have

$$0=\big(c(\alpha_M)-c(\beta_M)\big)d(M)\widehat{\varphi}_1(2n+2M-2|s|+|p|)\widehat{\varphi}_2(2n+2M-|s|), \tag{3.24}$$

for $M>M_0$. As $d(M)\neq 0$ and $c(\alpha_M)\neq c(\beta_M)$, it is easy to see that (3.22) holds.

Let $E_1=\{M\in\mathbb{N}: M\geq M_0 \text{ and } \widehat{\varphi}_1(2n+2M-2|s|+|p|)=0\}$ and $E_2=\{M\in\mathbb{N}: M\geq M_0 \text{ and } \widehat{\varphi}_2(2n+2M-|s|)=0\}$. Then $\{M\in\mathbb{N}: M\geq M_0\}=E_1\bigcup E_2$. Since

$$\sum_{M\geq M_0}\frac{1}{M}\leq\sum_{M\in E_1}\frac{1}{M}+\sum_{M\in E_2}\frac{1}{M}, \tag{3.25}$$

we know that at least one of the series $\sum_{M\in E_1}(1/M)$ and $\sum_{M\in E_2}(1/M)$ diverges. Hence it follows from Remark 2.2 that $\varphi_1=0$ or $\varphi_2=0$. □

4. Finite-Rank Semicommutator

On the unit ball $\mathbb{B}_n$ ($n\geq 2$), we will show that the semicommutator of two Toeplitz operators with some quasihomogeneous symbols is of finite rank if and only if it is zero.

Theorem 4.1. *Let p, s be two multi-indexes, and let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$ such that φ_1, $\xi^p\overline{\xi}^s\varphi_2$, and $\xi^p\overline{\xi}^s\varphi_1\varphi_2$ are T-functions. If the semicommutator $(T_{\varphi_1},T_{\xi^p\overline{\xi}^s\varphi_2}]$ has finite rank, then it is equal to zero.*

Proof. Let S be the semicommutator $(T_{\varphi_1}, T_{\xi^p\overline{\xi}^s\varphi_2}]$. If S is finite rank, using Proposition 3.5, there exists $\alpha_0 \succeq 0$ such that

$$S(z^\alpha) = 0 \quad \forall \alpha \succeq \alpha_0. \tag{4.1}$$

Therefore, Lemma 3.2 implies that

$$\begin{aligned}(2n + 2|\alpha| + 2|p| - 2|s|)\widehat{\varphi_1}(2n + 2|\alpha| + 2|p| - 2|s|)\widehat{\varphi_2}(2n + 2|\alpha| + |p| - |s|)\\ = \widehat{\varphi_1\varphi_2}(2n + 2|\alpha| + |p| - |s|)\end{aligned} \tag{4.2}$$

for all $\alpha \succeq \alpha_0$. Since

$$\frac{1}{(2n + 2|\alpha| + 2|p| - 2|s|)} = \widehat{r}^{|p|}(2n + 2|\alpha| + |p| - 2|s|), \tag{4.3}$$

the above equation is equivalent to

$$r^{|p|}\widehat{\varphi_1 * _M r^{|s|}}\varphi_2(2n + 2|\alpha| + |p| - 2|s|) = r^{|p|} * _M \widehat{r^{|s|}}\varphi_1\varphi_2(2n + 2|\alpha| + |p| - 2|s|). \tag{4.4}$$

Note that $(r^{|p|}\widehat{\varphi_1 * _M r^{|s|}}\varphi_2)$ and $(r^{|p|} * _M \widehat{r^{|s|}}\varphi_1\varphi_2)$ are both analytic on the right half-plane $\{z : \operatorname{Re} z > 2\}$ and the sequence $\{2n + 2|\alpha| + |p| - 2|s|\}_{\alpha\succeq\alpha_0}$ is arithmetic. Then Remark 2.2 implies that

$$r^{|p|}\varphi_1 * _M r^{|s|}\varphi_2 = r^{|p|} * _M r^{|s|}\varphi_1\varphi_2. \tag{4.5}$$

Hence, $T_{\varphi_1}T_{\xi^p\overline{\xi}^s\varphi_2} = T_{\xi^p\overline{\xi}^s\varphi_1\varphi_2}$. The proof is complete. □

Next we will consider when the semicommutator of two quasihomogeneous Toeplitz operators is a finite-rank operator.

