Research Article

Uniform Decay of Solutions for a Nonlinear Viscoelastic Wave Equation with Boundary Dissipation

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We consider a nonlinear viscoelastic wave equation $u_{tt}(t) - k_0 \Delta u(t) + \int_0^t g(t-s) \operatorname{div}(a(x) \nabla u(s)) ds + b(x)u_t = f(u)$, with nonlinear boundary damping in a bounded domain Ω . Under appropriate assumptions imposed on g and with certain initial data, we establish the general decay rate of the solution energy which is not necessarily of exponential or polynomial type. This work generalizes and improves earlier results in the literature.

1. Introduction

In this paper, we are concerned with the energy decay rate of the following viscoelastic problem with nonlinear boundary dissipation:

$$u_{tt}(t) - k_0 \Delta u(t) + \int_0^t g(t-s) \operatorname{div}(a(x) \nabla u(s)) ds + b(x) u_t = f(u), \quad \text{in } \Omega \times (0, \infty),$$

$$u = 0, \quad \text{on } \Gamma_0 \times (0, \infty),$$

$$k_0 \frac{\partial u}{\partial v} - \int_0^t g(t-s)(a(x) \nabla u(s)) \cdot v \, ds + h(u_t) = 0, \quad \text{on } \Gamma_1 \times (0, \infty),$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad x \in \Omega,$$
(1.1)

where $k_0 > 0$ and Ω is a bounded domain in \mathbb{R}^n $(n \ge 1)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint with meas $(\Gamma_0) > 0$, and ν is the unit outward normal to Γ . The relaxation function *g* is a positive and uniformly decaying function, *h* and *a* are functions satisfying some conditions given in (A2) and (A3), respectively, $b : \Omega \to R^+$ is a function, and $f(u) = |u|^{p-2}u$ with

$$1 \le p \le \frac{2}{n-2}, \quad n > 2, \qquad 1 \le p < \infty, \quad \text{if } n = 2.$$
 (1.2)

This type of equations usually arise in the theory of viscoelasticity. It is well known that viscoelastic materials have memory effects, which is due to the mechanical response influenced by the history of the materials themselves. As these materials have a wide application in the natural sciences, their dynamics are interesting and of great importance. From the mathematical point of view, their memory effects are modeled by integrodifferential equations. Hence, questions related to the behavior of the solutions for the PDE system have attracted considerable attention in recent years.

For example, Cavalcanti et al. [1] considered the following problem:

$$u_{tt} - \Delta u + \int_{0}^{t} g(t-s)\Delta u(s)ds + a(x)u_{t} + |u|^{\gamma}u = 0, \quad \text{in } \Omega \times (0,\infty),$$
$$u(x,0) = u_{0}(x), \quad u_{t}(x,0) = u_{1}(x), \quad x \in \Omega,$$
$$u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0,$$
(1.3)

where Ω is a bounded domain in \mathbb{R}^n $(n \ge 1)$ with a smooth boundary, $\gamma > 0$, and $a : \Omega \to \mathbb{R}^+$ is a function, which may be null on a part of Ω . The authors established an exponential decay estimate under the conditions that $a(x) \ge a_0 > 0$ on $\omega \subset \Omega$, with meas $(\omega) > 0$ and satisfying some geometry conditions and

$$-\xi_1 g(t) \le g'(t) \le -\xi_2 g(t), \quad t \ge 0.$$
(1.4)

Berrimi and Messaoudi [2] improved the result [1] by introducing a new function. They proved an exponential decay result under weaker conditions on both *a* and *g*. In fact, they allowed the function *a* to vanish on any part of Ω , and, consequently, the geometry condition imposed on a part of boundary is no longer needed. Later, the same authors [3] and Messaoudi [4] extended the result to a situation in which a source term is competing with the viscoelastic dissipation. In [5], Cavalcanti and Oquendo considered the following:

$$u_{tt}(t) - k_0 \Delta u(t) + \int_0^t g(t-s) \operatorname{div}(a(x)\nabla u(s))ds + b(x)h(u_t) + f(u) = 0,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \ t \ge 0.$$
(1.5)

Under some conditions on the relaxation function g, they improved the result of [1]. Indeed, they proved that the solution of (1.5) decays exponentially to zero when g is decaying exponentially and h is linear, and the solution decays polynomially to zero when g is decaying polynomially and h is nonlinear.

On considering the boundary stabilization, Cavalcanti et al. [6] considered the following problem:

$$u_{tt}(t) - \Delta u(t) + \int_{0}^{t} g(t-s)\Delta u(s)ds = 0, \quad \text{in } \Omega \times (0,\infty),$$

$$u = 0, \quad \text{on } \Gamma_{0} \times (0,\infty),$$

$$\frac{\partial u}{\partial \nu} - \int_{0}^{t} g(t-s)\frac{\partial u}{\partial \nu}(s)ds + h(u_{t}) = 0, \quad \text{on } \Gamma_{1} \times (0,\infty),$$

$$u(0) = u_{0}, \quad u_{t}(0) = u_{1}, \quad x \in \Omega.$$
(1.6)

The existence and uniform decay rate results were established under quite restrictive assumptions on damping term *h* and the kernel function *g*. Later, Cavalcanti et al. [7] generalized this result without imposing a growth condition on *h* and under a weaker assumption on *g*. Recently, Messaoudi and Mustafa [8] exploited some properties of convex functions [9] and the multiplier method to extend these results. They established an explicit and general decay rate result without imposing any restrictive growth assumption on the damping term *h* and greatly weakened the assumption on *g*. Very recently, problem (1.1) has been considered by Li et al. [10] with b(x) = 0 and $f(u) = -|u|^{\gamma}u$, $\gamma > 0$. They showed the global existence and uniqueness of global solution of problem (1.1) and established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function *g*. We refer the reader to related works [7, 11–16] dealing with boundary stabilization.

