Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 454820, 20 pages
doi:10.1155/2012/454820

Research Article

Weighted Composition Operators from Hardy Spaces into Logarithmic Bloch Spaces

Flavia Colonna[1] and Songxiao Li[2]

[1] *Department of Mathematical Sciences, George Mason University, Fairfax, VA 22030, USA*
[2] *Department of Mathematics, Jiaying University, Guangdong, Meizhou 514015, China*

Correspondence should be addressed to Flavia Colonna, fcolonna@gmu.edu

Received 2 February 2012; Accepted 18 March 2012

Academic Editor: Ruhan Zhao

The logarithmic Bloch space $\mathcal{B}_{\log}$ is the Banach space of analytic functions on the open unit disk $\mathbb{D}$ whose elements f satisfy the condition $\|f\| = \sup_{z\in\mathbb{D}}(1-|z|^2)\log(2/(1-|z|^2))|f'(z)| < \infty$. In this work we characterize the bounded and the compact weighted composition operators from the Hardy space H^p (with $1 \le p \le \infty$) into the logarithmic Bloch space. We also provide boundedness and compactness criteria for the weighted composition operator mapping H^p into the little logarithmic Bloch space defined as the subspace of $\mathcal{B}_{\log}$ consisting of the functions f such that $\lim_{|z|\to 1}(1-|z|^2)\log(2/(1-|z|^2))|f'(z)| = 0$.

1. Introduction

Let X and Y be Banach spaces of analytic functions on a domain Ω in $\mathbb{C}$, ψ an analytic function on Ω, and let φ be an analytic function mapping Ω into itself. The *weighted composition operator with symbols ψ and φ* from X to Y is the operator $W_{\psi,\varphi}$ with range in Y defined by

$$W_{\psi,\varphi}f = M_\psi C_\varphi f = \psi(f\circ\varphi), \quad \text{for } f \in X, \tag{1.1}$$

where M_ψ denotes the multiplication operator with symbol ψ, and C_φ denotes the composition operator with symbol φ.

Let $H(\mathbb{D})$ be the set of analytic functions on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $0 < p < \infty$ the *Hardy space H^p* is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1}\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\frac{d\theta}{2\pi} < \infty. \tag{1.2}$$

Let H^∞ denote the space of all $f \in H(\mathbb{D})$ for which $\|f\|_\infty = \sup_{z\in\mathbb{D}}|f(z)| < \infty$.

The *Bloch space* $\mathcal{B}$ on the open unit disk $\mathbb{D}$ is the Banach space consisting of the analytic functions f on $\mathbb{D}$ such that

$$\|f\|_\beta = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\left|f'(z)\right| < \infty. \tag{1.3}$$

The Bloch norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_\beta$. Using the Schwarz-Pick lemma, it is easy to see that the Hardy space H^∞ is contained in $\mathcal{B}$ and $\|f\|_\beta \le \|f\|_\infty$. The inclusion is proper, as the function $f(z) = \log(1+z)/(1-z)$ shows.

The *little Bloch space*, denoted by $\mathcal{B}_0$, is defined as the set of the analytic functions f on $\mathbb{D}$ such that $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$. It is well known that $\mathcal{B}_0$ is a closed separable subspace of $\mathcal{B}$. The interested reader is referred to [1] for more information on the Bloch space.

The *logarithmic Bloch space* $\mathcal{B}_{\log}$ is defined as the set of functions f on $\mathbb{D}$ such that

$$\|f\| = \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|f'(z)\right| < \infty. \tag{1.4}$$

It is a Banach space under the norm defined by $\|f\|_{\mathcal{B}_{\log}} = |f(0)| + \|f\|$. Clearly, if $f \in \mathcal{B}_{\log}$, then $\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0$, so $\mathcal{B}_{\log}$ is a subset of the little Bloch space.

The *little logarithmic Bloch space*, denoted by $\mathcal{B}_{\log,0}$, is defined as the subspace of $\mathcal{B}_{\log}$ whose elements f satisfy the condition

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\log\frac{2}{1-|z|}\left|f'(z)\right| = 0. \tag{1.5}$$

The space $\mathcal{B}_{\log}$ arises in connection to the study of certain operators with symbol. Arazy [2] proved that the multiplication operator M_ψ is bounded on the Bloch space if and only if $\psi \in \mathcal{B}_{\log} \cap H^\infty$. In [3], Brown and Shields extended this result to the little Bloch space.

The space $\mathcal{B}_{\log}$ also arises in the study of Hankel operators on the Bergman one space. The Bergman space A^1 on $\mathbb{D}$ is defined to be the set of analytic functions f on $\mathbb{D}$ whose modulus is Lebesgue integrable over $\mathbb{D}$.

The Hankel operator H_f on A^1 is defined as $H_f g = (I-P)(\overline{f}g)$, where I is the identity operator, and P is the standard Bergman projection from L^1 into A^1. In [4], Attele showed that H_f is bounded on A^1 if and only if $f \in \mathcal{B}_{\log}$.

The study of operators with symbol on the logarithmic Bloch space began with the characterizations of the bounded and the compact composition operators given in [5] by Yoneda. In [6], Galanopoulos extended these results to the weighted composition operators on $\mathcal{B}_{\log}$. He also introduced a class of Banach spaces $Q^p_{\log}$ $(p > 0)$ closely related to $\mathcal{B}_{\log}$ and studied the Taylor coefficients of the functions in $\mathcal{B}_{\log}$. In [7], Ye characterized the bounded and the compact weighted composition operators on the little logarithmic Bloch space $\mathcal{B}_{\log,0}$. See [8, 9] for the study of the weighted composition operators on Bloch spaces and weighted Bloch spaces.

In this paper, we characterize the bounded and the compact weighted composition operators from the Hardy space H^p (with $1 \le p \le \infty$) to the logarithmic Bloch space $\mathcal{B}_{\log}$ as well as to its subspace $\mathcal{B}_{\log,0}$. The paper consists of five sections. Specifically, in Section 2, we

consider the bounded weighted composition operators mapping H^∞ into $\mathcal{B}_{\log}$ and $\mathcal{B}_{\log,0}$. In particular, we show that

$$\|W_{\psi,\varphi}\| \sim \sup_{n\in\mathbb{N}\cup\{0\}} \|\psi\varphi^n\|_{\mathcal{B}_{\log}}, \tag{1.6}$$

where the notation $A \sim B$ stands for $c_1 A \le B \le c_2 A$, for some positive constants c_1 and c_2. In Section 3, we look at the issue of compactness of such operators.

In Section 4, we characterize the bounded and the compact weighted composition operators mapping H^p into $\mathcal{B}_{\log}$ in the case when $1 \le p < \infty$. Finally, in Section 5, we study the operators mapping H^p into $\mathcal{B}_{\log,0}$.

2. Boundedness of $W_{\psi,\varphi}$ from H^∞ into $\mathcal{B}_{\log}$ and $\mathcal{B}_{\log,0}$

In the following theorem, we give two characterizations of boundedness when the operator maps H^∞ into $\mathcal{B}_{\log}$.

Theorem 2.1. *Let ψ be an analytic function on $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*

(a) *The operator $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log}$ is bounded.*

(b) $\sup_{n\in\mathbb{N}\cup\{0\}} \|\psi\varphi^n\|_{\mathcal{B}_{\log}} < \infty$.

