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*Research Article*

# Properties of Toeplitz Operators on Some Holomorphic Banach Function Spaces

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We characterize complex measures $\mu$ on the unit ball of $\mathbb{C}^n$, for which the general Toeplitz operator $T_\mu^\alpha$ is bounded or compact on the analytic Besov spaces $B_p(\mathbb{B}_n)$, also on the minimal Möbius invariant Banach spaces $B_1(\mathbb{B}_n)$ in the unit ball $\mathbb{B}_n$.

## 1. Introduction

Let $\mathbb{B}_n$ be the unit ball of the $n$-dimensional complex Euclidean space $\mathbb{C}^n$. We denote the class of all holomorphic functions on the unit ball $\mathbb{B}_n$ by $\mathscr{H}(\mathbb{B}_n)$. The ball centered at $\mathbf{z}$ with radius $r$ will be denoted by $B(\mathbf{z}, r)$. For $\alpha > -1$, let $d\nu_\alpha(\mathbf{z}) = c_\alpha(1-|\mathbf{z}|^2)^\alpha d\nu$, where $d\nu$ is the normalized Lebesgue volume measure on $\mathbb{B}_n$ and $c_\alpha = \Gamma(n+\alpha+1)/n!\Gamma(\alpha+1)$ (where $\Gamma$ denotes the Gamma function) so that $\nu_\alpha(\mathbb{B}_n) \equiv 1$.

For any $\mathbf{z} = (z_1, z_2, \ldots, z_n)$, $\mathbf{w} = (w_1, w_2, \ldots, w_n) \in \mathbb{C}^n$, the inner product is defined by $\langle \mathbf{z}, \mathbf{w}\rangle = \sum_{k=1}^n z_k \overline{w}_k$. For $f \in \mathscr{H}(\mathbb{B}_n)$, we write

$$\nabla f(\mathbf{z}) = \left(\frac{\partial f(\mathbf{z})}{\partial z_1}, \frac{\partial f(\mathbf{z})}{\partial z_2}, \ldots, \frac{\partial f(\mathbf{z})}{\partial z_n}\right),$$
$$\Re f(\mathbf{z}) = \langle \nabla f, \overline{\mathbf{z}}\rangle = \sum_{j=1}^n z_j \frac{\partial f(\mathbf{z})}{\partial z_j}. \tag{1.1}$$

For $f \in \mathscr{H}(\mathbb{B}_n)$ and $\mathbf{z} \in \mathbb{B}_n$, set

$$Q_{f(\mathbf{z})} = \sup_{\mathbf{w}\in\mathbb{C}^n\setminus\{0\}} \frac{\left|\langle \nabla f(\mathbf{z}), \overline{\mathbf{w}}\rangle\right|}{\sqrt{H_{\mathbf{z}}(\mathbf{w},\mathbf{w})}}, \tag{1.2}$$

where $H_{\mathbf{z}}(\mathbf{w},\mathbf{w})$ is the Bergman metric on $\mathbb{B}_n$, that is,

$$H_{\mathbf{z}}(\mathbf{w},\mathbf{w}) = \left(\frac{n+1}{2}\right)\frac{\left(1-|\mathbf{z}|^2\right)|\mathbf{w}|^2 + |\langle \mathbf{w},\mathbf{z}\rangle|^2}{\left(1-|\mathbf{z}|^2\right)^2}. \tag{1.3}$$

For $1 < p < \infty$, the Besov spaces $B_p(\mathbb{B}_n)$ consists of all functions $f \in \mathscr{H}(\mathbb{B}_n)$ for which (see [1])

$$\|f\|^p_{B_p(\mathbb{B}_n)} := \int_{\mathbb{B}_n} Q^p_{f(\mathbf{z})}\, d\nu(\mathbf{z}) < \infty. \tag{1.4}$$

From [1], we know that for $n \geq 2$, the Besov space is nontrivial if and only if $p > 2n$.

The analytic Besov space is the minimal Möbius invariant Banach space $B_1(\mathbb{B}_n)$ (see [2]) defined by

$$\|f\|_{B_1(\mathbb{B}_n)} := \sum_{|m|=n+1} \sup_{\mathbf{z}\in\mathbb{B}_n} \int_{\mathbb{B}_n} \left|\frac{\partial^m f(\mathbf{z})}{\partial \mathbf{z}^m}\right| d\nu(\mathbf{z}) < \infty. \tag{1.5}$$

For $\alpha \geq 0$, a function $f \in \mathscr{H}(\mathbb{B}_n)$ is said to belong to the $\alpha$-Bloch spaces $\mathcal{B}^\alpha(\mathbb{B}_n)$ if (see [3])

$$b_\alpha = \sup_{\mathbf{z}\in\mathbb{B}_n} \left|\nabla f(\mathbf{z})\right|\left(1-|\mathbf{z}|^2\right)^\alpha < \infty. \tag{1.6}$$

The little Bloch space $\mathcal{B}^\alpha_0(\mathbb{B}_n)$ consists of all $f \in \mathcal{B}^\alpha(\mathbb{B}_n)$ such that

$$\lim_{|\mathbf{z}|\to 1} \left|\nabla f(\mathbf{z})\right|\left(1-|\mathbf{z}|^2\right)^\alpha = 0. \tag{1.7}$$

With the norm $\|f\|_{\mathcal{B}^\alpha(\mathbb{B}_n)} = |f(0)| + b_\alpha$, we know that $\mathcal{B}^\alpha(\mathbb{B}_n)$ becomes a Banach space. For $\alpha = 1$, the spaces $\mathcal{B}^1$ and $\mathcal{B}^1_0$ become the Bloch and the little Bloch space (see, e.g., [2]).