Motivated by previous works, it is interesting to investigate the global existence and uniform decay result of solutions to problem (1.1) when a forcing source term is competing with the viscoelastic dissipation and nonlinear boundary damping under the weaker assumption on both *b* and *g*. In fact, we will allow the function *b* to be null on any part of Ω (including Ω itself) and the kernel function *g* is not necessarily decaying in an exponential or polynomial fashion. Therefore, our result allows a larger class of relaxation functions and improves the results in [10, 13] where only the exponential and polynomial rate was considered.

The remainder of this paper is organized as follows. In Section 2, we provide assumptions that will be used later and mention the local existence result Theorem 2.1. In Section 3, we prove our stability result that is given in Theorem 3.7.

2. Preliminary Results

In this section, we give assumptions and preliminaries that will be needed throughout the paper. First, we introduce the following set:

$$H^{1}_{\Gamma_{0}} = \left\{ \left. u \in H^{1}(\Omega) : u \right|_{\Gamma_{0}} = 0 \right\},$$

$$(2.1)$$

and endow $H^1_{\Gamma_0}$ with the Hilbert structure induced by $H^1(\Omega)$. We have that $H^1_{\Gamma_0}$ is a Hilbert space. For simplicity, we denote $\|\cdot\|_q = \|\cdot\|_{L^q(\Omega)}$ and $\|\cdot\|_{q,\Gamma_1} = \|\cdot\|_{L^q(\Gamma_1)}$, $1 \le q \le \infty$. According to (1.2), we have the imbedding: $H^1_{\Gamma_0} \hookrightarrow L^{2(p+1)}(\Omega)$. Let B > 0 be the optimal constant of Sobolev imbedding which satisfies the following inequality:

$$\|u\|_{2(p+1)} \le B \|\nabla u\|_{2}, \quad \forall u \in H^{1}_{\Gamma_{0}},$$
(2.2)

and we use the Trace-Sobolev imbedding: $H^1_{\Gamma_0} \hookrightarrow L^k(\Gamma_1)$, $1 \le k < 2(n-1)/(n-2)$. In this case, the imbedding constant is denoted by B_1 , that is,

$$\|u\|_{k,\Gamma_1} \le B_1 \|\nabla u\|_2. \tag{2.3}$$

Next, we state the assumptions for problem (1.1) as follows.

(A1) $g: [0, \infty) \to (0, \infty)$ is a bounded C^1 function satisfying

$$g(0) > 0, \qquad k_0 - ||a||_{\infty} \int_0^\infty g(s) ds = l > 0,$$
 (2.4)

and there exists a nonincreasing positive differentiable function ξ such that

$$g'(t) \le -\xi(t)g(t), \quad \forall t \ge 0, \qquad \int_0^\infty \xi(s)ds = \infty.$$
 (2.5)

(A2) $h: R \to R$ is a nondecreasing function with

$$h(s)s \ge \alpha |s|^2, \quad \forall s \in R,$$
 (2.6)

$$|h(s)| \le \beta |s|, \quad \forall s \in R.$$

$$(2.7)$$

(A3) $a: \Omega \to R$ is a nonnegative functions and $a \in C^1(\overline{\Omega})$ such that

$$a(x) \ge a_0 > 0,$$
 (2.8)

$$|\nabla a(x)|^2 \le \alpha_1^2 |a(x)|, \tag{2.9}$$

for some positive constant α_1 .

By using the Galerkin method and procedure similar to that of [10, 16], we can have the following local existence result for problem (1.1).

Theorem 2.1. Let hypotheses (A1)–(A3) and (1.2) hold and assume that $u_0 \in H^1_{\Gamma_0} \cap H^2(\Omega)$, $u_1 \in H^1_{\Gamma_0}$. Then there exists a strong solution u of (1.1) satisfying

$$u \in L^{\infty}([0,T); H^{1}_{\Gamma_{0}} \cap H^{2}(\Omega)),$$

$$u_{t} \in L^{\infty}([0,T); H^{1}_{\Gamma_{0}}),$$

$$u_{tt} \in L^{\infty}([0,T); L^{2}(\Omega)).$$
(2.10)

Furthermore, if $u_0 \in H^1_{\Gamma_0}$, $u_1 \in L^2(\Omega)$, then there exists a weak solution u of (1.1) satisfying

$$u \in C([0,T); H^{1}_{\Gamma_{0}}) \cap C^{1}([0,T); L^{2}(\Omega)),$$
(2.11)

for some T > 0.

3. Global Existence and Energy Decay

In this section, we focus our attention on the uniform decay of weak solutions to problem (1.1). For this purpose, we define

$$J(u(t)) = \frac{1}{2} \int_{\Omega} \left(k_0 - a(x) \int_0^t g(s) ds \right) |\nabla u(t)|^2 dx + \frac{1}{2} \left(g \circ \nabla u \right)(t) - \frac{1}{p} ||u||_p^p,$$
(3.1)

and the energy function

$$E(t) = \frac{1}{2} ||u_t||_2^2 + J(u(t)), \quad \text{for } t \in [0, T),$$
(3.2)

where

$$(g \circ \nabla u)(t) = \int_{\Omega} \int_0^t g(t-s)a(x)|\nabla u(t) - \nabla u(s)|^2 ds \, dx.$$
(3.3)

Adopting the proof of [10], we still have the following results.