(c) $\psi \in \mathcal{B}_{\log}$ *and* $\sigma_{\psi,\varphi} = \sup_{z\in\mathbb{D}}((1-|z|^2)|\psi(z)\varphi'(z)|/1-|\varphi(z)|^2)\log(2/(1-|z|^2)) < \infty$.

Proof. (a) $\Rightarrow$ (b). For $n \in \mathbb{N}$, the function $p_n(z) = z^n$ is bounded and $\|p_n\|_\infty = 1$. Therefore, if $W_{\psi,\varphi}$ is bounded, then $\|\psi\varphi^n\|_{\mathcal{B}_{\log}} \le \|W_{\psi,\varphi}\|$.

(b) $\Rightarrow$ (c) Let C be an upper bound for $\|\psi\varphi^n\|_{\mathcal{B}_{\log}}$, $n \ge 0$. Taking $n = 0$, we deduce that $\|\psi\|_{\mathcal{B}_{\log}} \le C$, so $\psi \in \mathcal{B}_{\log}$.

For $N \in \mathbb{N}$ and $n \ge 2$, define the sets

$$\begin{aligned} E_N &= \left\{ z \in \mathbb{D} : |\varphi(z)| \le 1 - \frac{1}{N} \right\}, \\ \Delta_n &= \left\{ z \in \mathbb{D} : 1 - \frac{1}{n-1} \le |\varphi(z)| \le 1 - \frac{1}{n} \right\}. \end{aligned} \tag{2.1}$$

Fix an integer $N > 2$, and $z \in \mathbb{D}$. For $z \in E_N$, by the product rule, we have

$$\begin{aligned} \frac{\left(1-|z|^2\right)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\log\frac{2}{1-|z|^2} &\le \frac{\left(1-|z|^2\right)\left(\left|(\psi\varphi)'(z)\right| + |\psi'(z)\varphi(z)|\right)}{1-(1-1/N)^2}\log\frac{2}{1-|z|^2} \\ &\le \frac{1}{1-(1-1/N)^2}\left(\|\psi\varphi\| + \|\psi\|\right) \\ &\le \frac{2C}{1-(1-1/N)^2}. \end{aligned} \tag{2.2}$$

In the proof of Theorem 2 of [10], it was shown that

$$\inf_{z\in\Delta_n} n|\varphi(z)|^{n-1}(1-|\varphi(z)|) \geq \frac{1}{e}. \tag{2.3}$$

For $|\varphi(z)| > 1-1/N$, there exists $n > N$ such that $z \in \Delta_n$. So

$$\begin{aligned}
\frac{\left(1-|z|^2\right)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\log\frac{2}{1-|z|^2} &\leq \frac{\left(1-|z|^2\right)\left|\psi(z)n\varphi(z)^{n-1}\varphi'(z)\right|}{(1-|\varphi(z)|)n|\varphi(z)|^{n-1}}\log\frac{2}{1-|z|^2} \\
&\leq e\left[\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|(\psi\varphi^n)'(z)\right|\right. \\
&\qquad \left. + \left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|\psi'(z)\varphi(z)^n\right|\right] \\
&\leq 2eC.
\end{aligned} \tag{2.4}$$

From (2.2) and (2.4), we deduce that $\sigma_{\psi,\varphi}$ is finite.

(c) $\Rightarrow$ (a) Let $f \in H^\infty$ with $\|f\|_\infty \leq 1$ and pick $z \in \mathbb{D}$. Then

$$\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}|\psi'(z)f(\varphi(z))| \leq \|\psi\|, \tag{2.5}$$

and, since $\|f\|_\beta \leq \|f\|_\infty \leq 1$,

$$\begin{aligned}
&\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}|\psi(z)f'(\varphi(z))\varphi'(z)| \\
&\quad = \frac{\left(1-|z|^2\right)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\log\frac{2}{1-|z|^2}\left(1-|\varphi(z)|^2\right)|f'(\varphi(z))| \\
&\quad \leq \frac{\left(1-|z|^2\right)|\psi(z)\varphi'(z)|}{1-|\varphi(z)|^2}\log\frac{2}{1-|z|^2}.
\end{aligned} \tag{2.6}$$

Thus, by (2.5) and (2.6), we deduce that $\|W_{\psi,\varphi}f\|_{\mathcal{B}_{\log}} \leq \|\psi\|_{\mathcal{B}_{\log}} + \sigma_{\psi,\varphi}$, completing the proof. □

We next turn our attention to the weighted composition operators mapping into the little logarithmic Bloch space.

Theorem 2.2. *Let ψ be an analytic function on $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*

(a) *The operator $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log,0}$ is bounded.*

(b) *For each integer $n \geq 0$, $\psi\varphi^n \in \mathcal{B}_{\log,0}$ and $\sup_{n\in\mathbb{N}}\|\psi\varphi^n\|_{\mathcal{B}_{\log}} < \infty$.*

(c) *$\psi \in \mathcal{B}_{\log,0}$ and $\lim_{|z|\to 1}((1-|z|^2)|\psi(z)\varphi'(z)|/(1-|\varphi(z)|^2))\log(2/(1-|z|^2)) = 0$.*

Proof. (a) $\Rightarrow$ (b) is proved as in the case of the operator mapping into $\mathcal{B}_{\log}$.

(b) $\Rightarrow$ (c) Suppose that (b) holds. If $E_N = \mathbb{D}$ for some integer $N > 1$, then for all $z \in \mathbb{D}$, we have

$$\begin{aligned}
&\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} \\
&\quad\le \frac{1}{1-(1-1/N)^2}\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|\log\frac{2}{1-|z|^2} \\
&\quad\le \frac{1}{1-(1-1/N)^2}\left(1-|z|^2\right)\left(\left|(\psi\varphi)'(z)\right|+\left|\psi'(z)\varphi(z)\right|\right)\log\frac{2}{1-|z|^2} \\
&\quad\le \frac{1}{1-(1-1/N)^2}\left(1-|z|^2\right)\left|(\psi\varphi)'(z)\right|\log\frac{2}{1-|z|^2} \\
&\qquad+\frac{1}{1-(1-1/N)^2}\left(1-|z|^2\right)\left|\psi'(z)\right|\log\frac{2}{1-|z|^2} \\
&\quad\longrightarrow 0, \qquad \text{as } |z|\longrightarrow 1.
\end{aligned} \tag{2.7}$$

If E_N is properly contained in $\mathbb{D}$, then for $z \in \mathbb{D} \setminus E_N$, arguing as in the proof of (b) $\Rightarrow$ (c) in Theorem 2.1, there exists $k \ge N$ such that $z \in \Delta_k$, so that

$$\begin{aligned}
\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} &\le \frac{\left(1-|z|^2\right)\left|\psi(z)k\varphi(z)^{k-1}\varphi'(z)\right|}{(1-|\varphi(z)|)k|\varphi(z)|^{k-1}}\log\frac{2}{1-|z|^2} \\
&\le e\left[\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|\left(\psi\varphi^k\right)'(z)\right|\right. \\
&\qquad\left.+\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|\psi'(z)\right|\right] \\
&= I+II,
\end{aligned} \tag{2.8}$$

where $I = e(1-|z|^2)\log(2/(1-|z|^2))|(\psi\varphi^k)'(z)|$ and $II = e(1-|z|^2)\log(2/(1-|z|^2))|\psi'(z)|$.