Remark 4.2. If the semicommutator $(T_{\xi^p\varphi_1}, T_{\xi^s\varphi_2}]$ is of finite rank, following the same process as in Theorem 4.1, we can prove that it must be zero.

On the unit disk, Čučković and Louhichi [12] gave an example to show that there exists a nonzero finite rank semicommutator $(T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$, where f and g are radial functions. However, the situation on the unit ball $\mathbb{B}_n$ $(n \geq 2)$ is different. Let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$ $(n \geq 2)$ and p, s be two multi-indexes. Then we will prove that $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ is a finite-rank operator if and only if $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}] = 0$. Now, we begin with the case that p and s are not orthogonal.

Theorem 4.3. *Let p, s be two multi-indexes which are not orthogonal, and let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$ $(n \geq 2)$ such that $\xi^p\varphi_1$, $\overline{\xi}^s\varphi_2$ and $\xi^p\overline{\xi}^s\varphi_1\varphi_2$ are T-functions. If the semicommutator $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ has finite rank, then $\varphi_1 = 0$ or $\varphi_2 = 0$.*

Proof. Let S be the semicommutator $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$. If S is of finite rank, using Proposition 3.5, we can get that there exists α_0 such that

$$S(z^\alpha) = 0 \quad \forall \alpha \succeq \alpha_0. \tag{4.6}$$

Lemma 3.2 gives that

$$\begin{aligned}&\frac{\alpha!(n+|\alpha|-|s|)!}{(\alpha-s)!(n-1+|\alpha|)!}2\left(n+|\alpha|+\left|p\right|-|s|\right)\widehat{\varphi_1}\left(2n+2|\alpha|-2|s|+\left|p\right|\right)\widehat{\varphi_2}(2n+2|\alpha|-|s|)\\ &\quad=\frac{(\alpha+p)!(n+|\alpha|+\left|p\right|-|s|)!}{(\alpha+p-s)!(n-1+|\alpha|+\left|p\right|)!}\widehat{\varphi_1\varphi_2}\left(2n+2|\alpha|+\left|p\right|-|s|\right)\end{aligned} \tag{4.7}$$

for all $\alpha \succeq \alpha_0$.

Since p is not orthogonal to s, from the proof of Theorem 3.9 and using (4.7), we can get that there exists M_0 such that

$$\widehat{\varphi}_1\left(2n+2M-2|s|+\left|p\right|\right)\widehat{\varphi}_2(2n+2M-|s|)=0, \quad \forall M \geq M_0. \tag{4.8}$$

Analogous to the proof of Theorem 3.9, it is easy to get $\varphi_1 = 0$ or $\varphi_2 = 0$. □

Next, we will show that there exists no nontrivial finite-rank semicommutator $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ in the case that $p \perp s$.

Theorem 4.4. *Let p, s be two multi-indexes with $p \perp s$, and let φ_1 and φ_2 be two integrable radial functions on $\mathbb{B}_n$ such that $\xi^p\varphi_1$, $\overline{\xi}^s\varphi_2$ and $\xi^p\overline{\xi}^s\varphi_1\varphi_2$ are T-functions. The semicommutator $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ has finite rank if and only if $(T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}] = 0$.*

Proof. We only need to prove the necessity. Let $S = (T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ be of finite rank. Since $p \perp s$, it is easy to see that $\alpha + p \succeq s$ if and only if $\alpha \succeq s$ for multi-indexes α. By Lemma 3.2, the following statements hold:

(i) if $\alpha + p \not\succeq s$, then $S(z^\alpha) = 0$;