Lemma 3.1. For any $u \in C^1(0,T; H^1(\Omega))$, we have

$$\int_{\Omega} a(x) \int_{0}^{t} g(t-s) \nabla u(s) \nabla u_{t}(t) ds \, dx = -\frac{1}{2} \int_{\Omega} a(x) g(t) |\nabla u(t)|^{2} dx + \frac{1}{2} (g' \circ \nabla u) (t) -\frac{1}{2} \frac{d}{dt} \left[(g \circ \nabla u) (t) - \int_{\Omega} a(x) \int_{0}^{t} g(s) ds |\nabla u(t)|^{2} dx \right].$$
(3.4)

Lemma 3.2. Let u be the solution of (1.1), then, under assumptions (A1)-(A2), E(t) is a nonincreasing function on [0,T) and

$$E'(t) = -\int_{\Gamma_1} u_t h(u_t) d\Gamma + \frac{1}{2} (g' \circ \nabla u)(t) - \frac{1}{2} \int_{\Omega} a(x) g(t) |\nabla u(t)|^2 dx - \int_{\Omega} b(x) |u_t(t)|^2 dx \le 0.$$
(3.5)

Next, we define a functional F, which helps in establishing the desired results. Setting

$$F(x) = \frac{1}{2}x^2 - \frac{B^p}{pl^{p/2}}x^p, \quad x > 0.$$
(3.6)

Remark 3.3. As in [17], we can verify that the functional *F* is increasing in $(0, \lambda_0)$, decreasing in (λ_0, ∞) , and *F* has a maximum at $\lambda_0 = (l^{p/2}/B^p)^{1/(p-2)}$ with the following maximum value:

$$E_1 = \frac{p-2}{2p} l^{p/(p-2)} B^{-2p/(p-2)}.$$
(3.7)

Further, from (3.1), (3.2), (2.4), and the definition of *F* by (3.6), we have

$$E(t) \ge J(u(t)) \ge \frac{1}{2} l \|\nabla u(t)\|^2 + \frac{1}{2} (g \circ \nabla u)(t) - \frac{B^p}{p l^{p/2}} \left(\sqrt{l \|\nabla u\|_2^2 + (g \circ \nabla u)(t)} \right)^p$$

$$= F\left(\sqrt{l \|\nabla u\|_2^2 + (g \circ \nabla u)(t)} \right), \quad t \ge 0.$$
(3.8)

Lemma 3.4. Suppose that (A1)-(A2) and (1.2) hold. Assume further that $u_0 \in H^1_{\Gamma_0}$, $u_1 \in L^2(\Omega)$, and satisfy $\||\nabla u_0\|_2^2 < \lambda_0^2$ and $E(0) < E_1$. Then, it holds that

$$||\nabla u(t)||_{2}^{2} + (g \circ \nabla u)(t) < \lambda_{0}^{2},$$
(3.9)

for all $t \in [0,T)$. Moreover, one has $J(u(t)) < E_1$ and

$$|||\nabla u(t)||_{2}^{2} + (g \circ \nabla u)(t) \le \frac{2pE(t)}{p-2} < \frac{2pE(0)}{p-2} < \frac{2pE_{1}}{p-2},$$
(3.10)

for all $t \in [0, T)$.

Proof. Using (3.8) and considering E(t) is a nonincreasing function, we obtain

$$F\left(\sqrt{l\|\nabla u\|_{2}^{2} + (g \circ \nabla u)(t)}\right) \le E(t) \le E(0) < E_{1}, \quad t \in [0, T).$$
(3.11)

Further, from Remark 3.3, we observe that *F* is increasing in $(0, \lambda_0)$, decreasing in (λ_0, ∞) , and $F(\lambda) \rightarrow -\infty$ as $\lambda \rightarrow \infty$. Thus, as $E(0) < E_1$, there exist $\lambda'_1 < \lambda_0 < \lambda_1$ such that $F(\lambda'_1) = F(\lambda_1) = E(0)$, which together with $l \|\nabla u_0\|_2^2 < \lambda_0^2$ infer that

$$F\left(\sqrt{l\|\nabla u_0\|_2^2}\right) \le E(0) = F(\lambda_1'). \tag{3.12}$$

This implies that $l^{1/2} \| \nabla u_0 \|_2 \le \lambda'_1$. Next, we will prove that

$$\sqrt{l} \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \le \lambda_1'.$$
(3.13)

To establish (3.13), we argue by contradiction. Suppose that (3.13) does not hold, then there exists $t^* \in (0, T)$ such that

$$\sqrt{l} \|\nabla u(t^*)\|_2^2 + (g \circ \nabla u)(t^*) > \lambda_1'.$$
(3.14)

Case 1. If $\lambda'_1 < \sqrt{l \|\nabla u(t^*)\|_2^2 + (g \circ \nabla u)(t^*)} < \lambda_0$, then $F\left(\sqrt{l\|\nabla u(t^*)\|_2^2+\left(g\circ\nabla u\right)(t^*)}\right)>F(\lambda_1')=E(0)\geq E(t^*).$ (3.15)

This contradicts (3.11).