For every point $\mathbf{a} \in \mathbb{B}_n$, the Möbius transformation $\varphi_{\mathbf{a}} : \mathbb{B}_n \longrightarrow \mathbb{B}_n$ is defined by

$$\varphi_{\mathbf{a}}(\mathbf{z}) = \frac{\mathbf{a} - P_{\mathbf{a}}(\mathbf{z}) - S_{\mathbf{a}} Q_{\mathbf{a}}(\mathbf{z})}{1 - \langle \mathbf{z}, \mathbf{a}\rangle}, \qquad \mathbf{z} \in \mathbb{B}_n, \tag{1.8}$$

where $S_{\mathbf{a}} = \sqrt{1-|\mathbf{a}|^2}$, $P_{\mathbf{a}}(\mathbf{z}) = \mathbf{a}\langle \mathbf{z}, \mathbf{a}\rangle/|\mathbf{a}|^2$, $P_0 = 0$ and $Q_{\mathbf{a}} = I - P_{\mathbf{a}}$ (see, e.g., [2] or [4]). The map $\varphi_{\mathbf{a}}$ has the following properties that $\varphi_{\mathbf{a}}(0) = \mathbf{a}$, $\varphi_{\mathbf{a}}(\mathbf{a}) = 0$, $\varphi_{\mathbf{a}} = \varphi_{\mathbf{a}}^{-1}$ and

$$1 - \langle \varphi_{\mathbf{a}}(\mathbf{z}), \varphi_{\mathbf{a}}(\mathbf{w})\rangle = \frac{\left(1-|\mathbf{a}|^2\right)(1-\langle \mathbf{z}, \mathbf{w}\rangle)}{(1-\langle \mathbf{z}, \mathbf{a}\rangle)(1-\langle \mathbf{a}, \mathbf{w}\rangle)}, \tag{1.9}$$

where $\mathbf{z}$ and $\mathbf{w}$ are arbitrary points in $\mathbb{B}_n$. In particular,

$$1 - \left|\varphi_{\mathbf{a}}(\mathbf{z})\right|^2 = \frac{\left(1-|\mathbf{a}|^2\right)\left(1-|\mathbf{z}|^2\right)}{|1-\langle \mathbf{z}, \mathbf{a}\rangle|^2}. \tag{1.10}$$

The following result can be found in [3].

**Proposition 1.1.** *Let $f \in \mathscr{H}(\mathbb{B}_n)$, $2n < p < \infty$. Then $f \in B_p(\mathbb{B}_n)$ if and only if*

$$\iint_{\mathbb{B}_n} \left(\frac{|f(\mathbf{w}) - f(\mathbf{z})|}{|1-\langle \mathbf{z}, \mathbf{w}\rangle|}\right)^p \left(1-|\mathbf{z}|^2\right)^{p/2} \left(1-|\mathbf{w}|^2\right)^{p/2} d\mu(\mathbf{w})\, d\nu(\mathbf{z}) < \infty. \tag{1.11}$$

For $\alpha > -1$ and $0 < p < \infty$, the weighted Bergman spaces $A_\alpha^p(\mathbb{B}_n)$ consists of all functions $f \in \mathscr{H}(\mathbb{B}_n)$ for which

$$\|f\|_{A_\alpha^p}^p := \int_{\mathbb{B}_n} |f(\mathbf{z})|^p d\nu_\alpha(\mathbf{z}) < \infty. \tag{1.12}$$

It is clear that $A_\alpha^p = L^p(\mathbb{B}_n, d\nu_\alpha) \bigcap \mathscr{H}(\mathbb{B}_n)$ and $A_\alpha^p$ is a linear subspace of $L^p(\mathbb{B}_n, d\nu_\alpha)$. When $\alpha = 0$, we simply write $A^p(\mathbb{B}_n)$ for $A_0^p(\mathbb{B}_n)$. In the special case when $p = 2$, $A_\alpha^2(\mathbb{B}_n)$ is a Hilbert space. It is well known that for $\alpha > -1$ the Bergman kernel of $A_\alpha^2(\mathbb{B}_n)$ is given by

$$K^\alpha(\mathbf{z}, \mathbf{w}) = \frac{1}{(1-\langle \mathbf{z}, \mathbf{w}\rangle)^{n+1+\alpha}}, \qquad \mathbf{z}, \mathbf{w} \in \mathbb{B}_n. \tag{1.13}$$

For $\alpha > -1$, a complex measure $\mu$ such that

$$\left|\int_{\mathbb{B}_n} \left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w})\right| = \left|\int_{\mathbb{B}_n} d\mu_\alpha(\mathbf{w})\right| < \infty \tag{1.14}$$

define a Toeplitz operator as follows:

$$T_\mu^\alpha f(\mathbf{z}) = c_\alpha \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^\alpha f(\mathbf{w})}{(1-\langle \mathbf{z}, \mathbf{w}\rangle)^{n+\alpha+1}} d\mu(\mathbf{w}) = \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\, d\mu_\alpha(\mathbf{w})}{(1-\langle \mathbf{z}, \mathbf{w}\rangle)^{n+\alpha+1}}, \tag{1.15}$$

where $\mathbf{z} \in \mathbb{B}_n$ and $f \in L^1(\mathbb{B}_n, (1-|\mathbf{z}|^2)^\alpha d\mu)$.

For $\alpha, \beta > -1$, define the function $P_{\alpha,\beta}(f)(\mathbf{z})$, for $\mathbf{z} \in \mathbb{B}_n$ by:

$$P_{\alpha,\beta} f(\mathbf{z}) = c_\alpha \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\left(1 - |\mathbf{w}|^2\right)^\alpha}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\beta+1}} d\nu(\mathbf{w}) = \int_{\mathbb{B}_n} \frac{f(\mathbf{w}) d\nu_\alpha(\mathbf{w})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\beta+1}}. \tag{1.16}$$

In case $\beta = \alpha$ we write $P_\alpha$ instead of $P_{\alpha,\alpha}$ and we have that $P_\alpha(\mu)(\mathbf{z}) = T_\mu^\alpha(1)(\mathbf{z})$, where 1 stands for the constant function. For $\beta > \alpha$ the function $P_{\alpha,\beta}(\mu)$ is equivalent to the $(\beta - \alpha)$ fractional derivative of $P_\alpha(\mu)$. The Bergman projection $P_\alpha$ is the orthogonal form $L^2(\mathbb{B}_n, d\nu_\alpha)$ onto $A_\alpha^2(\mathbb{B}_n)$ defined by:

$$P_\alpha f(\mathbf{z}) = c_\alpha \int_{\mathbb{B}_n} K^\alpha(\mathbf{z}, \mathbf{w}) f(\mathbf{w}) d\nu_\alpha(\mathbf{w}). \tag{1.17}$$

The Bergman projection $P_\alpha$ naturally extends to an integral operator on $L^1(\mathbb{B}_n, d\nu_\alpha)$.