(ii) if $\alpha \succeq s$, then

$$\begin{aligned}S(z^\alpha) = \Bigg\{ &\frac{2(\alpha+p)!(n+|\alpha|+\left|p\right|-|s|)!}{(\alpha+p-s)!(n-1+|\alpha|+\left|p\right|)!}\widehat{\varphi_1\varphi_2}\left(2n+2|\alpha|+\left|p\right|-|s|\right)\\ &-\frac{2\alpha!(n+|\alpha|-|s|)!}{(\alpha-s)!(n-1+|\alpha|)!}2\left(n+|\alpha|+\left|p\right|-|s|\right)\widehat{\varphi_1}\left(2n+2|\alpha|-2|s|+\left|p\right|\right)\\ &\times\widehat{\varphi_2}(2n+2|\alpha|-|s|)\Bigg\}z^{\alpha+p-s}.\end{aligned} \tag{4.9}$$

Combining (i) and (ii) with the assumption that S has finite rank, we get that there exists $\alpha_0 \succeq s$ such that

$$S(z^\alpha) = 0 \quad \text{for } \alpha \succeq \alpha_0. \tag{4.10}$$

To finish the proof, we will prove that $S(z^\alpha) = 0$ for $\alpha \succeq s$.

Note that $\alpha!(\alpha + p - s)! = (\alpha + p)!(\alpha - s)!$ for $p \perp s$ and $\alpha \succeq s$. Then for each $\alpha \succeq s$, (ii) implies that $S(z^\alpha) = 0$ if and only if

$$\begin{aligned} &\frac{(n + |\alpha| - |s|)!}{(n - 1 + |\alpha|)!}\widehat{\varphi_1}(2n + 2|\alpha| - 2|s| + |p|)\widehat{\varphi_2}(2n + 2|\alpha| - |s|) \\ &\quad = \frac{(n + |\alpha| + |p| - |s| - 1)!}{2(n - 1 + |\alpha| + |p|)!}\widehat{\varphi_1\varphi_2}(2n + 2|\alpha| + |p| - |s|), \end{aligned} \tag{4.11}$$

that is,

$$\begin{aligned} &\frac{1}{\prod_{i=1}^{|s|-1}(2n + 2|\alpha| - 2i)}\widehat{\varphi}_1(2n + 2|\alpha| - 2|s| + |p|)\widehat{\varphi}_2(2n + 2|\alpha| - |s|) \\ &\quad = \frac{1}{\prod_{i=1}^{|s|}(2n + 2|\alpha| + 2|p| - 2i)}\widehat{\varphi_1\varphi_2}(2n + 2|\alpha| + |p| - |s|). \end{aligned} \tag{4.12}$$

Since $\widehat{\chi_{[0,1]}}(n) = 1/n$, the preceding equality is equivalent to

$$\begin{aligned} &\prod_{i=1}^{|s|-1}\widehat{\chi_{[0,1]}}(2n + 2|\alpha| - 2i)\widehat{\varphi}_1(2n + 2|\alpha| - 2|s| + |p|)\widehat{\varphi}_2(2n + 2|\alpha| - |s|) \\ &\quad = \prod_{i=1}^{|s|}\widehat{\chi_{[0,1]}}(2n + 2|\alpha| + 2|p| - 2i)\widehat{\varphi_1\varphi_2}(2n + 2|\alpha| + |p| - |s|) \end{aligned} \tag{4.13}$$

for all $\alpha \succeq s$.

By (4.10), we obtain that the equality (4.13) holds for all $\alpha \succeq \alpha_0$. It is easy to see that $\{2n + 2|\alpha|\}_{\alpha \succeq \alpha_0}$ is arithmetic. Therefore, by Remark 2.2, we have

$$\prod_{i=1}^{|s|-1}\widehat{\chi_{[0,1]}}(z - 2i)\widehat{\varphi}_1(z - 2|s| + |p|)\widehat{\varphi}_2(z - |s|) = \prod_{i=1}^{|s|}\widehat{\chi_{[0,1]}}(z + 2|p| - 2i)\widehat{\varphi_1\varphi_2}(z + |p| - |s|) \tag{4.14}$$

for all $\operatorname{Re} z \geq 2$.