Case 2. If $\sqrt{l \|\nabla u(t^*)\|_2^2 + (g \diamond \nabla u)(t^*)} \ge \lambda_0$, then by continuity of $\sqrt{l \|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)}$, there exists $0 < t_1 < t^*$ such that

$$\lambda_{1}' < \sqrt{l \|\nabla u(t_{1})\|_{2}^{2} + (g \circ \nabla u)(t_{1})} < \lambda_{0},$$
(3.16)

then

$$F\left(\sqrt{l\|\nabla u(t_1)\|_2^2 + (g \circ \nabla u)(t_1)}\right) > F(\lambda_1') = E(0) \ge E(t_1).$$
(3.17)

This is also a contradiction of (3.11). Thus, we have proved the inequality (3.13).

To prove (3.10), we note for $\lambda < \lambda_0$, such that

$$F(\lambda) = \lambda^2 \left(\frac{1}{2} - \frac{B^p}{pl^{p/2}}\lambda^{p-2}\right) \ge \lambda^2 \left(\frac{1}{2} - \frac{B^p}{pl^{p/2}}\lambda_0^{p-2}\right) \ge \frac{p-2}{2p}\lambda^2,$$
(3.18)

because of $\lambda_0 = (l^{p/2}/B^p)^{1/(p-2)}$. Thanks to $(l \| \nabla u(t) \|_2^2 + (g \circ \nabla u)(t))^{1/2} < \lambda_0$ by (3.9), we obtain

$$\frac{p-2}{2p} \left(l \|\nabla u\|_{2}^{2} + (g \circ \nabla u)(t) \right) \leq F \left(\sqrt{l \|\nabla u\|_{2}^{2} + (g \circ \nabla u)(t)} \right)$$

$$\leq J(u(t)) \leq E(t) < E(0) < E_{1}.$$
(3.19)

Therefore, we complete the proof of Lemma 3.4.

Theorem 3.5. Let $u_0 \in H^1_{\Gamma_0}$, $u_1 \in L^2(\Omega)$, and (A1)-(A2) and (1.2) hold. Assume further that $\||\nabla u_0\|_2^2 < \lambda_0^2$ and $E(0) < E_1$, then the problem (1.1) admits a global solution. Furthermore, for all $t \in [0, \infty)$, one has

$$||\nabla u(t)||_{2}^{2} + (g \circ \nabla u)(t) \le \frac{2pE(t)}{p-2} < \frac{2pE(0)}{p-2} < \frac{2pE_{1}}{p-2},$$
(3.20)

$$\|u\|_{p}^{p} \leq L\Big(l\|\nabla u\|_{2}^{2} + (g \circ \nabla u)(t)\Big),$$
(3.21)

with $L = (B^p/l)(2pE(0)/(p-2)l)^{(p-2)/2}$.

Proof. It follows from (3.19) and (3.8) that

$$\frac{1}{2} \|u_t\|_2^2 + \frac{p-2}{2p} \left(l \|\nabla u\|_2^2 + (g \circ \nabla u)(t) \right) \leq \frac{1}{2} \|u_t\|_2^2 + F\left(\sqrt{l \|\nabla u\|_2^2 + (g \circ \nabla u)(t)}\right) \\
\leq \frac{1}{2} \|u_t\|_2^2 + J(u(t)) \\
= E(t) < E(0) < E_1.$$
(3.22)

Thus, we have the inequality (3.20) and we also establish the boundedness of u_t in $L^2(\Omega)$ and the boundedness of u in $H^1_{\Gamma_0}$. Moreover, from (2.2) and (3.22), we also obtain the boundedness of u in $L^p(\Omega)$. Hence, it must have $T = \infty$.

Additionally, using (2.2) and (3.20), we obtain

$$\begin{aligned} \|u\|_{p}^{p} &\leq B^{p} \|\nabla u\|_{2}^{p} \leq \frac{1}{l} \left(B^{p} \left(\frac{2pE(0)}{(p-2)l} \right)^{(p-2)/2} \right) l \|\nabla u\|_{2}^{2} \\ &\leq L \Big(l \|\nabla u\|_{2}^{2} + (g \circ \nabla u)(t) \Big), \end{aligned}$$
(3.23)

for all $t \ge 0$.

Now, we will investigate the asymptotic behavior of the energy function E(t). First, we define some functionals and establish Lemma 3.6. Let

$$G(t) = E(t) + \varepsilon_1 \Phi(t) + \varepsilon_2 \Psi(t), \qquad (3.24)$$

where

$$\Phi(t) = \int_{\Omega} u_t u \, dx,\tag{3.25}$$

$$\Psi(t) = \int_{\Omega} a(x)u_t \int_0^t g(t-s)(u(s) - u(t))ds \, dx, \tag{3.26}$$

and ε_1 , ε_2 are some positive constants to be be specified later.