Toeplitz operators have been studied extensively on the Bergman spaces by many authors. For references, see [5, 6]. Boundedness and compactness of general Toeplitz operators $T_\mu^\alpha$ on the $\alpha$-Bloch $\mathcal{B}^\alpha(\mathbb{D})$ spaces have been investigated in [7] on the unit disk $\mathbb{D}$ for $0 < \alpha < \infty$. Also in [8], the authors extend the Toeplitz operator $T_\mu^\alpha$ to $\mathcal{B}^\alpha(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$ and completely characterize the positive Borel measure $\mu$ such that $T_\mu^\alpha$ is bounded or compact on $\mathcal{B}^\alpha(\mathbb{B}_n)$ with $1 \le \alpha < 2$. Recently, in [9], general Toeplitz operators $T_\mu^\alpha$ on the analytic Besov $B_p(\mathbb{D})$ spaces with $1 \le p < \infty$ have been investigated. Under a prerequisite condition, the authors characterized complex measure $\mu$ on the unit disk $\mathbb{D}$ for which $T_\mu^\alpha$ is bounded or compact on Besov space $B_p(\mathbb{D})$. For more details on several studies of different classes of Toeplitz operators we refer to [6, 10–16] and others.

In the present paper, we will extend the general Toeplitz operators $T_\mu^\alpha$ to $B_p(\mathbb{B}_n)$ in the unit ball o f $\mathbb{C}^n$ and completely characterize the positive Borel measure $\mu$ such that $T_\mu^\alpha$ is bounded or compact on the $B_p(\mathbb{B}_n)$ spaces with $2n < p < \infty$. The extension requires some different techniques from those used in [9].

Let $\beta(\cdot,\cdot)$ be the Bergman metric on $\mathbb{B}_n$. Denote the Bergman metric ball at $\mathbf{a}$ by $B(\mathbf{a}, r) = \{\mathbf{z} \in \mathbb{B}_n : \beta(\mathbf{a}, \mathbf{z}) < r$, where $\mathbf{a} \in \mathbb{B}_n$ and $r > 0\}$.

**Lemma 1.2** (see [2, Theorem 2.23]). *For fixed $r > 0$, there is a sequence $\{\mathbf{w}^{(j)}\} \in \mathbb{B}_n$ such that*

(i) $\bigcup_{j=1}^\infty B(\mathbf{w}^{(j)}, r) = \mathbb{B}_n$;

(ii) *there is a positive integer $N$ such that each $\mathbf{z} \in \mathbb{B}_n$ is contained in at most $N$ of the sets $B(\mathbf{w}^{(j)}, 2r)$.*

A positive Borel measure $\mu$ on the unit ball $\mathbb{B}_n$ is said to be a Carleson measure for $B_p(\mathbb{B}_n)$ if there exists $C > 0$ such that

$$\int_{\mathbb{B}_n} |f(\mathbf{z})|^p d\mu(\mathbf{z}) \le C \|f\|_{B_p(\mathbb{B}_n)}, \quad \forall f \in B_p(\mathbb{B}_n). \tag{1.18}$$

The following characterization of Carleson measures can be found in [2] or in [5]. A positive Borel measure $\mu$ on the unit ball $\mathbb{B}_n$ is said to be a Carleson measure for the Bergman space $A^p_\alpha(\mathbb{B}_n)$ if

$$\int_{\mathbb{B}_n} |f(\mathbf{z})|^p d\nu_\alpha(\mathbf{z}) \le C\|f\|^p_{A^p_\alpha(\mathbb{B}_n)}, \quad \forall f \in A^p_\alpha(\mathbb{B}_n). \tag{1.19}$$

It is well known that a positive Borel measure $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure if and only if

$$\sup_{\mathbf{w}^{(j)}\in\mathbb{B}_n} \frac{\mu(B(\mathbf{w}^{(j)}, r))}{\nu(B(\mathbf{w}^{(j)}, r))} < \infty, \tag{1.20}$$

where $\{\mathbf{w}^{(j)}\}$ is the sequence in Lemma 1.2. If $\mu$ satisfies that

$$\lim_{j\to\infty} \frac{\mu(B(\mathbf{w}^{(j)}, r))}{\nu(B(\mathbf{w}^{(j)}, r))} = 0, \tag{1.21}$$

then $\mu$ is called vanishing Carleson measure for $A^p(\mathbb{B}_n)$.

These two are special cases of a more general notion of Carleson measures on normed spaces of analytic functions.

In general, let $\mu$ be a positive measure on $\mathbb{B}_n$ and $X$ a Möbius invariant space. For $0 < p < 1$; then $\mu$ is an $(X, p)$-Carleson measure if there is a constant $C > 0$ so that (see [2])

$$\int_{\mathbb{B}_n} |f(\mathbf{z})|^p d\mu(\mathbf{z}) \le C\|f\|^p_X, \quad \forall f \in X. \tag{1.22}$$

Also, define

$$\|\nu\|_{X,p} = \sup_{f\in X, \|f\|_X\le 1} \int_{\mathbb{B}_n} |f(\mathbf{z})|^p d\mu(\mathbf{z}). \tag{1.23}$$

We say that $\mu$ is vanishing $(X, p)$-Carleson measure if for any sequence $\{f_n\} \in X$ with $\|f_n\|_X \le 1$ and such that $f_n \to 0$ uniformly on compact subset of $\mathbb{B}_n$, we have that

$$\lim_{n\to\infty} \int_{\mathbb{B}_n} |f_n(\mathbf{z})|^p d\mu(\mathbf{z}) = 0. \tag{1.24}$$

Throughout the paper, we will say that the expressions $A$ and $B$ are equivalent, and write $A \approx B$, whenever there exist positive constants $C_1$ and $C_2$ such that $C_1 A \le B \le C_2 A$. As usual, the letter $C$ will denote a positive constant, possibly different on each occurrence.

## 2. Bounded Toeplitz Operators on $B_p(\mathbb{B}_n)$ Spaces

We are going to work with Toeplitz operators acting on Besov spaces $B_p(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$.

We start with the following lemma.