By the assumption of $\psi\varphi^n \in \mathcal{B}_{\log,0}$, we have

$$I \le e\sup_{n\ge k}\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|(\psi\varphi^n)'(z)\right| \longrightarrow 0 \tag{2.9}$$

as $|z| \to 1$. On the other hand, since $\psi \in \mathcal{B}_{\log,0}$, $II \to 0$ as $|z| \to 1$. Therefore,

$$\lim_{|z|\to 1}\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} = 0. \tag{2.10}$$

(c) $\Rightarrow$ (a) Assume that (c) holds. To prove that $W_{\psi,\varphi}$ is bounded, it suffices to show that $W_{\psi,\varphi}f \in \mathcal{B}_{\log,0}$ for each $f \in H^\infty$, since the boundedness of the operator can be shown as in the proof of Theorem 2.1. Since $\psi \in \mathcal{B}_{\log,0}$, for $f \in H^\infty$ and $z \in \mathbb{D}$, we have

$$\left(1-|z|^2\right)\left|\psi'(z)f(\varphi(z))\right|\log\frac{2}{1-|z|^2} \le \left(1-|z|^2\right)\left|\psi'(z)\right|\log\frac{2}{1-|z|^2}\|f\|_\infty \longrightarrow 0, \tag{2.11}$$

as $|z| \to 1$. On the other hand, by (2.6),

$$\left(1-|z|^2\right)\left|\psi(z)f'(\varphi(z))\varphi'(z)\right|\log\frac{2}{1-|z|^2} \le \frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2}\|f\|_\infty \longrightarrow 0, \tag{2.12}$$

as $|z| \to 1$. Hence,

$$\left(1-|z|^2\right)\left|\left(W_{\psi,\varphi}f\right)'(z)\right|\log\frac{2}{1-|z|^2} \longrightarrow 0 \tag{2.13}$$

as $|z| \to 1$, completing the proof. □

In Section 3, we shall prove that all bounded weighted composition operators from H^∞ into $\mathcal{B}_{\log,0}$ are compact.

3. Compactness of $W_{\psi,\varphi}$ from H^∞ into $\mathcal{B}_{\log}$ and $\mathcal{B}_{\log,0}$

The following criterion for compactness follows by a standard argument similar, for example, to that outlined in Proposition 3.11 of [11].

Lemma 3.1. *Let ψ be analytic on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, $1 \le p \le \infty$. The operator $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in\mathbb{N}}$ in H^p which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$.*

The proof of the following result is similar to the proof of Lemma 1 of [12]. Hence we omit it.

Lemma 3.2. *A closed set K in $\mathcal{B}_{\log,0}$ is compact if and only if it is bounded and satisfies the following:*

$$\lim_{|z|\to 1}\sup_{f\in K}\left(1-|z|^2\right)\left|f'(z)\right|\log\frac{2}{1-|z|^2} = 0. \tag{3.1}$$

We now introduce two one-parameter families of functions which will be used to characterize the compactness of the operators under consideration.

Fix $a \in \mathbb{D}$ and, for $z \in \mathbb{D}$, define

$$f_a(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{1/2}, \qquad g_a(z) = \frac{1-|a|^2}{1-\overline{a}z}. \tag{3.2}$$

Theorem 3.3. *Let ψ be analytic on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, and assume that $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log}$ is bounded. Then the following conditions are equivalent:*

(a) $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log}$ *is compact.*

(b) $\lim_{|\varphi(w)|\to 1}\|W_{\psi,\varphi} f_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$ *and* $\lim_{|\varphi(w)|\to 1}\|W_{\psi,\varphi} g_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$.

(c) $\lim_{|\varphi(w)|\to 1}((1-|w|^2)|\psi(w)\varphi'(w)|/1-|\varphi(w)|^2)\log(2/(1-|w|^2)) = 0$ *and* $\lim_{|\varphi(w)|\to 1}(1-|w|^2)|\psi'(w)|\log(2/(1-|w|^2)) = 0$.

(d) $\lim_{n\to\infty}\|\psi\varphi^n\|_{\mathcal{B}_{\log}} = 0$ *and* $\lim_{|\varphi(w)|\to 1}(1-|w|^2)|\psi'(w)|\log(2/(1-|w|^2)) = 0$.

Proof. We begin by showing that (a), (b), and (c) are equivalent.

(a) $\Rightarrow$ (b) Suppose that $W_{\psi,\varphi}$ is compact and that $\{w_n\}$ is a sequence in $\mathbb{D}$ such that $|\varphi(w_n)| \to 1$ as $n \to \infty$. Since the sequences $\{f_{\varphi(w_n)}\}$ and $\{g_{\varphi(w_n)}\}$ are bounded in H^∞ and converge to 0 uniformly on compact subsets of $\mathbb{D}$, by Lemma 3.1, it follows that $\|W_{\psi,\varphi} f_{\varphi(w_n)}\|_{\mathcal{B}_{\log}} \to 0$ and $\|W_{\psi,\varphi} g_{\varphi(w_n)}\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$.

(b) $\Rightarrow$ (c) Assume that (b) holds. Fix $w \in \mathbb{D}$. A straightforward calculation shows that

$$\begin{aligned}
\left(\psi\left(f_{\varphi(w)} \circ \varphi\right)\right)'(w) &= \psi'(w) + \frac{\psi(w)\overline{\varphi(w)}\varphi'(w)}{2\left(1-\left|\varphi(w)\right|^2\right)}, \\
\left(\psi\left(g_{\varphi(w)} \circ \varphi\right)\right)'(w) &= \psi'(w) + \frac{\psi(w)\overline{\varphi(w)}\varphi'(w)}{1-\left|\varphi(w)\right|^2}.
\end{aligned} \tag{3.3}$$

Eliminating $\psi'(w)$, we obtain that

$$\frac{\psi(w)\overline{\varphi(w)}\varphi'(w)}{2\left(1-\left|\varphi(w)\right|^2\right)} = \left(\psi\left(g_{\varphi(w)} \circ \varphi\right)\right)'(w) - \left(\psi\left(f_{\varphi(w)} \circ \varphi\right)\right)'(w). \tag{3.4}$$

Thus, for $|\varphi(w)| > r \in (0,1)$,

$$\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2} \le \frac{2}{r}\left(\left\|W_{\psi,\varphi} f_{\varphi(w)}\right\|_{\mathcal{B}_{\log}} + \left\|W_{\psi,\varphi} g_{\varphi(w)}\right\|_{\mathcal{B}_{\log}}\right). \tag{3.5}$$

Taking the limit as $|\varphi(w)| \to 1$, we deduce that

$$\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2} \longrightarrow 0. \tag{3.6}$$

On the other hand, using (3.3), we obtain that

$$\left(1-|w|^2\right)\left|\psi'(w)\right|\log\frac{2}{1-|w|^2}\leq\left\|W_{\psi,\varphi}g_{\varphi(w)}\right\|_{\mathcal{B}_{\log}}+\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2}. \tag{3.7}$$

Taking the limit as $|\varphi(w)|\to 1$, we obtain that

$$\left(1-|w|^2\right)\left|\psi'(w)\right|\log\frac{2}{1-|w|^2}\longrightarrow 0, \tag{3.8}$$

proving (c).