Lemma 3.6. There exist two positive constants β_1 and β_2 such that the relation

$$\beta_1 E(t) \le G(t) \le \beta_2 E(t) \tag{3.27}$$

holds, for $\varepsilon_1, \varepsilon_2 > 0$ *small enough.*

Proof. By Hölder's inequality, Young's inequality, (2.2), and (2.8), we deduce that

$$\begin{split} |\Phi(t)| &\leq \frac{1}{2} \|u_t\|_2^2 + \frac{B^2}{2} \|\nabla u\|_2^2, \\ |\Psi(t)| &\leq \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} \int_{\Omega} \left(a(x) \int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \\ &\leq \frac{1}{2} \|u_t\|_2^2 + \frac{\|a\|_{\infty}^2}{2} \int_0^t g(s)ds \int_{\Omega} \int_0^t g(t-s)|u(t)-u(s)|^2 ds \, dx \\ &\leq \frac{1}{2} \|u_t\|_2^2 + \frac{(k_0-l)\|a\|_{\infty}B^2}{2a_0} (g \circ \nabla u)(t). \end{split}$$
(3.28)

Hence, taking (3.24) and (3.28) into account, we have

$$G(t) = E(t) + \varepsilon_{1}\Phi(t) + \varepsilon_{2}\Psi(t)$$

$$\leq E(t) + c_{1}||u_{t}||_{2}^{2} + c_{2}||\nabla u||_{2}^{2} + c_{3}(g \circ \nabla u)(t), \qquad (3.29)$$

$$G(t) \geq E(t) - c_{4}(||u_{t}||_{2}^{2} + ||\nabla u||_{2}^{2} + (g \circ \nabla u)(t)),$$

where $c_1 = (\varepsilon_1 + \varepsilon_2)/2$, $c_2 = \varepsilon_1 B^2/2$, $c_3 = (k_0 - l) ||a||_{\infty} B^2 \varepsilon_2/2a_0$, and $c_4 = \max(c_1, c_2, c_3)$. Thus, using (3.22) and selecting $\varepsilon_1, \varepsilon_3 > 0$ small enough, there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le G(t) \le \beta_2 E(t). \tag{3.30}$$

Theorem 3.7. Let (A1)–(A3) and (1.2) hold. Assume that $u_0 \in H^1_{\Gamma_0}$, $u_1 \in L^2(\Omega)$, $|||\nabla u_0||_2^2 < \lambda_0^2$, and $E(0) < E_1$. Then, for any $t_0 > 0$, there exist two positive constants K and k such that the solution of (1.1) satisfies

$$E(t) \le K e^{-k \int_{t_0}^t \xi(s) ds}, \quad \text{for } t \ge t_0.$$
(3.31)

Proof. First, we estimate the derivative of G(t). From (3.25) and using (1.1), we have

$$\Phi'(t) = \|u_t\|_2^2 - k_0 \|\nabla u\|_2^2 + \int_{\Omega} \nabla u(t) a(x) \int_0^t g(t-s) \nabla u(s) ds \, dx$$

$$- \int_{\Gamma_1} h(u_t) u \, d\Gamma - \int_{\Omega} b(x) u_t u \, dx + \|u\|_p^p.$$
(3.32)

The third, the fourth, and the fifth terms on the right-hand side of (3.32) can be estimated as follows. From Hölder's inequality, Young's inequality, and (2.4), for $\eta > 0$, we have

$$\int_{\Omega} \nabla u(t)a(x) \int_{0}^{t} g(t-s)\nabla u(s)ds \, dx$$

$$\leq \frac{k_{0}}{2} \|\nabla u\|_{2}^{2} + \frac{1}{2k_{0}} \int_{\Omega} \left(a(x) \int_{0}^{t} g(t-s)(\nabla u(s) - \nabla u(t) + \nabla u(t))ds\right)^{2} dx \qquad (3.33)$$

$$\leq \left[\frac{k_{0}}{2} + \frac{1}{2k_{0}}(1+\eta)(k_{0}-l)^{2}\right] \|\nabla u\|_{2}^{2} + \frac{1}{2k_{0}}\left(1+\frac{1}{\eta}\right)(k_{0}-l)(g \circ \nabla u)(t).$$

Employing Hölder's inequality, Young's inequality, (2.2), (2.3), and (2.7), for $\delta_1, \delta_2 > 0$, we see that

$$\left| \int_{\Gamma_{1}} h(u_{t}) u \, d\Gamma \right| \leq \delta_{1} B_{*}^{2} \|\nabla u\|_{2}^{2} + \frac{\beta^{2}}{4\delta_{1}} \int_{\Gamma_{1}} u_{t}^{2} \, d\Gamma,$$

$$\int_{\Omega} b(x) u_{t} u \, dx \leq B^{2} \|b\|_{\infty} \delta_{2} \|\nabla u\|_{2}^{2} + \frac{1}{4\delta_{2}} \int_{\Omega} b(x) u_{t}^{2} \, dx.$$
(3.34)

A substitution of (3.33)-(3.34) into (3.32) yields

$$\Phi'(t) \leq \|u_t\|_2^2 - \left(\frac{k_0}{2} - \frac{1}{2k_0}(1+\eta)(k_0-l)^2 - \delta_1 B_*^2 - B^2 \|b\|_{\infty} \delta_2\right) \|\nabla u\|_2^2 + \frac{1}{2k_0} \left(1 + \frac{1}{\eta}\right) (k_0 - l) \left(g \circ \nabla u\right)(t) + \frac{\beta^2}{4\delta_1} \int_{\Gamma_1} u_t^2 \, d\Gamma + \frac{1}{4\delta_2} \int_{\Omega} b(x) u_t^2 \, dx + \|u\|_p^p.$$
(3.35)