**Lemma 2.1.** *Let $0 < p < \infty$, $-1 < \alpha, t < \infty$. If*

$$P_{0,\alpha} f(\mathbf{z}) = \int_{\mathbb{B}_n} \frac{f(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+1}} d\nu(\mathbf{z}), \tag{2.1}$$

*then $P_{0,\alpha}$ is a bounded operator from $L^p(\mathbb{B}_n, d\nu_t)$ into $A^p_{t+p\alpha}(\mathbb{B}_n)$ if and only if $-p\alpha < t+1 < p$.*

*Proof.* Let

$$\begin{aligned} Tf(\mathbf{z}) &= \left(1 - |\mathbf{w}|^2\right)^{\alpha} P_{0,\alpha} f(\mathbf{z}) \\ &= \left(1 - |\mathbf{w}|^2\right)^{\alpha} \int_{\mathbb{B}_n} \frac{f(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+1}} d\nu(\mathbf{z}). \end{aligned} \tag{2.2}$$

By Theorem 2.10 in [2], we know that $T$ is bounded on $L^p(\mathbb{B}_n, d\nu_t)$ if and only if $-p\alpha < t+1 < p$. However, it is obvious that $P_{0,\alpha}$ is bounded from $L^p(\mathbb{B}_n, d\nu_t)$ into $A^p_{t+p\alpha}(\mathbb{B}_n)$ if and only if $T$ is bounded on $L^p(\mathbb{B}_n, d\nu_t)$. □

**Theorem 2.2.** *Let $2n < p < \infty, \alpha > -1$ and let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T^\alpha_\mu$ is bounded on $B_p(\mathbb{B}_n)$ spaces if and only if $P_\alpha(\mu)(\mathbf{w})$ is a $(B_p(\mathbb{B}_n), p)$-Carleson measure.*

*Proof.* Let $2n < p, q < \infty$ where $1/p + 1/q = 1$ and let $\alpha > -1$. We know that the dual spaces of $B_p(\mathbb{B}_n)$ are $B_q(\mathbb{B}_n)$ under the paring

$$\langle f, g \rangle = f(0)\overline{g(0)} + \int_{\mathbb{B}_n} \Re f(\mathbf{z}) \overline{\Re g(\mathbf{z})} d\nu(\mathbf{z}), \quad f \in B_p(\mathbb{B}_n),\ g \in B_q(\mathbb{B}_n). \tag{2.3}$$

To prove the boundedness of $T^\alpha_\mu$, it suffices to show that

$$\left| \left\langle T^\alpha_\mu(f), g \right\rangle \right| \le C \|f\|_{B_p(\mathbb{B}_n)} \|g\|_{B_q(\mathbb{B}_n)}, \tag{2.4}$$

for all $f \in B_p(\mathbb{B}_n)$ and $g \in B_q(\mathbb{B}_n)$, where $C$ is a positive constant that does not depend on $f$ or $g$.

Now we define $G(\mathbf{w})$ by the following:

$$G(\mathbf{w}) = \mathbf{w} P_{0,\alpha+1} \Re g(\mathbf{w}) = \mathbf{w} \int_{\mathbb{B}_n} \frac{\Re g(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+2}} d\nu(\mathbf{z}). \tag{2.5}$$

Then

$$\begin{aligned}\left\langle T_\mu^\alpha f, g\right\rangle &= T_\mu^\alpha f(0)\overline{g(0)} + \int_{\mathbb{B}_n} T_\mu^\alpha(\Re f)(\mathbf{z})\overline{\Re g(\mathbf{z})}\, d\nu(\mathbf{z}) \\ &= T_\mu^\alpha f(0)\overline{g(0)} + c_\alpha \int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^\alpha \Re f(\mathbf{w})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\mu(\mathbf{w})\right)\overline{\Re g(\mathbf{z})} d\nu(\mathbf{z}).\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}\overline{f(w)} - \overline{f(0)} = \overline{f(\mathbf{w})} - P_\alpha\left(\overline{f}\right)(\mathbf{w}) &= \overline{f(\mathbf{w})} - c_\alpha \int_{\mathbb{B}_n} \frac{\overline{f(\mathbf{z})}\left(1-|\mathbf{z}|^2\right)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu(\mathbf{z}) \\ &= c_\alpha \int_{\mathbb{B}_n} \frac{\left(\overline{f(\mathbf{w})} - \overline{f(\mathbf{z})}\right)}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu_\alpha(\mathbf{z}),\end{aligned} \tag{2.7}$$

we have

$$\begin{aligned}T_\mu^\alpha f(0) &= \int_{\mathbb{B}_n} f(\mathbf{w}) d\mu_\alpha(\mathbf{w}) \\ &= f(0)\int_{\mathbb{B}_n} d\mu_\alpha(\mathbf{w}) + c_\alpha^2 \iint_{\mathbb{B}_n} \frac{\left(\overline{f(\mathbf{w})} - \overline{f(\mathbf{z})}\right)}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu_\alpha(\mathbf{z}) d\mu_\alpha(\mathbf{w}).\end{aligned} \tag{2.8}$$

This implies

$$\left|T_\mu^\alpha f(0)\right| \le C\left|f(0)\right| + c_\alpha^2 \iint_{\mathbb{B}_n} \frac{\left|f(\mathbf{w}) - f(\mathbf{z})\right|}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}} d\nu_\alpha(\mathbf{z}) d\mu_\alpha(\mathbf{w}). \tag{2.9}$$

By Proposition 1.1, we have

$$\begin{aligned}&\iint_{\mathbb{B}_n} \frac{\left|f(\mathbf{w}) - f(\mathbf{z})\right|}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}} d\nu_\alpha(\mathbf{z}) d\mu_\alpha(\mathbf{w}) \\ &\quad = \left(\int_{\mathbb{B}_n}\left(1-|\mathbf{z}|^2\right)^{p\alpha-p/2}\int_{\mathbb{B}_n} \frac{\left|f(\mathbf{w}) - f(\mathbf{z})\right|^p\left(1-|\mathbf{z}|^2\right)^{p/2}\left(1-|\mathbf{w}|^2\right)^{p/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^p}\right. \\ &\qquad \left.\times \frac{\left(1-|\mathbf{w}|^2\right)^{p\alpha-p/2}}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{p(n+\alpha)}} d\mu(\mathbf{w}) d\nu(\mathbf{z})\right)^{1/p} \\ &\quad \le C\|f\|_{B_p(\mathbb{B}_n)}\int_{\mathbb{B}_n}\left(1-|\mathbf{z}|^2\right)^{\alpha-(1/2)}\int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-(1/2)}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}).\end{aligned} \tag{2.10}$$