In particular if $z = 2n + 2|\alpha|$, with $\alpha \succeq s$, we have

$$\begin{aligned}&\prod_{i=1}^{|s|-1}\widehat{\chi_{[0,1]}}(2n+2|\alpha|-2i)\widehat{\varphi}_1\big(2n+2|\alpha|-2|s|+|p|\big)\widehat{\varphi}_2(2n+2|\alpha|-|s|)\\&\quad=\prod_{i=1}^{|s|}\widehat{\chi_{[0,1]}}(2n+2|\alpha|+2|p|-2i)\widehat{\varphi_1\varphi_2}(2n+2|\alpha|+|p|-|s|).\end{aligned}\tag{4.15}$$

It follows that the equality (4.13) holds for all $\alpha \succeq s$. So $S(z^\alpha) = 0$ for all $\alpha \succeq s$. Hence, the proof is complete. □

Example 4.5. Let p, s be two multi-indexes, φ is a bounded radial function and a_p, $b_s \in \mathbb{C}$. If $(T_\varphi, T_{a_p z^p}]$ is of finite rank, using Theorem 4.1, we obtain that $(T_\varphi, T_{a_p z^p}] = 0$. If p is not orthogonal to s and $(T_{a_p z^p}, T_{b_s \overline{z}^s}]$ is of finite rank, so it follows from Theorem 4.3 that $a_p = 0$ or $b_s = 0$. But if $p \perp s$, there exist $a_{p_0} \neq 0$ and $b_{s_0} \neq 0$ such that $(T_{a_{p_0} z^p}, T_{b_{s_0} \overline{z}^s}] = 0$. In particular, suppose $p = (0,0)$, $s = (2,2)$, $a_{p_0} = b_{s_0} = 1$, a direct calculation gives that $p \perp s$ and $T_{z^p} T_{\overline{z}^s} = T_{z^p \overline{z}^s}$, that is, $(T_{z^p}, T_{\overline{z}^s}] = 0$.

5. Finite-Rank Commutators

In this section, let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$. We now pass to investigate the commutator of two quasihomogeneous Toeplitz operators and consider when $[T_{\varphi_1}, T_{\xi^p \overline{\xi}^s \varphi_2}]$, $[T_{\xi^p \varphi_1}, T_{\xi^s \varphi_2}]$, or $[T_{\xi^p \varphi_1}, T_{\overline{\xi}^s \varphi_2}]$ have finite rank, respectively.

Theorem 5.1. *Let p, s be two multi-indexes with $p \perp s$, and let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$ such that φ_1 and $\xi^p \overline{\xi}^s \varphi_2$ are T-functions. If φ_1 is not a constant, then $[T_{\varphi_1}, T_{\xi^p \overline{\xi}^s \varphi_2}]$ is of finite rank if and only if $|p| = |s|$ or $\varphi_2 = 0$.*

Proof. Let S be the commutator $[T_{\varphi_1}, T_{\xi^p \overline{\xi}^s \varphi_2}]$. By Lemma 3.2, $S(z^\alpha) = 0$ if and only if

$$\begin{aligned}\widehat{\varphi}_2(2n+2|\alpha|+|p|-|s|)\big\{&(n+|\alpha|+|p|-|s|)\widehat{\varphi}_1(2n+2|\alpha|+2|p|-2|s|)\\&-(n+|\alpha|)\widehat{\varphi}_1(2n+2|\alpha|)\big\}=0,\end{aligned}\tag{5.1}$$

for $\alpha \succeq s$. If S is of finite rank, using Proposition 3.5, there exists $\alpha_0 \succeq p + s$ such that

$$S(z^\alpha) = 0 \quad \text{for } \alpha \succeq \alpha_0. \tag{5.2}$$

Since $p \perp s$, by Theorem 2.1 and following the same process as in Theorem 4.4, we get $S = 0$. Using Theorem 4.4 in [1], we have $S = 0$ if and only if $|p| = |s|$ or $\varphi_2 = 0$.

Conversely, if $|p| = |s|$ or $\varphi_2 = 0$, then we can easily show that $S(z^\alpha) = 0$ for each multi-index α, which implies that T_{φ_1} and $T_{\xi^p \overline{\xi}^s \varphi_2}$ commute. □

Remark 5.2. The same as in Theorem 5.1, we can easily prove if the commutator $[T_{\xi^p\varphi_1}, T_{\xi^s\varphi_2}]$ is of finite rank, then it must be a zero operator.

Next, we give some examples.