Letting $\eta = l/(k_0 - l) > 0$ and $\delta_1 = l/(8B_*^2)$, $\delta_2 = l/8B^2 ||b||_{\infty}$ in the above inequality, we obtain

$$\Phi'(t) \leq -\frac{l}{4} \|\nabla u\|_{2}^{2} + \|u_{t}\|_{2}^{2} + \frac{(k_{0} - l)}{2l} (g \circ \nabla u)(t) + \frac{2\beta^{2}B_{*}^{2}}{l} \int_{\Gamma_{1}} u_{t}^{2} d\Gamma + \frac{2B^{2} \|b\|_{\infty}}{l} \int_{\Omega} b(x)u_{t}^{2} dx + \|u\|_{p}^{p}.$$
(3.36)

Next, we estimate $\Psi'(t)$. Taking the derivative of $\Psi(t)$ in (3.26) and using (1.1) to obtain

$$\Psi'(t) = \int_{\Omega} k_0 a(x) \nabla u(t) \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds \, dx$$

+
$$\int_{\Omega} k_0 \nabla u(t) \cdot \nabla a(x) \int_0^t g(t-s) (u(t) - u(s)) ds \, dx$$

-
$$\int_{\Omega} \left(\int_0^t g(t-s) a(x) \nabla u(s) \cdot \nabla a(x) ds \right) \left(\int_0^t g(t-s) (u(t) - u(s)) ds \right) dx$$

$$-\int_{\Omega} a(x) \left(\int_{0}^{t} g(t-s)a(x)\nabla u(s)ds \right) \left(\int_{0}^{t} g(t-s)(\nabla u(t) - \nabla u(s))ds \right) dx \\ +\int_{\Omega} a(x)b(x)u_{t} \int_{0}^{t} g(t-s)(u(t) - u(s))ds dx \\ +\int_{\Gamma_{1}} a(x)h(u_{t}) \int_{0}^{t} g(t-s)(u(t) - u(s))ds d\Gamma \\ -\int_{\Omega} a(x)|u|^{p-2}u \int_{0}^{t} g(t-s)(u(t) - u(s))ds dx \\ -\int_{\Omega} a(x)u_{t} \int_{0}^{t} g'(t-s)(u(t) - u(s))ds dx - \left(\int_{0}^{t} g(s)ds \right) \int_{\Omega} a(x) \left| u_{t}^{2} \right| dx.$$
(3.37)

As in deriving (3.36), in what follows we will estimate the right-hand side of (3.37). Using Young's inequality, Hölder's inequality, (2.4), and (2.9), for $\delta > 0$, we have

$$\begin{split} \left| \int_{\Omega} k_{0}a(x)\nabla u(t) \int_{0}^{t} g(t-s)(\nabla u(t) - \nabla u(s))ds \, dx \right| \\ &\leq k_{0}^{2}\delta \|\nabla u\|_{2}^{2} + \frac{1}{4\delta} \int_{\Omega} \left(a(x) \int_{0}^{t} g(t-s)(\nabla u(t) - \nabla u(s))ds \right)^{2} dx \\ &\leq k_{0}^{2}\delta \|\nabla u\|_{2}^{2} + \frac{\|a\|_{\infty}}{4\delta} \int_{0}^{t} g(s)ds \int_{\Omega} a(x) \int_{0}^{t} g(t-s)|\nabla u(t) - \nabla u(s)|^{2} ds \, dx \\ &\leq k_{0}^{2}\delta \|\nabla u\|_{2}^{2} + \frac{k_{0} - l}{4\delta} (g \circ \nabla u)(t), \end{split}$$
(3.38)
$$\int_{\Omega} k_{0}\nabla u(t) \cdot \nabla a(x) \int_{0}^{t} g(t-s)(u(t) - u(s))ds \, dx \\ &\leq k_{0}a_{1} \int_{\Omega} |\nabla u| \sqrt{a(x)} \left(\int_{0}^{t} g(s)ds \right)^{1/2} \left(\int_{0}^{t} g(t-s)(u(t) - u(s))^{2} ds \right)^{1/2} dx \\ &\leq k_{0}^{2}a_{1}^{2}\delta \|\nabla u\|_{2}^{2} + \frac{(k_{0} - l)B^{2}}{4\delta a_{0}} (g \circ \nabla u)(t). \end{split}$$

Again, exploiting (2.9), Young's inequality, Hölder's inequality, and (2.4), we obtain

$$\left| \int_{\Omega} \left(\int_{0}^{t} g(t-s)a(x)\nabla u(s) \cdot \nabla a(x)ds \right) \left(\int_{0}^{t} g(t-s)(u(t)-u(s))ds \right) dx \right|$$

$$\leq \alpha_{1}^{2}\delta \int_{\Omega} a^{2}(x) \left(\int_{0}^{t} g(t-s)|\nabla u(s)|ds \right)^{2} dx$$

$$+ \frac{1}{4\delta} \int_{\Omega} a(x) \left(\int_{0}^{t} g(t-s)(u(t)-u(s))ds \right)^{2} dx$$

$$\leq 2\alpha_{1}^{2}\delta(k_{0}-l)\int_{\Omega}a(x)\int_{0}^{t}g(t-s)\left(|\nabla u(s)-\nabla u(t)|^{2}+|\nabla u(t)|^{2}\right)ds\,dx$$

$$+\frac{1}{4\delta}\int_{\Omega}a(x)\left(\int_{0}^{t}g(t-s)(u(t)-u(s))ds\right)^{2}dx$$

$$\leq 2\alpha_{1}^{2}\delta(k_{0}-l)^{2}\|\nabla u\|_{2}^{2}+\left(2\alpha_{1}^{2}\delta(k_{0}-l)+\frac{(k_{0}-l)B^{2}}{4\delta a_{0}}\right)(g\circ\nabla u)(t),$$