(c) $\Rightarrow$ (a) Suppose that (c) holds. Let $\{f_n\}$ be a bounded sequence in H^∞ converging to 0 uniformly on compact subsets of $\mathbb{D}$. Set $C=\sup_{n\in\mathbb{N}}\|f_n\|_\infty$. Then, given $\varepsilon>0$, there exists $r\in(0,1)$ such that for $|\varphi(w)|>r$,

$$\begin{gathered}\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2}<\frac{\varepsilon}{2C},\\ \left(1-|w|^2\right)\left|\psi'(w)\right|\log\frac{2}{1-|w|^2}<\frac{\varepsilon}{2C}.\end{gathered} \tag{3.9}$$

Then, for $w\in\mathbb{D}$, noting that $(1-|\varphi(w)|^2)|f_n'(\varphi(w))|\leq\|f_n\|_\infty\leq C$, we obtain that

$$\begin{aligned}&\left(1-|w|^2\right)\left|\left(\psi\left(f_n\circ\varphi\right)\right)'(w)\right|\log\frac{2}{1-|w|^2}\\ &\quad\leq\left(1-|w|^2\right)\left(\left|\psi'(w)f_n\left(\varphi(w)\right)\right|+\left|\psi(w)f_n'\left(\varphi(w)\right)\varphi'(w)\right|\right)\log\frac{2}{1-|w|^2}\\ &\quad\leq\left(1-|w|^2\right)\left|\psi'(w)\right|\left\|f_n\right\|_\infty\log\frac{2}{1-|w|^2}\\ &\qquad+\frac{\left(1-|w|^2\right)\left(1-\left|\varphi(w)\right|^2\right)\left|\psi(w)f_n'\left(\varphi(w)\right)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2}\\ &\quad\leq C\left[\left(1-|w|^2\right)\left|\psi'(w)\right|+\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\right]\log\frac{2}{1-|w|^2}.\end{aligned} \tag{3.10}$$

Thus, for $|\varphi(w)| > r$, we have

$$\left(1-|w|^2\right)\left|(\psi(f_n\circ\varphi))'(w)\right|\log\frac{2}{1-|w|^2}<\varepsilon. \tag{3.11}$$

On the other hand, for $|\varphi(w)| \le r$,

$$\left(1-|w|^2\right)\left|(\psi(f_n\circ\varphi))'(w)\right|\log\frac{2}{1-|w|^2}\le \|\psi\|\left|f_n(\varphi(w))\right|+\sigma_{\psi,\varphi}\left|f_n'(\varphi(w))\right|. \tag{3.12}$$

Thus, by the uniform convergence to 0 of f_n and f_n' on compact sets, we see that (3.11) holds also in this case for n sufficiently large. Hence, $\|\psi(f_n\circ\varphi)\| \le \varepsilon$ for n sufficiently large. Since $|\psi(0)f_n(\varphi(0))| \to 0$, we deduce that $\|\psi(f_n\circ\varphi)\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$. Therefore, $W_{\psi,\varphi}$ is compact.

(a) $\Rightarrow$ (d) Suppose that $W_{\psi,\varphi}$ is compact. Since the sequence $\{p_n\}$ defined by $p_n(z) = z^n$, $z \in \mathbb{D}$ is bounded in H^∞ and converges to 0 uniformly on compact subsets, by Lemma 3.1, it follows that $\|\psi\varphi^n\|_{\mathcal{B}_{\log}} = \|W_{\psi,\varphi}p_n\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$. The second condition in (d) follows from the equivalence of (a) and (c).

(d) $\Rightarrow$ (c) Fix $\varepsilon > 0$ and choose $N > 2$ such that $\|\psi\varphi^n\|_{\mathcal{B}_{\log}} < \varepsilon/2$ for all $n \ge N$ and $(1-|w|^2)|\psi'(w)|\log(2/(1-w|^2)) < \varepsilon/2$ for $|\varphi(w)| > 1-1/N$.

For $|\varphi(w)| > 1-1/N$, there exists $n > N$ such that $z \in \Delta_n$. Using the product rule, we may write the following

$$\psi(w)n\varphi(w)^{n-1}\varphi'(w) = \left(\psi\varphi^n\right)'(w)-\psi'(w)\varphi(w)^n, \tag{3.13}$$

so that

$$\begin{aligned}&\left(1-|w|^2\right)\left|\psi(w)n\varphi(w)^{n-1}\varphi'(w)\right|\log\frac{2}{1-|w|^2}\\&\quad\le \left\|\psi\varphi^n\right\|_{\mathcal{B}_{\log}}+\left(1-|w|^2\right)\left|\psi'(w)\varphi(w)^n\right|\log\frac{2}{1-|w|^2}.\end{aligned} \tag{3.14}$$

The left-hand side of (3.14) can be written as

$$\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\left(1-\left|\varphi(w)\right|^2\right)n\left|\varphi(w)\right|^{n-1}\log\frac{2}{1-|w|^2}, \tag{3.15}$$

which, by (2.3), is bounded below by

$$\frac{1}{e}\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2}. \tag{3.16}$$

Thus, from (3.14), we deduce that

$$\begin{aligned}\frac{1}{e}\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{1-\left|\varphi(w)\right|^2}\log\frac{2}{1-|w|^2}\\ \leq \left\|\psi\varphi^n\right\|_{\mathcal{B}_{\log}}+\left(1-|w|^2\right)\left|\psi'(w)\right|\log\frac{2}{1-|w|^2}<\varepsilon,\end{aligned}\tag{3.17}$$

proving (c). The equivalence of statements (a)–(d) is now established. □

Next, we characterize the compact weighted composition operators from H^∞ into $\mathcal{B}_{\log,0}$.

Theorem 3.4. *Let ψ be an analytic function on $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*

(a) *The operator $W_{\psi,\varphi}: H^\infty \to \mathcal{B}_{\log,0}$ is compact.*

(b) *For each integer $n \geq 0$, $\psi\varphi^n \in \mathcal{B}_{\log,0}$ and $\lim_{n\to\infty}\|\psi\varphi^n\|_{\mathcal{B}_{\log}} = 0$.*

(c) *$\psi \in \mathcal{B}_{\log,0}$ and*

$$\lim_{|z|\to 1}\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2}=0.\tag{3.18}$$

Proof. (a) ⇒ (b) is immediate.