Example 5.3. Suppose that $f(z) = a_p z^s$, where $a_p \neq 0$ and $s \neq 0$. Let $g = \xi^k \varphi \neq 0$ be a T-function, where φ is a radial function. Then $[T_f, T_g]$ is of finite rank if and only if T_f and T_g commute. By Theorem 4.9 in [1], we can also get $[T_f, T_g]$ is of finite rank if and only if g is a monomial.

On the unit disk, if the commutator $[T_{e^{ip\theta}f}, T_{e^{-is\theta}g}]$ has finite rank N, then N is at most equal to the quasihomogeneous degree s and a nonzero finite rank commutator has been given in [12]. On the unit ball $\mathbb{B}_n (n \geq 2)$, we will show that the commutator $[T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ has finite rank if and only if $T_{\xi^p\varphi_1}$ commutes with $T_{\overline{\xi}^s\varphi_2}$ if and only if $\varphi_1 = 0$ or $\varphi_2 = 0$.

Theorem 5.4. *Let p, s be two nonzero multi-indexes, and let φ_1, φ_2 be two integrable radial functions on $\mathbb{B}_n$ $(n \geq 2)$ such that $\xi^p\varphi_1$ and $\overline{\xi}^s\varphi_2$ are T-functions. If the commutator $[T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$ has finite rank, then $\varphi_1 = 0$ or $\varphi_2 = 0$.*

Proof. Let S denote the commutator $[T_{\xi^p\varphi_1}, T_{\overline{\xi}^s\varphi_2}]$. Applying Lemma 3.2, we get

$$\begin{aligned} S(z^\alpha) = 4\Bigg(& \frac{(\alpha)!(n+|\alpha|-|s|)!(n+|\alpha|+|p|-|s|)}{(\alpha-s)!(n-1+|\alpha|)!}\widehat{\varphi}_1(2n+2|\alpha|+|p|-2|s|)\widehat{\varphi}_2(2n+2|\alpha|-|s|) \\ & - \frac{(\alpha+p)!(n+|\alpha|+|p|-|s|)!(n+|\alpha|+|p|)}{(\alpha+p-s)!(n-1+|\alpha|+|p|)!} \\ & \times\widehat{\varphi}_1(2n+2|\alpha|+|p|)\widehat{\varphi}_2(2n+2|\alpha|+2|p|-|s|)\Bigg) z^{\alpha+p-s}, \end{aligned} \tag{5.3}$$

for $\alpha \geq s$. If S is finite rank, using Proposition 3.5, there exists $\alpha_0 \geq s$ such that

$$S(z^\alpha) = 0 \quad \text{for } \alpha \geq \alpha_0. \tag{5.4}$$

If $p \perp s$, then we have $\alpha!(\alpha+p-s)! = (\alpha+p)!(\alpha-s)!$. Combining (5.4) and (5.3), we have

$$\begin{aligned} & \frac{(n+|\alpha|-|s|)!(n+|\alpha|+|p|-|s|)}{(n-1+|\alpha|)!}\widehat{\varphi}_1(2n+2|\alpha|+|p|-2|s|)\widehat{\varphi}_2(2n+2|\alpha|-|s|) \\ & = \frac{(n+|\alpha|+|p|-|s|)!(n+|\alpha|+|p|)}{(n-1+|\alpha|+|p|)!}\widehat{\varphi}_1(2n+2|\alpha|+|p|)\widehat{\varphi}_2(2n+2|\alpha|+2|p|-|s|), \end{aligned} \tag{5.5}$$

for $\alpha \geq \alpha_0$. Analogous to the proof of Theorem 4.8 in [1], it is not difficult to get that $\varphi_1 = 0$ or $\varphi_2 = 0$.