$$\int_{\Omega}a(x)\left(\int_{0}^{t}g(t-s)a(x)\nabla u(s)ds\right)\left(\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))ds\right)dx$$

$$\leq \delta\int_{\Omega}a^{2}(x)\left(\int_{0}^{t}g(t-s)|\nabla u(s)|ds\right)^{2}dx$$

$$+\frac{1}{4\delta}\int_{\Omega}a^{2}(x)\left(\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))ds\right)^{2}dx$$

$$\leq 2\delta(k_{0}-l)^{2}\|\nabla u\|_{2}^{2}+\left(2\delta+\frac{1}{4\delta}\right)(k_{0}-l)(g\circ\nabla u)(t),$$

$$\left|\int_{\Omega}a(x)b(x)u_{t}\int_{0}^{t}g(t-s)(u(t)-u(s))ds\,dx\right|$$

$$\leq \delta\|b\|_{\infty}\int_{\Omega}b(x)u_{t}^{2}dx+\frac{(k_{0}-l)\|a\|_{\infty}B^{2}}{4a_{0}\delta}(g\circ\nabla u)(t).$$
(3.39)

Utilizing Hölder's inequality, Young's inequality, (2.3), and (2.7), the sixth term on the right-hand side of (3.37) can be estimated as

$$\left| \int_{\Gamma_1} a(x)h(u_t) \int_0^t g(t-s)(u(t)-u(s))ds \, d\Gamma \right| \le \delta\beta^2 \int_{\Gamma_1} u_t^2 \, d\Gamma + \frac{(k_0-l)\|a\|_{\infty} B_*^2}{4a_0\delta} (g \circ \nabla u)(t).$$
(3.40)

As for the seventh and the eighth terms on the right-hand side of (3.37), using Hölder's inequality, Young's inequality, (2.2), (3.20), and (2.4), we obtain

$$\begin{aligned} \left| \int_{\Omega} a(x) |u|^{p-2} u \int_{0}^{t} g(t-s)(u(t)-u(s)) ds \, dx \right| \\ &\leq \delta ||u||_{2(p-1)}^{2(p-1)} + \frac{(k_{0}-l)||a||_{\infty}B^{2}}{4a_{0}\delta} (g \circ \nabla u)(t) \\ &\leq \delta B^{2(p-1)} \left(\frac{2pE(0)}{l(p-2)} \right)^{p-2} ||\nabla u||_{2}^{2} + \frac{(k_{0}-l)||a||_{\infty}B^{2}}{4a_{0}\delta} (g \circ \nabla u)(t), \end{aligned}$$
(3.41)
$$\left| \int_{\Omega} a(x) u_{t} \int_{0}^{t} g'(t-s)(u(t)-u(s)) ds \, dx \right| \leq \delta ||u_{t}||_{2}^{2} - \frac{g(0)||a||_{\infty}^{2}B^{2}}{4a_{0}\delta} (g' \circ \nabla u)(t). \end{aligned}$$

Combining these estimates (3.38)-(3.41), (3.37) becomes

$$\Psi'(t) \leq -\left(a_0 \int_0^t g(s) ds - \delta\right) \|u_t\|_2^2 + \delta c_5 \|\nabla u\|_2^2 + c_6 (g \circ \nabla u)(t) - \frac{g(0) \|u\|_{\infty}^2 B^2}{4a_0 \delta} (g' \circ \nabla u)(t) + \delta \|b\|_{\infty} \int_{\Omega} b(x) u_t^2 dx + \delta \beta^2 \int_{\Gamma_1} u_t^2 d\Gamma,$$
(3.42)

where $c_5 = k_0^2 (\alpha_1^2 + 1) + 2(\alpha_1^2 + 1)(k_0 - l)^2 + B^{2(p-1)} (2pE(0)/l(p-2))^{p-2}$ and $c_6 = (k_0 - l)(B^2/2\delta a_0 + 2\alpha_1^2\delta + (2\delta + 1/2\delta) + ||a||_{\infty} (B_*^2 + 3B^2)/4a_0\delta)$. Hence, we conclude from (3.24), (3.5), (3.36), (3.42), and (2.6) that

$$\begin{aligned} G'(t) &= E'(t) + \varepsilon_1 \Phi'(t) + \varepsilon_2 \Psi'(t) \\ &\leq -\left(\varepsilon_2 \left(a_0 g_0 - \delta\right) - \varepsilon_1\right) \|u_t\|_2^2 - \left(\frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta c_5\right) \|\nabla u\|_2^2 \\ &+ \left(\varepsilon_2 c_6 + \frac{(k_0 - l)\varepsilon_1}{2l}\right) (g \circ \nabla u)(t) - \left(1 - \frac{2\varepsilon_1 B^2 \|b\|_{\infty}}{l} - \varepsilon_2 \delta \|b\|_{\infty}\right) \int_{\Omega} b(x) u_t^2 dx \\ &- \left(\alpha - \frac{2B_*^2 \varepsilon_1 \beta^2}{l} - \varepsilon_2 \delta \beta^2\right) \int_{\Gamma_1} |u_t|^2 d\Gamma - \left(\frac{1}{2} - \varepsilon_2 \frac{g(0) \|a\|_{\infty}^2 B^2}{4a_0 \delta}\right) (-g' \circ \nabla u)(t) \\ &+ \varepsilon_1 \|u\|_p^p, \quad \forall t \ge t_0, \end{aligned}$$

$$(3.43)$$

where we have used the fact that for any $t_0 > 0$,

$$\int_{0}^{t} g(s)ds \ge \int_{0}^{t_{0}} g(s)ds = g_{0}, \quad \forall t \ge t_{0},$$
(3.44)

because *g* is positive and continuous with g(0) > 0. At this point, we choose $\delta > 0$ small enough so that

$$\frac{4\delta c_5}{l} < \frac{a_0 g_0}{2} < a_0 g_0 - \delta.$$
(3.45)