(b) ⇒ (c) It suffices to show that if (b) holds, then

$$\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2}\longrightarrow 0\tag{3.19}$$

as $|z| \to 1$. Fix $\varepsilon > 0$ and let $N > 2$ be an integer such that $\|\psi\varphi^n\|_{\mathcal{B}_{\log}} < \varepsilon/e$ for all $n \geq N$. Observe that, since for $z \in \mathbb{D}$,

$$\left|\psi(z)\varphi'(z)\right| \leq \left|(\psi\varphi)'(z)\right| + \left|\psi'(z)\varphi(z)\right| \leq \left|(\psi\varphi)'(z)\right| + \left|\psi'(z)\right|,\tag{3.20}$$

and, by assumption, the functions $\psi\varphi$ and ψ are in $\mathcal{B}_{\log,0}$, we have

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|\log\frac{2}{1-|z|^2}=0.\tag{3.21}$$

Therefore, there is $r \in (0,1)$, such that for $|z| > r$,

$$\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|\log\frac{2}{1-|z|^2}<\frac{(2N-1)\varepsilon}{N^2}.\tag{3.22}$$

Thus, if $z \in E_N$, and $|z| > r$, then

$$\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} \le \frac{N^2}{2N-1}\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|\log\frac{2}{1-|z|^2} < \varepsilon. \tag{3.23}$$

On the other hand, if $z \notin E_N$, then there exists $n > N$ such that $z \in \Delta_n$, so, as shown in the proof of (d) implies (c) of Theorem 3.3, we have

$$\begin{aligned}\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} &\le e\left[\left\|\psi\varphi^n\right\|_{\mathcal{B}_{\log}} + \left(1-|z|^2\right)\left|\psi'(z)\right|\log\frac{2}{1-|z|^2}\right] \\ &< \varepsilon + e\left(1-|z|^2\right)\left|\psi'(z)\right|\log\frac{2}{1-|z|^2} \longrightarrow \varepsilon,\end{aligned} \tag{3.24}$$

as $|z| \to 1$. Since ε is arbitrary, the result follows.

(c) $\Rightarrow$ (a) Let $\{f_n\}$ be a bounded sequence in H^∞ converging to 0 uniformly on compact subsets, and let $C = \sup_{n\in\mathbb{N}}\|f_n\|_\infty$. We wish to show that $W_{\psi,\varphi}f_n \in \mathcal{B}_{\log,0}$ and $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$. As shown in the proof of (c) implies (a) of Theorem 3.3, for $z \in \mathbb{D}$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left(1-|z|^2\right)\left|\left(W_{\psi,\varphi}f_n\right)'(z)\right|\log\frac{2}{1-|z|^2} &\le C\left(1-|z|^2\right)\left|\psi'(z)\right|\log\frac{2}{1-|z|^2} \\ &\quad + C\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{1-\left|\varphi(z)\right|^2}\log\frac{2}{1-|z|^2} \longrightarrow 0,\end{aligned} \tag{3.25}$$

as $|z| \to 1$. Thus, $W_{\psi,\varphi}f_n \in \mathcal{B}_{\log,0}$. The convergence to 0 of $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_{\log}}$ is proved as in the case of the operator mapping into $\mathcal{B}_{\log}$. □

From Theorems 2.2 and 3.4, we obtain the following result.

Corollary 3.5. *Let ψ be an analytic function on $\mathbb{D}$, and let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*

(a) *The operator $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log,0}$ is bounded.*

(b) *The operator $W_{\psi,\varphi} : H^\infty \to \mathcal{B}_{\log,0}$ is compact.*

(c) *For each integer $n \ge 0$, $\psi\varphi^n \in \mathcal{B}_{\log,0}$ and $\lim_{n\to\infty}\|\psi\varphi^n\|_{\mathcal{B}_{\log}} = 0$.*

(d) *$\psi \in \mathcal{B}_{\log,0}$ and $\lim_{|z|\to 1}((1-|z|^2)|\psi(z)\varphi'(z)|/(1-|\varphi(z)|^2))\log(2/(1-|z|^2)) = 0$.*

In the special cases when φ is the identity, respectively, ψ is identically 1, we obtain the following results.

Corollary 3.6. *Let ψ be analytic on $\mathbb{D}$. The following statements are equivalent:*

(a) $M_\psi : H^\infty \to \mathcal{B}_{\log}$ *is bounded,*

(b) $M_\psi : H^\infty \to \mathcal{B}_{\log,0}$ *is bounded,*

(c) ψ *is identically 0.*

Corollary 3.7. *Let φ be an analytic self map of $\mathbb{D}$. Then the following statements are equivalent:*

(a) $C_\varphi : H^\infty \to \mathcal{B}_{\log}$ *is bounded,*

(b) $\sup_{n\in\mathbb{N}}\|\varphi^n\|_{\mathcal{B}_{\log}} < \infty$,

(c) $\sup_{z\in\mathbb{D}}((1-|z|^2)|\varphi'(z)|/(1-|\varphi(z)|^2))\log(2/(1-|z|^2)) < \infty$.

Corollary 3.8. *Let φ be an analytic self map of $\mathbb{D}$. Then the following statements are equivalent:*

(a) $C_\varphi : H^\infty \to \mathcal{B}_{\log}$ *is compact,*

(b) $\lim_{n\to\infty}\|\varphi^n\|_{\mathcal{B}_{\log}} = 0$,

(c) $\lim_{|\varphi(z)|\to 1}((1-|z|^2)|\varphi'(z)|/(1-|\varphi(z)|^2))\log(2/(1-|z|^2)) = 0$.

Corollary 3.9. *Let φ be an analytic self map of $\mathbb{D}$. Then the following statements are equivalent:*

(a) $C_\varphi : H^\infty \to \mathcal{B}_{\log,0}$ *is bounded,*

(b) $C_\varphi : H^\infty \to \mathcal{B}_{\log,0}$ *is compact,*

(c) $\varphi \in \mathcal{B}_{\log,0}$ *and* $\lim_{n\to\infty}\|\varphi^n\|_{\mathcal{B}_{\log}} = 0$,

(d) $\lim_{|z|\to 1}((1-|z|^2)|\varphi'(z)|/(1-|\varphi(z)|^2))\log(2/(1-|z|^2)) = 0$.

4. $W_{\psi,\varphi}$ from H^p $(1 \le p < \infty)$ into $\mathcal{B}_{\log}$

We begin this section with two useful point evaluation estimates that will be needed to prove our results.

Lemma 4.1 (See [11]). *Let $0 < p < \infty$. Then for any $f \in H^p$, $z \in \mathbb{D}$,*

$$|f(z)| \le \frac{\|f\|_{H^p}}{\left(1-|z|^2\right)^{1/p}}. \tag{4.1}$$

Lemma 4.2 (See [13]). *Let $0 < p < \infty$. Then for any $f \in H^p$, $z \in \mathbb{D}$,*

$$|f'(z)| \le C\frac{\|f\|_{H^p}}{\left(1-|z|^2\right)^{1+1/p}}. \tag{4.2}$$

Fix $1 \le p < \infty$ and $a \in \mathbb{D}$. For $z \in \mathbb{D}$, define the functions

$$f_a(z) = \frac{\left(1-|a|^2\right)^{2-1/p}}{(1-\overline{a}z)^2}, \qquad g_a(z) = \frac{\left(1-|a|^2\right)^2}{(1-\overline{a}z)^{2+1/p}}. \tag{4.3}$$

Then $f_a, g_a \in H^p$ and the norms $\|f_a\|_{H^p}$ and $\|g_a\|_{H^p}$ are bounded by constants only dependent of p. In addition, a straightforward calculation shows that

$$\begin{aligned} f_a(a) &= g_a(a) = \frac{1}{\left(1-|a|^2\right)^{1/p}}, \\ f_a'(a) &= \frac{2\overline{a}}{\left(1-|a|^2\right)^{1+1/p}}, \\ g_a'(a) &= \frac{(2+1/p)\overline{a}}{\left(1-|a|^2\right)^{1+1/p}}. \end{aligned} \tag{4.4}$$

We use these two families of functions to characterize the bounded and the compact weighted composition operators from H^p to $\mathcal{B}_{\log}$.