On the other hand, if p is not orthogonal to s, then (5.3) and (5.4) imply that

$$\widehat{\varphi}_1(2n+2|\alpha|+|p|)\widehat{\varphi}_2(2n+2|\alpha|+2|p|-|s|) = C_\alpha \widehat{\varphi}_1(2n+2|\alpha|+|p|-2|s|)\widehat{\varphi}_2(2n+2|\alpha|-|s|), \tag{5.6}$$

for $\alpha \succeq \alpha_0$, where

$$C_\alpha = \frac{\alpha!(\alpha+p-s)!(n+|\alpha|-|s|)!(n-1+|\alpha|+|p|)!}{(\alpha+p)!(\alpha-s)!(n-1+|\alpha|)!(n+|\alpha|+|p|-|s|-1)!(n+|\alpha|+|p|)}. \tag{5.7}$$

Following the same process as in Theorem 3.9, we get $\varphi_1 = 0$ or $\varphi_2 = 0$, as desired. □

Acknowledgment

The authors thank the referee for several suggestions that improved the paper. This research is supported by NSFC, Item Number: 10971020.

References

[1] Z.-H. Zhou and X.-T. Dong, "Algebraic properties of Toeplitz operators with radial symbols on the Bergman space of the unit ball," *Integral Equations and Operator Theory*, vol. 64, no. 1, pp. 137–154, 2009.

[2] I. Louhichi, E. Strouse, and L. Zakariasy, "Products of Toeplitz operators on the Bergman space," *Integral Equations and Operator Theory*, vol. 54, no. 4, pp. 525–539, 2006.

[3] S. Grudsky, A. Karapetyants, and N. Vasilevski, "Toeplitz operators on the unit ball in $\mathbb{C}^n$ with radial symbols," *Journal of Operator Theory*, vol. 49, no. 2, pp. 325–346, 2003.

[4] A. Brown and P. R. Halmos, "Algebraic properties of Toeplitz operators," *Journal für die Reine und Angewandte Mathematik*, vol. 213, pp. 89–102, 1963.

[5] P. Ahern and Ž. Čučković, "A theorem of Brown-Halmos type for Bergman space Toeplitz operators," *Journal of Functional Analysis*, vol. 187, no. 1, pp. 200–210, 2001.

[6] P. Ahern, "On the range of the Berezin transform," *Journal of Functional Analysis*, vol. 215, no. 1, pp. 206–216, 2004.

[7] I. Louhichi and L. Zakariasy, "On Toeplitz operators with quasihomogeneous symbols," *Archiv der Mathematik*, vol. 85, no. 3, pp. 248–257, 2005.

[8] S. Axler, Sun-Y.A. Chang, and D. Sarason, "Products of Toeplitz operators," *Integral Equations and Operator Theory*, vol. 1, no. 3, pp. 285–309, 1978.

[9] X. Ding and D. Zheng, "Finite rank commutator of Toeplitz operators or Hankel operators," *Houston Journal of Mathematics*, vol. 34, no. 4, pp. 1099–1119, 2008.

[10] K. Guo, S. Sun, and D. Zheng, "Finite rank commutators and semicommutators of Toeplitz operators with harmonic symbols," *Illinois Journal of Mathematics*, vol. 51, no. 2, pp. 583–596, 2007.

[11] D. H. Luecking, "Finite rank Toeplitz operators on the Bergman space," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1717–1723, 2008.

[12] Ž. Čučković and I. Louhichi, "Finite rank commutators and semicommutators of quasihomogeneous Toeplitz operators," *Complex Analysis and Operator Theory*, vol. 2, no. 3, pp. 429–439, 2008.

[13] B. Zhang, Y. Shi, and Y. Lu, "Algebraic properties of Toeplitz operators on the polydisk," *Abstract and Applied Analysis*, vol. 2011, Article ID 962313, 18 pages, 2011.

[14] R. Remmert, *Classical Topics in Complex Function Theory*, vol. 172, Springer, New York, NY, USA, 1998.

[15] T. Le, "Finite-rank products of Toeplitz operators in several complex variables," *Integral Equations and Operator Theory*, vol. 63, no. 4, pp. 547–555, 2009.

[16] N. Vasilevski, "Parabolic quasi-radial quasi-homogeneous symbols and commutative algebras of Toeplitz operators," in *Topics in Operator Theory. Volume 1. Operators, Matrices and Analytic Functions*, vol. 202, pp. 553–568, Birkhäuser, Basel, Switzerland, 2010.

[17] N. Vasilevski, "Quasi-radial quasi-homogeneous symbols and commutative Banach algebras of Toeplitz operators," *Integral Equations and Operator Theory*, vol. 66, no. 1, pp. 141–152, 2010.