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Research Article

Invariant and Absolute Invariant Means of Double Sequences

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We examine some properties of the invariant mean, define the concepts of strong σ-convergence and absolute σ-convergence for double sequences, and determine the associated sublinear functionals. We also define the absolute invariant mean through which the space of absolutely σ-convergent double sequences is characterized.

1. Introduction and Preliminaries

For the following notions, we refer to [1, 2].

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if

$$\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty. \tag{1.1}$$

The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

A double sequence $x = (x_{jk})$ is said to *converge to the limit L in Pringsheim's sense* (shortly, *p-convergent to L*) if for every $\varepsilon > 0$ there exists an integer N such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case L is called the p-limit of x. If in addition $x \in \mathcal{M}_u$, then x is said to be *boundedly convergent to L in Pringsheim's sense* (shortly, *bp-convergent to L*).

A double sequence $x = (x_{jk})$ is said to *converge regularly to L* (shortly, *r-convergent to L*) if x is p-convergent and the limits $x_j := \lim_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that in this case the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the p-limit of x.

In general, for any notion of convergence ν, the space of all ν-convergent double sequences will be denoted by $\mathcal{C}_\nu$ and the limit of a ν-convergent double sequence x by ν-$\lim_{j,k} x_{jk}$, where $\nu \in \{p, bp, r\}$.

Let Ω denote the vector space of all double sequences with the vector space operations defined coordinatewise. Vector subspaces of Ω are called *double sequence spaces*.

All considered double sequence spaces are supposed to contain

$$\operatorname{span}\left\{\mathbf{e^{jk}} \mid j,\ k \in \mathbb{N}\right\}, \tag{1.2}$$

where

$$\mathbf{e^{jk}_{il}} = \begin{cases} 1, & \text{if } (j,k) = (i,\ell), \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

We denote the pointwise sums $\sum_{j,k} \mathbf{e^{jk}}$, $\sum_j \mathbf{e^{jk}}$ $(k \in \mathbb{N})$, and $\sum_k \mathbf{e^{jk}}$, $(j \in \mathbb{N})$ by $\mathbf{e}$, $\mathbf{e^k}$ and $\mathbf{e_j}$, respectively.

Let E be the space of double sequences converging with respect to a convergence notion ν, F a double sequence space, and $A = (a_{mnjk})$ a 4-dimensional matrix of scalars. Define the set

$$F_A^{(\nu)} := \left\{ x \in \Omega \mid [Ax]_{mn} := \nu\text{-}\sum_{j,k} a_{mnjk} x_{jk} \text{ exists and } Ax := ([Ax]_{mn})_{m,n} \in F \right\}. \tag{1.4}$$

Then we say that A maps the space E into the space F if $E \subset F_A^{(\nu)}$ and denote by (E, F) the set of all 4-dimensional matrices A which map E into F.

We say that a 4-dimensional matrix A is $\mathcal{C}_\nu$-*conservative* if $\mathcal{C}_\nu \subset \mathcal{C}_{\nu A}^{(\nu)}$, and $\mathcal{C}_\nu$-*regular* if in addition

$$\nu\text{-}\lim Ax := \nu\text{-}\lim_{m,n} [Ax]_{mn} = \nu\text{-}\lim_{m,n} x_{mn} \quad (x \in \mathcal{C}_\nu), \tag{1.5}$$

where

$$\mathcal{C}_{\nu A}^{(\nu)} := \left\{ x \in \Omega \mid [Ax]_{mn} := \nu\text{-}\sum_{j,k} a_{mnjk} x_{jk} \text{ exists and } Ax := ([Ax]_{mn})_{m,n} \in \mathcal{C}_\nu \right\}. \tag{1.6}$$

Matrix transformations for double sequences are considered by various authors, namely, [3–5].

Let σ be a one-to-one mapping from the set $\mathbb{N}_0 = \{0, 1, 2, \ldots.\}$ into itself. A continuous linear functional φ on l_∞ is said to be an *invariant mean* or a σ-*mean* (see [6, 7]) if and only if (i) $\varphi(x) \geq 0$ when the sequence $x = (x_k)$ has $x_k \geq 0$ for all k, (ii) $\varphi(e) = 1$, where $e = (1, 1, 1, \ldots)$, and (iii) $\varphi(x) = \varphi(x_{\sigma(k)})$ for all $x \in l_\infty$.