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{4\varepsilon_2 c_5 \delta}{l} < \varepsilon_1 < \frac{a_0 g_0}{2} \varepsilon_2 \tag{3.46}$$

will make

$$k_1 = \frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta c_5 > 0,$$

$$k_2 = \varepsilon_2 (a_0 g_0 - \delta) - \varepsilon_1 > 0.$$
(3.47)

Then, we choose ε_1 and ε_2 so small that (3.27) and (3.45) remain valid, further

$$k_{3} = 1 - \frac{2\epsilon_{1}B^{2}||b||_{\infty}}{l} - \epsilon_{2}\delta||b||_{\infty} > 0$$

$$k_{4} = \alpha - \frac{2B_{*}^{2}\epsilon_{1}\beta^{2}}{l} - \epsilon_{2}\delta\beta^{2} > 0$$

$$k_{5} = \frac{1}{2} - \epsilon_{2}\frac{g(0)||a||_{\infty}^{2}B^{2}}{4a_{0}\delta} > 0.$$
(3.48)

Hence, for all $t \ge t_0$, we arrive at

$$G'(t) \leq -k_1 \|\nabla u\|_2^2 - k_2 \|u_t\|_2^2 + c_7 (g \circ \nabla u)(t) + c_8 (g' \circ \nabla u)(t) - k_3 \int_{\Omega} b(x) u_t^2 dx - k_4 \int_{\Gamma_1} |u_t|^2 d\Gamma + \varepsilon_1 \|u\|_{p+1}^{p+1},$$
(3.49)

which yields (if needed, one can choose ε_1 sufficiently small)

$$G'(t) \le -c_9 E(t) + c_{10} (g \circ \nabla u)(t), \tag{3.50}$$

where c_i , i = 7, 8, 9, 10 are some positive constants. It follows from (3.50), (2.5), and (3.5) that

$$\xi(t)G'(t) \leq -c_9\xi(t)E(t) + c_{10}\xi(t)(g \circ \nabla u)(t)$$

$$\leq -c_9\xi(t)E(t) - c_{10}(g' \circ \nabla u)(t)$$

$$\leq -c_9\xi(t)E(t) - 2c_{10}E'(t), \quad \text{for } t \geq t_0.$$
(3.51)

That is,

$$L'(t) \le -c_9\xi(t)E(t) \le -k\xi(t)L(t), \text{ for } t \ge t_0,$$
 (3.52)

where $L(t) = \xi(t)G(t) + 2c_{10}E(t)$ is equivalent to E(t) by Lemma 3.6 and k is a positive constant. A integration of (3.52) leads to

$$L(t) \le L(t_0)e^{-k\int_{t_0}^t \xi(s)ds}, \quad \text{for } t \ge t_0.$$
(3.53)

Again, employing L(t) is equivalent to E(t) leads to

$$E(t) \le K e^{-k \int_{t_0}^t \dot{\xi}(s) ds}, \quad \text{for } t \ge t_0,$$
 (3.54)

where *K* is a positive constant. This completes the proof.

Remark 3.8. We illustrate the energy decay rate given by Theorem 3.7 through the following examples which are introduced in [18].

(i) If

$$\xi(t) = \alpha, \quad \alpha > 0, \tag{3.55}$$

then (3.54) gives the exponential decay estimate

$$E(t) \le K e^{-k\alpha t}.\tag{3.56}$$

Similarly, if

$$\xi(t) = \alpha (1+t)^{-1}, \quad \alpha > 0,$$
 (3.57)

then we obtain the polynomial decay estimate

$$E(t) \le K(1+t)^{-\alpha k}$$
. (3.58)

(ii) If

$$g(t) = \alpha e^{-\alpha_1 (\ln(1+t))^{\nu}},$$
 (3.59)

with α , α_1 , $\nu > 1$, then (2.5) holds for

$$\xi(t) = \frac{\alpha_1 \nu (\ln(1+t))^{\nu-1}}{1+t}.$$
(3.60)

Thus (3.54) gives the estimate

$$E(t) \le K e^{-ka_1(\ln(1+t))^{\nu}}.$$
 (3.61)

(iii) If

$$g(t) = \frac{\alpha}{(2+t)^{\nu} (\ln(2+t))^{\alpha_1}},$$
(3.62)

where $\alpha > 0$ and $\nu > 1$ and $\alpha_1 \in R$ (or $\nu = 1$ and $\alpha_1 > 1$), then for

$$\xi(t) = \frac{\nu(\ln(2+t)) + \alpha_1}{(2+t)(\ln(2+t))^{\alpha_1}},\tag{3.63}$$

we obtain from (3.54) that

$$E(t) \le \frac{K}{\left[(2+t)^{\nu} (\ln(2+t))^{\alpha_1}\right]^k}.$$
(3.64)

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