Since $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, taking $\alpha - 1/2 > -1$, then as in [8] (see also Proposition 1.4.10 of [4]), we get

$$\left(1-|\mathbf{z}|^2\right)^{-1/2}\int_{\mathbb{B}_n}\frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-1/2}}{|1-\langle\mathbf{z},\mathbf{w}\rangle|^{n+\alpha}}d\mu(\mathbf{w})\le C. \tag{2.11}$$

Then,

$$\begin{aligned}\left|T_\mu^\alpha f(0)\right| &\le C\left|f(0)\right| + C\|f\|_{B_p(\mathbb{B}_n)}\int_{\mathbb{B}_n}d\nu_\alpha(\mathbf{z})\\ &\le C\|f\|_{B_p(\mathbb{B}_n)}.\end{aligned} \tag{2.12}$$

Therefore

$$\left|T_\mu^\alpha f(0)g(0)\right|\le C\|f\|_{B_q(\mathbb{B}_n)}\|g\|_{B_q(\mathbb{B}_n)}. \tag{2.13}$$

By Fubini's Theorem we have

$$\begin{aligned}\left\langle T_\mu^\alpha f, g\right\rangle &= \int_{\mathbb{B}_n}T_\mu^\alpha(\Re f)(\mathbf{z})\overline{\Re g(\mathbf{z})}d\nu(\mathbf{z})\\ &= c_\alpha\int_{\mathbb{B}_n}\left(\int_{\mathbb{B}_n}\frac{\left(1-|\mathbf{w}|^2\right)^\alpha\Re f(\mathbf{w})}{(1-\langle\mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}}d\mu(\mathbf{w})\right)\overline{\Re g(\mathbf{z})}d\nu(\mathbf{z})\\ &= c_\alpha\int_{\mathbb{B}_n}f(\mathbf{w})\left(1-|\mathbf{w}|^2\right)^\alpha\left(\overline{\mathbf{w}}\int_{\mathbb{B}_n}\frac{\overline{\Re g(\mathbf{z})}d\nu(\mathbf{z})}{(1-\langle\mathbf{z},\mathbf{w}\rangle)^{n+\alpha+2}}\right)d\mu(\mathbf{w})\\ &= c_\alpha\int_{\mathbb{B}_n}f(\mathbf{w})\overline{G(\mathbf{w})}\left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}).\end{aligned} \tag{2.14}$$

Using the operator $P_\alpha$, divide the integral

$$\int_{\mathbb{B}_n}f(\mathbf{w})\overline{G(\mathbf{w})}\left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}) = \int_{\mathbb{B}_n}\overline{f(\mathbf{w})}G(\mathbf{w})\left(1-|\mathbf{w}|^2\right)^\alpha d\overline{\mu}(\mathbf{w}), \tag{2.15}$$

we have

$$\begin{aligned}\left\langle T_\mu^\alpha f, g\right\rangle &= c_\alpha\int_{\mathbb{B}_n}\left[(I-P_\alpha)\left(f\overline{G}\right)\right](\mathbf{w})\left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w})\\ &\quad + c_\alpha\int_{\mathbb{B}_n}P_\alpha\left(f\overline{G}\right)(\mathbf{w})\left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w})\\ &= I_1 + I_2,\end{aligned} \tag{2.16}$$

where $I$ is the identity operator, and

$$(I-P_\alpha)\left(f\overline{G}\right)(\mathbf{w}) = f(\mathbf{w})\overline{G(\mathbf{w})} - c_\alpha \int_{\mathbb{B}_n} \frac{f(\mathbf{z})\overline{G(\mathbf{z})}\left(1-|\mathbf{z}|^2\right)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu(\mathbf{z})$$
$$= c_\alpha \int_{\mathbb{B}_n} \frac{(f(\mathbf{w})-f(\mathbf{z}))\overline{G(\mathbf{z})}\left(1-|\mathbf{z}|^2\right)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu(\mathbf{z}). \tag{2.17}$$

By Proposition 1.1, we have

$$|I_1| = c_\alpha \left| \int_{\mathbb{B}_n} \left[(I-P_\alpha)\left(f\overline{G}\right)\right](\mathbf{w})\left(1-|\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}) \right|$$
$$= c_\alpha^2 \left| \iint_{\mathbb{B}_n} \frac{(f(\mathbf{w})-f(\mathbf{z}))\overline{G(\mathbf{z})}\left(1-|\mathbf{z}|^2\right)^\alpha \left(1-|\mathbf{w}|^2\right)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}} d\nu(\mathbf{z}) d\mu(\mathbf{w}) \right|$$
$$= c_\alpha^2 \int_{\mathbb{B}_n} |G(\mathbf{z})|\left(1-|\mathbf{z}|^2\right)^\alpha \int_{\mathbb{B}_n} \frac{|f(\mathbf{w})-f(\mathbf{z})|\left(1-|\mathbf{w}|^2\right)^\alpha}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}} d\mu(\mathbf{w}) d\nu(\mathbf{z})$$
$$= c_\alpha^2 \Bigg( \int_{\mathbb{B}_n} |G(\mathbf{z})|^p \left(1-|\mathbf{z}|^2\right)^{p\alpha-p/2}$$
$$\times \int_{\mathbb{B}_n} \frac{|f(\mathbf{w})-f(\mathbf{z})|^p \left(1-|\mathbf{z}|^2\right)^{p/2}\left(1-|\mathbf{w}|^2\right)^{p/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^p}$$
$$\cdot \frac{\left(1-|\mathbf{w}|^2\right)^{p\alpha-p/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{p(n+\alpha)}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \Bigg)^{1/p}$$
$$\leq C\|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} |G(\mathbf{z})|\left(1-|\mathbf{z}|^2\right)^{\alpha-1/2} \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-1/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}). \tag{2.18}$$