We say that a sequence $x = (x_k)$ is σ-*convergent* to the limit L if $\varphi(x) = L$ for all σ-means φ. We denote by V_σ the set of all σ-convergent sequences $x = (x_k)$. Clearly $c \subset V_\sigma$. Note that a σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits, that is to say, if and only if $\sigma^k(n) \neq n$, for all $n \geq 0,\ k \geq 1$ (see [8]).

Recently, the concept of invariant mean for double sequences was defined in [9].

Let σ be a one-to-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional φ_2 on $\mathcal{M}_u$ is said to be an *invariant mean* or a *σ-mean* if and only if (i) $\varphi_2(x) \geq 0$ if $x \geq 0$ (i.e., $x_{jk} \geq 0$ for all j,k); (ii) $\varphi_2(E) = 1$, where $E = (e_{jk})$, $e_{jk} = 1$ for all j,k, and (iii) $\varphi_2(x) = \varphi_2((x_{\sigma(j),\sigma(k)})) = \varphi_2((x_{\sigma(j),k})) = \varphi_2((x_{j,\sigma(k)}))$.

If $\sigma(n) = n+1$ then σ-mean is reduced to the Banach limit for double sequences [10].

The idea of σ-convergence for double sequences has recently been introduced in [11] and further studied in [9, 12–16].

A double sequence $x = (x_{jk})$ of real numbers is said to be *σ-convergent* to a number L if and only if $x \in \mathcal{V}_\sigma$, where

$$\begin{aligned}
&\mathcal{V}_\sigma = \left\{ x \in \mathcal{M}_u : \lim_{p,\,q\to\infty} \tau_{pqst}(x) = L \text{ uniformly in } s,t;\ L = \sigma - \lim x \right\},\\
&\tau_{pqst} := \tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{\sigma^j(s),\sigma^k(t)}.\\
&\tau_{0qst} := \tau_{0qst}(x) = \frac{1}{(q+1)} \sum_{k=0}^{q} x_{s,\sigma^k(t)},\\
&\tau_{p0st} := \tau_{p0st}(x) = \frac{1}{(p+1)} \sum_{j=0}^{p} x_{\sigma^j(s),t},
\end{aligned} \tag{1.7}$$

$\tau_{0,0,s,t} = x_{st}$ and $\tau_{-1,q,s,t} = \tau_{p,-1,s,t} = \tau_{-1,-1,s,t} = 0$.

Note that $\mathcal{C}_{bp} \subset \mathcal{V}_\sigma \subset \mathcal{M}_u$.

Throughout this paper limit of a double sequence means *bp*-limit.

For $\sigma(n) = n+1$, the set $\mathcal{V}_\sigma$ is reduced to the set f_2 of almost convergent double sequences [17]. A double sequence $x = (x_{jk})$ of real numbers is said to be *almost convergent* to a number L if and only if

$$\lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} x_{j+s,k+t} = L \text{ uniformly in } s,t. \tag{1.8}$$

The concept of almost convergence for single sequences was introduced by Lorentz [18].

Remark 1.1. In view of the following example, it may be remarked that this does not exclude the possibility that every boundedly convergent double sequence might have a uniquely determined σ-mean not necessarily equal to its *bp*-limit.

For example, let $\sigma(n) = 0$ for all n. Then it is easily seen that any bounded double sequence (and hence, in particular, any boundedly convergent double sequence) has σ-mean x_{00}.

In this paper we examine some properties of the invariant mean and define the concepts of absolute σ-convergence and strong σ-convergence for double sequences analogous to the case of single sequences [8, 19]. We further define the absolute invariant mean through which the space of absolutely σ-convergent double sequences is characterized.

2. Strong and Absolute σ-Convergence

In this section we define the concepts of strong σ-convergence and absolute σ-convergence for double sequences. These concepts for single sequences were studied in [8, 19–21].