Theorem 4.3. *Let $1 \le p < \infty$, ψ analytic on $\mathbb{D}$ and let φ be an analytic self-map of $\mathbb{D}$. Then the following conditions are equivalent:*

(a) $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ *is bounded,*

(b) $\psi\varphi \in \mathcal{B}_{\log}$, $A = \sup_{w\in\mathbb{D}} \|W_{\psi,\varphi} f_{\varphi(w)}\|_{\mathcal{B}_{\log}} < \infty$ *and* $B = \sup_{w\in\mathbb{D}} \|W_{\psi,\varphi} g_{\varphi(w)}\|_{\mathcal{B}_{\log}} < \infty$,

(c)

$$\begin{aligned} x_{\psi,\varphi} &= \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|\psi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1/p}} \log\frac{2}{1-|z|^2} < \infty, \\ y_{\psi,\varphi} &= \sup_{z\in\mathbb{D}} \frac{\left(1-|z|^2\right)|\psi(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^{1+1/p}} \log\frac{2}{1-|z|^2} < \infty. \end{aligned} \tag{4.5}$$

Proof. (a) $\Rightarrow$ (b) Assume that $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is bounded. Then $\psi\varphi \in \mathcal{B}_{\log}$ and for each $w \in \mathbb{D}$,

$$\begin{aligned} \left\|W_{\psi,\varphi} f_{\varphi(w)}\right\|_{\mathcal{B}_{\log}} &\le \|W_{\psi,\varphi}\| \left\|f_{\varphi(w)}\right\|_{H^p} \le C\|W_{\psi,\varphi}\|, \\ \left\|W_{\psi,\varphi} g_{\varphi(w)}\right\|_{\mathcal{B}_{\log}} &\le \|W_{\psi,\varphi}\| \left\|g_{\varphi(w)}\right\|_{H^p} \le C\|W_{\psi,\varphi}\|, \end{aligned} \tag{4.6}$$

for some constant C, so A and B are finite.

(b) $\Rightarrow$ (c) Suppose that $\psi\varphi \in \mathcal{B}_{\log}$, and the quantities A and B are finite. From (4.4), for $w \in \mathbb{D}$, we have

$$\left(\psi\left(f_{\varphi(w)} \circ \varphi\right)\right)'(w) = \frac{\psi'(w)}{\left(1-|\varphi(w)|^2\right)^{1/p}} + \frac{2\psi(w)\varphi'(w)\overline{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}, \tag{4.7}$$

whence

$$\begin{aligned}
&\frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}}\log\frac{2}{1-|w|^2}\\
&\quad\le\left(1-|w|^2\right)\log\frac{2}{1-|w|^2}\left|\left(\psi\left(f_{\varphi(w)}\circ\varphi\right)\right)'(w)\right|+2\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2}\\
&\quad\le\left\|W_{\psi,\varphi}f_{\varphi(w)}\right\|_{\mathcal{B}_{\log}}+2\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2}\\
&\quad\le A+2\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2}.
\end{aligned}\tag{4.8}$$

Moreover,

$$\left(\psi\left(g_{\varphi(w)}\circ\varphi\right)\right)'(w)=\frac{\psi'(w)}{\left(1-|\varphi(w)|^2\right)^{1/p}}+\frac{(2+1/p)\psi(w)\varphi'(w)\overline{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}.\tag{4.9}$$

Therefore, subtracting (4.7) from (4.9) and taking the modulus, we obtain

$$\frac{1}{p}\frac{|\psi(w)\varphi'(w)\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\le\left|\left(\psi\left(f_{\varphi(w)}\circ\varphi\right)\right)'(w)\right|+\left|\left(\psi\left(g_{\varphi(w)}\circ\varphi\right)\right)'(w)\right|,\tag{4.10}$$

which yields that

$$\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)\varphi(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2}\le p(A+B).\tag{4.11}$$

Consequently, from (4.8), we deduce that

$$\frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}}\log\frac{2}{1-|w|^2}\le(1+2p)A+2pB.\tag{4.12}$$

Taking the supremum over all $w\in\mathbb{D}$, we see that $x_{\psi,\varphi}$ is finite.

Fix $r \in (0,1)$. If $|\varphi(w)| > r$, then from (4.11) we have

$$\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{\left(1-\left|\varphi(w)\right|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2} < \frac{p(A+B)}{r}. \tag{4.13}$$

On the other hand, since $\psi\varphi \in \mathcal{B}_{\log}$, if $|\varphi(w)| \le r$, then,

$$\begin{aligned}\frac{\left(1-|w|^2\right)\left|\psi(w)\varphi'(w)\right|}{\left(1-\left|\varphi(w)\right|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2} &\le \frac{\left(1-|w|^2\right)\left|(\psi\varphi)'(w)\right|}{(1-r^2)^{1+1/p}}\log\frac{2}{1-|w|^2} \\ &\quad + \frac{\left(1-|w|^2\right)\left|\psi'(w)\varphi(w)\right|}{(1-r^2)\left(1-\left|\varphi(w)\right|^2\right)^{1/p}}\log\frac{2}{1-|w|^2} \\ &\le \frac{\|\psi\varphi\|}{(1-r^2)^{1+1/p}} + \frac{x_{\psi,\varphi}}{1-r^2} < \infty.\end{aligned} \tag{4.14}$$

Taking the supremum over all $w \in \mathbb{D}$, it follows that $y_{\psi,\varphi}$ is finite as well.

(c) $\Rightarrow$ (a) Suppose that $x_{\psi,\varphi}$ and $y_{\psi,\varphi}$ are finite. For arbitrary z in $\mathbb{D}$ and $f \in H^p$, by Lemmas 4.1 and 4.2, we have

$$\begin{aligned}&\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|\left(W_{\psi,\varphi}f\right)'(z)\right| \\ &\quad\le \left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left(\left|\psi'(z)\right|\left|f(\varphi(z))\right| + \left|f'(\varphi(z))\right|\left|\psi(z)\varphi'(z)\right|\right) \\ &\quad\le \left(\frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}}\log\frac{2}{1-|z|^2} + C\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}}\log\frac{2}{1-|z|^2}\right)\|f\|_{H^p} \\ &\quad\le \left(x_{\psi,\varphi} + Cy_{\psi,\varphi}\right)\|f\|_{H^p}.\end{aligned} \tag{4.15}$$

Taking the supremum over all $z \in \mathbb{D}$ and applying Lemma 4.1, we obtain that

$$\begin{aligned}\|W_{\psi,\varphi}f\|_{\mathcal{B}_{\log}} &= \left|\psi(0)f(\varphi(0))\right| + \|W_{\psi,\varphi}f\| \\ &\le \left(\frac{|\psi(0)|}{\left(1-\left|\varphi(0)\right|^2\right)^{1/p}} + x_{\psi,\varphi} + Cy_{\psi,\varphi}\right)\|f\|_{H^p}.\end{aligned} \tag{4.16}$$