By Lemma 2.1, the operator $P_{0,\alpha}$ is bounded from $L^p(\mathbb{B}_n, d\nu_t)$ into $A^p_{t+p\alpha}(\mathbb{B}_n)$ whenever $-p\alpha < t+1 < p$. Since $g \in B_q(\mathbb{B}_n)$ if and only if $\Re g \in A^q_{q-2}(\mathbb{B}_n)$, and we have from above, $P_{0,\alpha+1}$ maps $A^q_{q-2}(\mathbb{B}_n)$ boundedly into $A^q_{(q-2)+q(\alpha+1)}(\mathbb{B}_n)$, whenever $-q(\alpha+1) < (q-2)+1 < q$, or $q > 1/(\alpha+2)$, which is always true if $\alpha > -1$. Thus $G(\mathbf{w}) \in A^q_{q(\alpha+2)-2}(\mathbb{B}_n)$. It can easily seen that $G \in A^1_\alpha(\mathbb{B}_n)$ and that $\|G\|_{A^1_\alpha(\mathbb{B}_n)} \le C\|G\|_{A^q_{q(\alpha+2)-2}(\mathbb{B}_n)} \le C\|g\|_{B_q(\mathbb{B}_n)}$. Thus

$$\begin{aligned}
|I_1| &\le C\|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} |G(\mathbf{z})|\left(1-|\mathbf{z}|^2\right)^\alpha \left( \left(1-|\mathbf{z}|^2\right)^{-1/2} \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-1/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha}} d\mu(\mathbf{w}) \right) d\nu(\mathbf{z}) \\
&\le C\|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} \|G\|_{A^1_\alpha} \left(1-|\mathbf{z}|^2\right)^{-1/2} \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-1/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\le C\|f\|_{B_p(\mathbb{B}_n)} \int_{\mathbb{B}_n} \|g\|_{B_q(\mathbb{B}_n)} \left(1-|\mathbf{z}|^2\right)^{-1/2} \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{w}|^2\right)^{\alpha-1/2}}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}).
\end{aligned} \tag{2.19}$$

By (2.11), we get

$$|I_1| \le C\|f\|_{B_p(\mathbb{B}_n)} \|g\|_{B_q(\mathbb{B}_n)}. \tag{2.20}$$

Next consider $I_2$, we have

$$\begin{aligned}
|I_2| &= c_\alpha \left| \int_{\mathbb{B}_n} P_\alpha\left(f\overline{G}\right)(\mathbf{z}) d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha^2 \left| \iint_{\mathbb{B}_n} \frac{f(\mathbf{w})\overline{G(\mathbf{w})}\left(1-|\mathbf{w}|^2\right)^\alpha}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}} d\nu(\mathbf{w}) d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha \int_{\mathbb{B}_n} |f(\mathbf{w})||G(\mathbf{w})|\left(1-|\mathbf{w}|^2\right)^\alpha c_\alpha \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{z}|^2\right)^\alpha d\mu(\mathbf{z})}{|1-\langle \mathbf{z},\mathbf{w}\rangle|^{n+\alpha+1}} d\nu(\mathbf{w}) \\
&\le C \int_{\mathbb{B}_n} \|G\|_{A^1_\alpha(\mathbb{B}_n)} |f(\mathbf{w})| P_\alpha(\mu)(\mathbf{w}) d\nu(\mathbf{w}) \\
&\le C \int_{\mathbb{B}_n} \|g\|_{B_q(\mathbb{B}_n)} |f(\mathbf{w})| P_\alpha(\mu)(\mathbf{w}) d\nu(\mathbf{w}).
\end{aligned} \tag{2.21}$$

Therefore, $T^\alpha_\mu$ is bounded on $B_p(\mathbb{B}_n)$ if and only if

$$\int_{\mathbb{B}_n} |f(\mathbf{w})| P_\alpha(\mu)(\mathbf{w}) d\nu(\mathbf{w}) \le C\|f\|_{B_p(\mathbb{B}_n)} \tag{2.22}$$

if and only if the measure $P_\alpha(\mu)(\mathbf{w})$ is a $(B_p(\mathbb{B}_n), p)$-Carleson measure. □

Now, we will characterize boundedness of Toeplitz operators on the minimal Möbius invariant Banach spaces of holomorphic functions $B_1(\mathbb{B}_n)$ in the unit ball of $\mathbb{C}^n$.

**Theorem 2.3.** *Let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T_\mu^\alpha$ is bounded on $B_1(\mathbb{B}_n)$ spaces if and only if*

$$\sum_{|m|=n+1} \left| \frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w}) \right| d\nu(\mathbf{w}) \tag{2.23}$$

*is a $(B_1(\mathbb{B}_n), 1)$-Carleson measure.*

*Proof.* We will use the fact that the dual spaces of $B_1(\mathbb{B}_n)$ are the Bloch space $\mathcal{B}(\mathbb{B}_n)$ under the paring

$$\langle f, g \rangle = \int_{\mathbb{B}_n} \Re f(\mathbf{z}) \overline{\Re g(\mathbf{z})} d\nu(\mathbf{z}), \qquad f \in B_1(\mathbb{B}_n), \ g \in \mathcal{B}(\mathbb{B}_n). \tag{2.24}$$

Similarly, as in the proof of Theorem 2.2, by duality, we have that $T_\mu^\alpha$ is bounded on $B_1(\mathbb{B}_n)$ spaces if and only if

$$\left| \left\langle T_\mu^\alpha(f), g \right\rangle \right| = c_\alpha \left| \int_{\mathbb{B}_n} f(\mathbf{w}) \overline{G(\mathbf{w})} \left(1 - |\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}) \right| \leq C \|f\|_{B_1(\mathbb{B}_n)} \|g\|_{\mathcal{B}(\mathbb{B}_n)}, \tag{2.25}$$

for all $f \in B_1(\mathbb{B}_n)$ and $g \in \mathcal{B}(\mathbb{B}_n)$, where

$$G(\mathbf{w}) = \mathbf{w} P_{0,\alpha+1} \Re g(\mathbf{w}) = \mathbf{w} \int_{\mathbb{B}_n} \frac{\Re g(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+2}} d\nu(\mathbf{z}). \tag{2.26}$$