Remark 2.1. In [9], it was shown that the sublinear functional V defined on $\mathcal{M}_u$ dominates and generates the σ-means, where $V : \mathcal{M}_u \to \mathbb{R}$ is defined by

$$V(x) = \inf_{p=(p_{jk})\in\mathcal{V}_{0\sigma}} \limsup_{j,k}(x_{jk} + p_{jk}). \tag{2.1}$$

Now we investigate the sublinear functional which generates the space $[\mathcal{V}_\sigma]$ of strongly σ-convergent double sequences defined in [22] as

$$[\mathcal{V}_\sigma] = \left\{ x = (x_{jk}) \in \mathcal{M}_u : \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} \left| x_{\sigma^j(s),\sigma^k(t)} - L \right| = 0, \text{ uniformly in } s,t \right\}. \tag{2.2}$$

Definition 2.2. We define $\Psi : \mathcal{M}_u \to \mathbb{R}$ by

$$\Psi(x) = \limsup_{p,q} \sup_{s,t} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} \left| x_{\sigma^j(s),\sigma^k(t)} \right|. \tag{2.3}$$

Let $\{\mathcal{M}_u, \Psi\}$ denote the set of all linear functionals Φ on $\mathcal{M}_u$ such that $\Phi(x) \le \Psi(x)$ for all $x = (x_{jk}) \in \mathcal{M}_u$. By Hahn-Banach Theorem, the set $\{\mathcal{M}_u, \Psi\}$ is nonempty.

If there exists $L \in \mathbb{R}$ such that

$$\Phi(x - L\mathbf{e}) = 0 \quad \forall \Phi \in \{\mathcal{M}_u, \Psi\}, \tag{$*$}$$

then we say that x is $\{\mathcal{M}_u, \Psi\}$-*convergent* to L and in this case we write $\{\mathcal{M}_u, \Psi\}$-lim $x = L$.

We are now ready to prove the following result.

Theorem 2.3. $[\mathcal{V}_\sigma]$ *is the set of all* $\{\mathcal{M}_u, \Psi\}$*-convergent sequences.*

Proof. Let $x \in [\mathcal{V}_\sigma]$. Then for each $\epsilon > 0$, there exist p_0, q_0 such that for $p > p_0, q > q_0$ and all s, t,

$$\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p}\sum_{k=0}^{q} \left| x_{\sigma^j(s),\sigma^k(t)} - L \right| < \epsilon, \tag{2.4}$$

and this implies that $\Psi(x - L\mathbf{e}) \leq \epsilon$. In a similar manner, we can prove that $\Psi(L\mathbf{e} - x) \leq \epsilon$. Hence $|\Phi(x - L\mathbf{e})| \leq \Psi(x - L\mathbf{e}) \leq \epsilon$ for all $\Phi \in \{\mathcal{M}_u, \Psi\}$. Therefore $\Phi(x - L\mathbf{e}) = 0$ for all $\Phi \in \{\mathcal{M}_u, \Psi\}$ and this implies that by (2.6) $x \in [\mathcal{V}_\sigma]$ implies that x is $\{\mathcal{M}_u, \Psi\}$-convergent.

Conversely, suppose that x is $\{\mathcal{M}_u, \Psi\}$-convergent, that is,

$$\Phi(x - L\mathbf{e}) = 0 \quad \forall \Phi \in \{\mathcal{M}_u, \Psi\}. \tag{2.5}$$

Since Ψ is sublinear functional on $\mathcal{M}_u$, by Hahn-Banach Theorem, there exists $\Phi_0 \in \{\mathcal{M}_u, \Psi\}$ such that $\Phi_0(x - L\mathbf{e}) = \Psi(x - L\mathbf{e})$. Hence $\Psi(x - L\mathbf{e}) = 0$; since $\Psi(x) = \Psi(-x)$, it follows that $x \in [\mathcal{V}_\sigma]$. This completes the proof of the theorem. □

Now we define the concept of absolute σ-convergence for double sequences.

Put

$$\phi_{pqst}(x) = \tau_{pqst}(x) - \tau_{p-1,q,s,t}(x) - \tau_{p,q-1,s,t}(x) + \tau_{p-1,q-1,s,t}(x). \tag{2.6}$$