The boundedness of the operator $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ follows by taking the supremum over all $f \in H^p$. □

Theorem 4.4. *Let $1 \le p < \infty$, ψ analytic on $\mathbb{D}$, φ an analytic self-map of $\mathbb{D}$, and assume that $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is bounded. Then the following conditions are equivalent:*

(a) *$W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is compact,*

(b) *$\lim_{|\varphi(w)|\to 1}\|W_{\psi,\varphi} f_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$ and $\lim_{|\varphi(w)|\to 1}\|W_{\psi,\varphi} g_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$,*

(c)

$$\lim_{|\varphi(w)|\to 1} \frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}} \log\frac{2}{1-|w|^2} = 0,$$
$$\lim_{|\varphi(w)|\to 1} \frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}} \log\frac{2}{1-|w|^2} = 0. \tag{4.17}$$

Proof. (a) $\Rightarrow$ (b) Suppose that $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is compact. Let $\{w_n\}$ be a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty}|\varphi(w_n)| = 1$. Observe that the sequences $\{f_{\varphi(w_n)}\}$ and $\{g_{\varphi(w_n)}\}$ are bounded in H^p and converge to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 3.1, it follows that $\|W_{\psi,\varphi} f_{\varphi(w_n)}\|_{\mathcal{B}_{\log}} \to 0$ and $\|W_{\psi,\varphi} g_{\varphi(w_n)}\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$, proving (b).

(b) $\Rightarrow$ (c) Assume that the limits in (b) are 0. Using the inequality (4.10), we obtain that

$$\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}} \log\frac{2}{1-|w|^2} \le \frac{p\left(\left\|W_{\psi,\varphi} f_{\varphi(w)}\right\|_{\mathcal{B}_{\log}} + \left\|W_{\psi,\varphi} g_{\varphi(w)}\right\|_{\mathcal{B}_{\log}}\right)}{|\varphi(w)|} \longrightarrow 0 \tag{4.18}$$

as $|\varphi(w)| \to 1$. Moreover, using (4.8), we deduce that

$$\frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}} \log\frac{2}{1-|w|^2} \longrightarrow 0 \tag{4.19}$$

as $|\varphi(w)| \to 1$.

(c) $\Rightarrow$ (a) Suppose that (c) holds. Let $\{f_n\}$ be a bounded sequence in H^p converging to 0 uniformly on compact subsets of $\mathbb{D}$. Set $C = \sup_{n\in\mathbb{N}}\|f_n\|_{H^p}$. Then, given $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}} \log\frac{2}{1-|w|^2} < \frac{\varepsilon}{2C},$$
$$\frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}} \log\frac{2}{1-|w|^2} < \frac{\varepsilon}{2C}, \tag{4.20}$$

for $|\varphi(w)| > r$. Therefore, again by Lemmas 4.1 and 4.2, and (4.20), for $|\varphi(w)| > r$, we have

$$\begin{aligned}
&\left(1-|w|^2\right)\log\frac{2}{1-|w|^2}\left|\left(\psi\left(f_n\circ\varphi\right)\right)'(w)\right| \\
&\quad\le \left(1-|w|^2\right)\log\frac{2}{1-|w|^2}\left(\left|\psi'(w)f_n(\varphi(w))\right|+\left|\psi(w)f_n'(\varphi(w))\varphi'(w)\right|\right) \\
&\quad\le \|f_n\|_{H^p}\left(\frac{\left(1-|w|^2\right)|\psi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1/p}}\log\frac{2}{1-|w|^2}+\frac{\left(1-|w|^2\right)|\psi(w)\varphi'(w)|}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}\log\frac{2}{1-|w|^2}\right) \\
&\quad< \varepsilon.
\end{aligned} \tag{4.21}$$

On the other hand, for $|\varphi(w)| \le r$, by the uniform convergence to 0 of f_n and f_n' on compact sets, we have

$$\left(1-|w|^2\right)\log\frac{2}{1-|w|^2}\left|\left(\psi\left(f_n\circ\varphi\right)\right)'(w)\right| \le x_{\psi,\varphi}\left|f_n(\varphi(w))\right|+y_{\psi,\varphi}\left|f'(\varphi(w))\right| \longrightarrow 0, \tag{4.22}$$

as $n \to \infty$. Since $|\psi(0)f_n(\varphi(0))| \to 0$, we conclude that $\|W_{\psi,\varphi}f_n\|_{\mathcal{B}_{\log}} \to 0$ as $n \to \infty$. Consequently, by Lemma 3.1, the operator $W_{\psi,\varphi}$ is compact. □

As a consequence of Theorems 4.3 and 4.4, noting that for $w \in \mathbb{D}$,

$$\left(C_\varphi g_{\varphi(w)}\right)'(w) = \frac{1}{2}\left(2+\frac{1}{p}\right)\left(C_\varphi f_{\varphi(w)}\right)'(w) = \frac{(2+1/p)\varphi'(w)\overline{\varphi(w)}}{\left(1-|\varphi(w)|^2\right)^{1+1/p}}, \tag{4.23}$$

we obtain the following characterizations of the bounded and the compact composition operators from H^p into $\mathcal{B}_{\log}$.

Corollary 4.5. *Let φ be an analytic self-map of $\mathbb{D}$, and $1 \le p < \infty$. The following statements are equivalent:*

(a) $C_\varphi : H^p \to \mathcal{B}_{\log}$ *is bounded,*

(b) $\sup_{w\in\mathbb{D}}\|C_\varphi f_{\varphi(w)}\|_{\mathcal{B}_{\log}} < \infty$,

(c) $\sup_{w\in\mathbb{D}}\|C_\varphi g_{\varphi(w)}\|_{\mathcal{B}_{\log}} < \infty$,

(d) $\sup_{w\in\mathbb{D}}((1-|w|^2)|\varphi'(w)|/(1-|\varphi(w)|^2)^{1+1/p})\log(2/(1-|w|^2)) < \infty$.

Corollary 4.6. *Let φ be an analytic self-map of $\mathbb{D}$, and $1 \le p < \infty$. If $C_\varphi : H^p \to \mathcal{B}_{\log}$ is bounded, then the following statements are equivalent:*

(a) $C_\varphi : H^p \to \mathcal{B}_{\log}$ *is compact,*

(b) $\lim_{|\varphi(w)|\to 1}\|C_\varphi f_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$,

(c) $\lim_{|\varphi(w)|\to 1}\|C_\varphi g_{\varphi(w)}\|_{\mathcal{B}_{\log}} = 0$,

(d) $\lim_{|\varphi(w)|\to 1}((1-|w|^2)|\varphi'(w)|/(1-|\varphi(w)|^2)^{1+1/p})\log(2/(1-|w|^2)) = 0$.

5. $W_{\psi,\varphi}$ from H^p $(1 \le p < \infty)$ into $\mathcal{B}_{\log,0}$

In this section, we characterize the boundedness and the compactness of the weighted composition operators $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log,0}$. Arguing as in the proof of Lemma 4.2 of [14], we easily get the following two lemmas.