Using the fact that

$$\left| \int_{\mathbb{B}_n} \frac{\Re g(\mathbf{z})}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} d\nu(\mathbf{z}) \right| \approx \left| \int_{\mathbb{B}_n} \frac{\left(1 - |\mathbf{z}|^2\right) \Re g(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+3}} d\nu(\mathbf{z}) \right|, \tag{2.27}$$

for $g \in \mathcal{B}(\mathbb{B}_n)$, we have that $|G(\mathbf{w})|(1 - |\mathbf{w}|^2)^{\alpha+1} < \infty$, which means that $G \in \mathcal{B}^{\alpha+2}(\mathbb{B}_n)$. Now using the operator $P_{\alpha+1}$, we have

$$\begin{aligned} \left\langle T_\mu^\alpha f, g \right\rangle &= c_\alpha \int_{\mathbb{B}_n} \left[ (I - P_{\alpha+1}) \left( f \overline{G} \right) \right] (\mathbf{w}) \left(1 - |\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}) \\ &\quad + c_\alpha \int_{\mathbb{B}_n} P_{\alpha+1} \left( f \overline{G} \right) (\mathbf{w}) \left(1 - |\mathbf{w}|^2\right)^\alpha d\mu(\mathbf{w}) \\ &= I_1 + I_2, \end{aligned} \tag{2.28}$$

$$(I - P_{\alpha+1}) \left( f \overline{G} \right) (\mathbf{w}) = c_{\alpha+2} \int_{\mathbb{B}_n} \frac{(f(\mathbf{w}) - f(\mathbf{z})) \overline{G(\mathbf{z})} \left(1 - |\mathbf{z}|^2\right)^{\alpha+1}}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+2}} d\nu(\mathbf{z}).$$

By Proposition 1.1, we have

$$
\begin{aligned}
|I_1| &= c_\alpha \left| \int_{\mathbb{B}_n} \left[ (I - P_{\alpha+1}) \left( f\overline{G} \right) \right](\mathbf{w}) \left( 1 - |\mathbf{w}|^2 \right)^\alpha d\mu(\mathbf{w}) \right| \\
&= c_\alpha c_{\alpha+1} \left| \iint_{\mathbb{B}_n} \frac{(f(\mathbf{w}) - f(\mathbf{z}))\overline{G(\mathbf{z})} \left( 1 - |\mathbf{z}|^2 \right)^{\alpha+2} \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} d\nu(\mathbf{z}) d\mu(\mathbf{w}) \right| \\
&= c_\alpha c_{\alpha+1} \int_{\mathbb{B}_n} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^{\alpha+1} \int_{\mathbb{B}_n} \frac{|f(\mathbf{w}) - f(\mathbf{z})| \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\leq C \int_{\mathbb{B}_n} \|f\|_{B_p(\mathbb{B}_n)} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^{\alpha+1/2} \int_{\mathbb{B}_n} \frac{\left( 1 - |\mathbf{w}|^2 \right)^{\alpha-1/2}}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\leq C \int_{\mathbb{B}_n} \|f\|_{B_p(\mathbb{B}_n)} \|G\|_{A^1_{\alpha+1}(\mathbb{B}_n)} \left( 1 - |\mathbf{z}|^2 \right)^{-1/2} \int_{\mathbb{B}_n} \frac{\left( 1 - |\mathbf{w}|^2 \right)^{\alpha-1/2}}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+1}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\leq C \|f\|_{B_1(\mathbb{B}_n)} \|g\|_{\mathcal{B}(\mathbb{B}_n)}.
\end{aligned}
\tag{2.29}
$$

Next consider $I_2$, notice first that

$$
\begin{gathered}
P_\alpha(\mu)(\mathbf{w}) = c_\alpha \int_{\mathbb{B}_n} \frac{d\mu_\alpha(\mathbf{z})}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+1}}; \\
\sum_{|m|=n+1} \left| \frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w}) \right| \approx \left| \int_{\mathbb{B}_n} \frac{(\overline{\mathbf{z}})^m d\mu_\alpha(\mathbf{z})}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} \right|.
\end{gathered}
\tag{2.30}
$$

Thus,

$$
\begin{aligned}
|I_2| &= c_\alpha \left| \int_{\mathbb{B}_n} \overline{(\mathbf{z})^m P_{\alpha+1} \left( f\overline{G} \right)(\mathbf{z})} d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha c_{\alpha+1} \left| \int_{\mathbb{B}_n} (\overline{\mathbf{z}})^m \int_{\mathbb{B}_n} \frac{f(\mathbf{w})\overline{G(\mathbf{w})} \left( 1 - |\mathbf{w}|^2 \right)^{\alpha+1}}{(1 - \langle \mathbf{z}, \mathbf{w} \rangle)^{n+\alpha+2}} d\nu(\mathbf{w}) d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha \int_{\mathbb{B}_n} |f(\mathbf{w})| |G(\mathbf{w})| \left( 1 - |\mathbf{w}|^2 \right)^{\alpha+1} \left( c_{\alpha+1} \int_{\mathbb{B}_n} \frac{(\overline{\mathbf{z}})^m \left( 1 - |\mathbf{z}|^2 \right)^\alpha d\mu(\mathbf{z})}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+2}} \right) d\nu(\mathbf{w}) \\
&= c_\alpha \int_{\mathbb{B}_n} |f(\mathbf{w})| |G(\mathbf{w})| \left( 1 - |\mathbf{w}|^2 \right)^{\alpha+1} \sum_{|m|=n+1} \left| \frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w}) \right| d\nu(\mathbf{w}).
\end{aligned}
\tag{2.31}
$$

It is known that $(A^1(\mathbb{B}_n))^* = \mathcal{B}^{\beta+1}(\mathbb{B}_n)$ under the paring

$$\langle F, H\rangle_\beta = c_\beta \int_{\mathbb{B}_n} F(\mathbf{w})\overline{H(\mathbf{w})}\left(1-|\mathbf{w}|^2\right)^\beta d\mu(\mathbf{w}), \qquad F \in A^1(\mathbb{B}_n),\ H \in \mathcal{B}^{\beta+1}(\mathbb{B}_n). \tag{2.32}$$

Since $G \in \mathcal{B}^{\alpha+2}(\mathbb{B}_n)$, $g \in \mathcal{B}(\mathbb{B}_n)$ for by the above duality we get

$$\begin{aligned}\sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} |I_2| &\approx C \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} \int_{\mathbb{B}_n} \|G\|_{A^1_{\alpha+1}(\mathbb{B}_n)} |f(\mathbf{w})| \sum_{|m|=n+1} \left|\frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w})\right| d\nu(\mathbf{w}) \\ &\le C \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} \int_{\mathbb{B}_n} |f(\mathbf{w})| \sum_{|m|=n+1} \left|\frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w})\right| d\nu(\mathbf{w}).\end{aligned} \tag{2.33}$$