Thus simplifying further, we get

$$\begin{aligned} \phi_{pqst}(x) &= \frac{1}{p(p+1)} \sum_{m=1}^{p} m \left[\frac{1}{q(q+1)} \sum_{n=1}^{q} n \left(x_{\sigma^m(s),\sigma^n(t)} - x_{\sigma^m(s),\sigma^{n-1}(t)} \right) \right] \\ &= \frac{1}{p(p+1)q(q+1)} \\ &\quad \times \sum_{m=1}^{p} \sum_{n=1}^{q} mn \left[x_{\sigma^m(s),\sigma^n(t)} - x_{\sigma^{m-1}(s),\sigma^n(t)} - x_{\sigma^m(s),\sigma^{n-1}(t)} + x_{\sigma^{m-1}(s),\sigma^{n-1}(t)} \right]. \end{aligned} \tag{2.7}$$

Now we write

$$\phi_{pqst}(x) = \begin{cases} \dfrac{1}{p(p+1)q(q+1)} \\ \quad \times \displaystyle\sum_{m=1}^{p} \sum_{n=1}^{q} mn \left[x_{\sigma^m(s),\sigma^n(t)} - x_{\sigma^{m-1}(s),\sigma^n(t)} - x_{\sigma^m(s),\sigma^{n-1}(t)} + x_{\sigma^{m-1}(s),\sigma^{n-1}(t)} \right], & p, q \geq 1, \\ \dfrac{1}{q(q+1)} \displaystyle\sum_{n=1}^{q} n \left[x_{s,\sigma^n(t)} - x_{s,\sigma^{n-1}(t)} \right], & p = 0, q \geq 1, \\ \dfrac{1}{p(p+1)} \displaystyle\sum_{n=1}^{p} m \left[x_{\sigma^m(s),t} - x_{\sigma^m(s),t} \right], & p \geq 1, q = 0, \end{cases} \tag{2.8}$$

and $\phi_{00st}(x) = x_{st}$.

In [9], the following was defined.

Definition 2.4. A double sequence $x = (x_{jk}) \in \mathcal{M}_u$ is said to be *absolutely σ-almost convergent* if and only if

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\phi_{pqst}(x)\right| \quad \text{converges uniformly in } s,t. \tag{2.9}$$

By $\mathcal{W}_\sigma$, we denote the space of all absolutely σ-almost convergent double sequences.

Now we define the following.

Definition 2.5. A double sequence $x = (x_{jk}) \in \mathcal{M}_u$ is said to be *absolutely σ-convergent* if and only if

(i) $\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|\phi_{pqst}(x)|$ converges uniformly in s,t;

(ii) $\lim_{p,q\to\infty}\tau_{pqst}(x)$, which must exist, should take the same value for all s,t.

By $\mathcal{BV}_\sigma$, we denote the space of all absolutely σ-convergent double sequences. It is easy to prove that both $\mathcal{W}_\sigma$ and $\mathcal{BV}_\sigma$ are Banach spaces with the norm

$$\|x\| = \sup_{s,t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\left|\phi_{pqst}(x)\right|. \tag{2.10}$$

Note that $\mathcal{BV}_\sigma \subset \mathcal{W}_\sigma \subset \mathcal{V}_\sigma$.

Remark 2.6. It is easy to see that the assertion (i) implies that $(\tau_{pqst}(x))$ (as a double sequence in p,q) converges uniformly in s,t, but it may converge to a different limit for different values of s,t. This point did not arise in Banach limit case in which $\sigma(n) = n+1$. In this case if we assume only that $\lim_{p,q\to\infty}\tau_{pqst}(x) = \ell$ for some value of s,t; then we must have $\lim_{p,q\to\infty}\tau_{pqst}(x) = \ell$ for any other s,t (but not necessarily uniformly in s,t). So if, as a special case, we assume uniform convergence, the value to $\tau_{pqst}(x)$ converges must be same for all s,t. This need not be in the general case. For example, consider $\sigma(n) = n+2$. Define the sequence $x = (x_{jk})$ by

$$x_{jk} = \begin{cases} 1, & \text{if } j \text{ is odd, } \forall k, \\ 0, & \text{if } j \text{ is even, } \forall k. \end{cases} \tag{2.11}$$

Then for all $p,q \geq 0$

$$\tau_{pqst}(x) = \begin{cases} 1, & \text{if } s \text{ is odd, } \forall t, \\ 0, & \text{if } s \text{ is even, } \forall t, \\ 0, & \text{otherwise,} \end{cases} \tag{2.12}$$

so that $\phi_{pqst}(x) = 0$ for all $p,q \geq 1$ (in particular, $\phi_{1111}(x) = x_{\sigma(1),\sigma(1)} - x_{11} = x_{3,3} - x_{11} = 1-1 = 0$, since $\sigma(1) = 1+2 = 3$). Thus (i) certainly holds, but the value of $\lim_{p,q\to\infty}\tau_{pqst}(x)$ is 1 when

s is odd and 0 when s is even (for all t). Moreover, it shows that the inclusion $\mathcal{BV}_\sigma \subset \mathcal{W}_\sigma$ is proper.