Lemma 5.1. *Suppose that φ is an analytic self-map of the unit disk, ψ analytic on $\mathbb{D}$, and $1 \le p < \infty$. Then,*

$$\lim_{|z|\to 1}\frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}}\log\frac{2}{1-|z|^2} = 0 \tag{5.1}$$

if and only if $\psi \in \mathcal{B}_{\log,0}$ and

$$\lim_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}}\log\frac{2}{1-|z|^2} = 0. \tag{5.2}$$

Lemma 5.2. *Suppose that φ is an analytic self-map of the unit disk, ψ analytic on $\mathbb{D}$, and $1 \le p < \infty$. Then,*

$$\lim_{|z|\to 1}\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}}\log\frac{2}{1-|z|^2} = 0 \tag{5.3}$$

if and only if $\lim_{|z|\to 1}(1-|z|)^2\log(2/(1-|z|^2))|\psi(z)\varphi'(z)| = 0$ and

$$\lim_{|\varphi(z)|\to 1}\frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}}\log\frac{2}{1-|z|^2} = 0. \tag{5.4}$$

The proof of the following theorem is a straightforward adaptation of the proof of Theorem 4.4 in [14]. We omit the details.

Theorem 5.3. *Let φ be an analytic self-map of the unit disk, $\psi \in H(\mathbb{D})$, and $1 \le p < \infty$. Then, the operator $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log,0}$ is bounded if and only if $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is bounded, $\psi \in \mathcal{B}_{\log,0}$, and*

$$\lim_{|z|\to 1}\left(1-|z|^2\right)\log\frac{2}{1-|z|^2}\left|\psi(z)\varphi'(z)\right| = 0. \tag{5.5}$$

We are now ready to prove the main result of this section.

Theorem 5.4. *Suppose that φ is an analytic self-map of the unit disk, $\psi \in H(\mathbb{D})$, and $1 \le p < \infty$. Then, $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}} \log \frac{2}{1-|z|^2} = 0,$$
$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}} \log \frac{2}{1-|z|^2} = 0. \tag{5.6}$$

Proof. First, we assume that $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log,0}$ is compact, and hence bounded. By Theorem 5.3, it follows that $\psi \in \mathcal{B}_{\log,0}$ and

$$\lim_{|z|\to 1} (1-|z|)^2 \log \frac{2}{1-|z|^2} \left|\psi(z)\varphi'(z)\right| = 0. \tag{5.7}$$

Since by assumption $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log}$ is compact, from Theorem 4.4, we know that

$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}} \log \frac{2}{1-|z|^2} = 0,$$
$$\lim_{|\varphi(z)|\to 1} \frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}} \log \frac{2}{1-|z|^2} = 0. \tag{5.8}$$

Applying Lemmas 5.1 and 5.2 yields the desired result.

Conversely, suppose that conditions (5.6) hold. From the proof of Theorem 4.3, we have that

$$\begin{aligned}
&\left(1-|z|^2\right) \log \frac{2}{1-|z|^2} \left|\left(W_{\psi,\varphi} f\right)'(z)\right| \\
&\quad \le \left(\frac{\left(1-|z|^2\right)\left|\psi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1/p}} \log \frac{2}{1-|z|^2} + C \frac{\left(1-|z|^2\right)\left|\psi(z)\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}} \log \frac{2}{1-|z|^2} \right) \|f\|_{H^p}.
\end{aligned} \tag{5.9}$$

Taking the supremum over all $f \in H^p$ such that $\|f\|_{H^p} \le 1$, then letting $|z| \to 1$, we obtain that

$$\lim_{|z|\to 1} \sup_{\|f\|_{H^p}\le 1} \left(1-|z|^2\right) \log \frac{2}{1-|z|^2} \left|\left(W_{\psi,\varphi}(f)\right)'(z)\right| = 0, \tag{5.10}$$

from which, by Lemma 3.2, we deduce that the operator $W_{\psi,\varphi} : H^p \to \mathcal{B}_{\log,0}$ is compact. □

From Theorems 5.3 and 5.4, we obtain the following corollary.

Corollary 5.5. *Suppose that φ is an analytic self-map of the unit disk and $1 \leq p < \infty$. Then, the following statements hold.*

(i) $C_\varphi : H^p \to \mathcal{B}_{\log,0}$ *is bounded if and only if* $C_\varphi : H^p \to \mathcal{B}_{\log}$ *is bounded and* $\varphi \in \mathcal{B}_{\log,0}$.

(ii) $C_\varphi : H^p \to \mathcal{B}_{\log,0}$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left|\varphi'(z)\right|}{\left(1-\left|\varphi(z)\right|^2\right)^{1+1/p}} \log \frac{2}{1-|z|^2} = 0. \tag{5.11}$$

Acknowledgments

The authors wish to express their gratitude to the referees for their careful reading of the paper and for their helpful suggestions. S. Li is supported by the Guangdong Natural Science Foundation (no. 10451401501004305), The National Natural Science Foundation of China (no. 11001107), and the Foundation for Distinguished Young Talents in Higher Education of Guangdong, China (no. LYM11117).

References

[1] J. M. Anderson, J. Clunie, and Ch. Pommerenke, "On Bloch functions and normal functions," *Journal für die Reine und Angewandte Mathematik*, vol. 270, pp. 12–37, 1974.
[2] J. Arazy, *Multipliers of Bloch Functions*, vol. 54, University of Haifa Mathematics, 1982.
[3] L. Brown and A. L. Shields, "Multipliers and cyclic vectors in the Bloch space," *The Michigan Mathematical Journal*, vol. 38, no. 1, pp. 141–146, 1991.
[4] K. R. M. Attele, "Toeplitz and Hankel operators on Bergman one space," *Hokkaido Mathematical Journal*, vol. 21, no. 2, pp. 279–293, 1992.
[5] R. Yoneda, "The composition operators on weighted Bloch space," *Archiv der Mathematik*, vol. 78, no. 4, pp. 310–317, 2002.
[6] P. Galanopoulos, "On $B_{\log}$ to $Q^p_{\log}$ pullbacks," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 712–725, 2008.
[7] S. Ye, "A weighted composition operator on the logarithmic Bloch space," *Bulletin of the Korean Mathematical Society*, vol. 47, no. 3, pp. 527–540, 2010.
[8] S. Ohno and R. Zhao, "Weighted composition operators on the Bloch space," *Bulletin of the Australian Mathematical Society*, vol. 63, no. 2, pp. 177–185, 2001.
[9] S. Stević and R. P. Agarwal, "Weighted composition operators from logarithmic Bloch-type spaces to Bloch-type spaces," *Journal of Inequalities and Applications*, Article ID 964814, 21 pages, 2009.
[10] H. Wulan, D. Zheng, and K. Zhu, "Compact composition operators on BMOA and the Bloch space," *Proceedings of the American Mathematical Society*, vol. 137, no. 11, pp. 3861–3868, 2009.
[11] C. C. Cowen and B. D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, Fla, USA, 1995.
[12] K. Madigan and A. Matheson, "Compact composition operators on the Bloch space," *Transactions of the American Mathematical Society*, vol. 347, no. 7, pp. 2679–2687, 1995.
[13] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, NY, USA, 1981.
[14] S. Li and S. Stević, "Weighted composition operators from Bergman-type spaces into Bloch spaces," *Indian Academy of Sciences. Proceedings. Mathematical Sciences*, vol. 117, no. 3, pp. 371–385, 2007.