3. Absolute Invariant Mean

Remark 3.1. It may be remarked that we have a class of linear continuous functionals φ_2 on $\mathcal{M}_u$ (which we call the set of invariant means) such that φ_2 is uniquely determined if and only if $x \in \mathcal{V}_\sigma$, that is, the largest set which determines φ_2 uniquely is $\mathcal{V}_\sigma$. Now we are going to deal with the similar situation which prevails for $\mathcal{BV}_\sigma$.

As an immediate consequence, we have the following.

Theorem 3.2. There does not exist a class of continuous linear functionals φ_2 on $\mathcal{M}_u$ such that φ_2 is uniquely determined if and only if $x \in \mathcal{BV}_\sigma$.

Proof. We first note that $\mathcal{BV}_\sigma$ is not closed in $\mathcal{M}_u$ (which follows from the case $\sigma(n) = n+1$ for single sequences which is proved in [23]). Given the value of $\varphi_2(x)$ for $x \in \mathcal{BV}_\sigma$, its value for $x \in \mathrm{cl}(\mathcal{BV}_\sigma)$ is determined by continuity. So if $\varphi_2(x)$ is unique for $x \in \mathcal{BV}_\sigma$, it must be unique in the set $\mathrm{cl}(\mathcal{BV}_\sigma)$, which is larger than $\mathcal{BV}_\sigma$. □

Remark 3.3. As in Remark 2.1, it is easy to see that the sublinear functional

$$\lambda(x) = \lim \sup_{p,q} \sup_{s,t} \tau_{pqst}(x) \tag{3.1}$$

both dominates and generates the functional φ_2 which is a σ-mean if and only if

$$-\lambda(-x) \leq \varphi_2(x) \leq \lambda(x). \tag{3.2}$$

It follows from (3.2) that φ_2 is unique σ-mean if and only if

$$\mathcal{V}_\sigma = \{x \in \mathcal{M}_u : \lambda(x) = -\lambda(-x)\}. \tag{3.3}$$

In the same vein, we seek a characterization of a class of linear functionals ψ_2 on $\mathcal{M}_u$ to define absolute invariant mean in terms of a suitable sublinear functional Q on $\mathcal{M}_u$.

Definition 3.4. A linear functional ψ_2 on $\mathcal{M}_u$ is an absolute invariant mean ($\mathcal{AIM}$) if and only if $-Q(-x) \leq \psi_2(x) \ \leq Q(x)$ and is unique $\mathcal{AIM}$ if and only if

$$\mathrm{cl}(\mathcal{BV}_\sigma) = \{x \in \mathcal{M}_u : Q(x) = -Q(-x)\}, \tag{3.4}$$

where

$$Q(x) = \lim \sup_{p,q} \sup_{s,t} \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} \left|\phi_{ijst}(x)\right| < \infty. \tag{3.5}$$

We have the following result.

Theorem 3.5. *One has*

$$\mathcal{BV}_\sigma = \{x \in \mathcal{M}_u : Q(x) = 0\}. \tag{3.6}$$

Proof. Since Q is a sublinear functional on $\mathcal{M}_u$, it follows from Hahn-Banach Theorem that there exists a continuous linear functional μ on $\mathcal{M}_u$ such that

$$\mu(x) \leq Q(x) \quad \forall x \in \mathcal{M}_u, \tag{3.7}$$

and this limit is unique if and only if $Q(x) = -Q(-x) = -Q(x)$, that is, if and only if $Q(x) = 0$ for all $x \in \mathcal{M}_u$. That is, if and only if

$$\lim_{p,q} \sum_{i=p}^{\infty} \sum_{j=q}^{\infty} \left| \phi_{ijst}(x) \right| = 0 \text{ uniformly in } s, t, \tag{3.8}$$

that is, if and only if $x \in \mathcal{BV}_\sigma$. □

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