Therefore, $T^\alpha_\mu$ is bounded on $B_p(\mathbb{B}_n)$ if and only if

$$\int_{\mathbb{B}_n} |f(\mathbf{w})| \sum_{|m|=n+1} \left|\frac{\partial^m}{\partial \mathbf{w}^m} P_\alpha(\mu)(\mathbf{w})\right| d\nu(\mathbf{w}\ ) \le C\|f\|_{B_p(\mathbb{B}_n)} \tag{2.34}$$

if and only if the measure $\sum_{|m|=n+1} |(\partial^m/\partial \mathbf{w}^m) P_\alpha(\mu)(\mathbf{w})| d\nu(\mathbf{w})$ is a $(B_p(\mathbb{B}_n), p)$-Carleson measure. □

## 3. Compact Toeplitz Operators on $B_p(\mathbb{B}_n)$ Spaces

In this section we will characterize compact Toeplitz operators on $B_p(\mathbb{B}_n)$ spaces in the unit ball of $\mathbb{C}^n$. We need the following lemma.

**Lemma 3.1.** *Let $0 < p < \infty$, $-1 < \alpha$ and $T^\alpha_\mu$ be bounded linear operator from $B_p(\mathbb{B}_n)$ into $B_p(\mathbb{B}_n)$ in the unit ball. Then $T^\alpha_\mu$ is compact on $B_p(\mathbb{B}_n)$ spaces if and only if $\|T^\alpha_\mu f_j\|_{B_p(\mathbb{B}_n)} \to 0$ as $j \to \infty$ wheneve r $\{f_j\}$ is a bounded sequence in $B_p(\mathbb{B}_n)$ that converges to 0 uniformly on $\mathbb{B}_n$.*

*Proof.* This lemma can be proved by Montel's Theorem. □

**Theorem 3.2.** *Let $2n < p < \infty, \alpha > -1$ and let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a vanishing $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T^\alpha_\mu$ is compact on $B_p(\mathbb{B}_n)$ spaces if and only if $P_\alpha(\mu)(\mathbf{w})$ is a vanishing $(B_p(\mathbb{B}_n), p)$-Carleson measure.*

*Proof.* Let $2n < p,\ q < \infty$ where $1/p + 1/q = 1$ and let $\{f_j\}$ be a sequence in $B_p(\mathbb{B}_n)$ satisfying $\|f_j\|_{B_p(\mathbb{B}_n)} \le 1$ and such that $f_j$ converges to 0 uniformly as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, and let $g \in B_q(\mathbb{B}_n)$. By duality, we have that $T^\alpha_\mu$ is compact on $B_p(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|g\|_{Bq(\mathbb{B}_n)}\le 1} \left|\left\langle T^\alpha_\mu(f_j), g\right\rangle\right| = 0. \tag{3.1}$$

As in the proof of Theorem 2.2,

$$\begin{aligned}\left\langle T_\mu^\alpha(f_j), g\right\rangle &= T_\mu^\alpha f_j(0)\overline{g(0)} + \int_{\mathbb{B}_n} T_\mu^\alpha(\Re f_j)(\mathbf{z})\overline{\Re g(\mathbf{z})}d\nu(\mathbf{z}) \\ &= T_\mu^\alpha f_j(0)\overline{g(0)} + c_\alpha\int_{\mathbb{B}_n} f_j(\mathbf{w})\overline{G(\mathbf{w})}d\mu_\alpha(\mathbf{w}),\end{aligned} \tag{3.2}$$

where

$$G(\mathbf{w}) = \mathbf{w}P_{0,\alpha+1}\Re g(\mathbf{w}) = \mathbf{w}\int_{\mathbb{B}_n}\frac{\Re g(\mathbf{z})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+2}}d\nu(\mathbf{z}). \tag{3.3}$$

Also as in the proof of Theorem 2.2,

$$\left|T_\mu^\alpha f(0)\right| \le C\|f\|_{B_p(\mathbb{B}_n)}. \tag{3.4}$$

Since $|\int_{\mathbb{B}_n} d\mu_\alpha(\mathbf{w})| < \infty$ and $\mu$ is a vanishing $(A^p(\mathbb{B}_n),p)$-Carleson measure, and $f_j$ converges to 0 uniformly as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, we get that

$$T_\mu^\alpha f(0) \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.5}$$

Thus $T_\mu^\alpha$ is compact on $B_p(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty}\sup_{\|g\|_{Bq(\mathbb{B}_n)}\le 1}\left|\int_{\mathbb{B}_n} f_j(\mathbf{w})\overline{G(\mathbf{w})}d\mu_\alpha(\mathbf{w})\right| = 0. \tag{3.6}$$

Using the operator $P_\alpha$, we have that

$$\begin{aligned}\int_{\mathbb{B}_n} f_j(\mathbf{w})\overline{G(\mathbf{w})}d\mu_\alpha(\mathbf{w}) &= c_\alpha\int_{\mathbb{B}_n}\left[(I-P_\alpha)\left(f_j\overline{G}\right)\right](\mathbf{z})d\mu_\alpha(\mathbf{z}) + c_\alpha\int_{\mathbb{B}_n} P_\alpha\left(f_j\overline{G}\right)(\mathbf{z})d\mu_\alpha(\mathbf{z}). \\ &= J_1 + J_2.\end{aligned} \tag{3.7}$$

For $0 < r < 1$ and $r\mathbb{B}_n = \{\mathbf{z} \in \mathbb{C}^n, |\mathbf{z}| \le r\}$, we have

$$\begin{aligned}
|J_1| &= c_\alpha \left| \int_{\mathbb{B}_n} \left[ (I - P_\alpha)\left( f_j \overline{G} \right) \right](\mathbf{w}) \left( 1 - |\mathbf{w}|^2 \right)^\alpha d\mu(\mathbf{w}) \right| \\
&= c_\alpha^2 \left| \iint_{\mathbb{B}_n} \frac{\left( f_j(\mathbf{w}) - f_j(\mathbf{z}) \right) \overline{G(\mathbf{z})} \left( 1 - |\mathbf{z}|^2 \right)^\alpha \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{\left( 1 - \langle \mathbf{z}, \mathbf{w} \rangle \right)^{n+\alpha+1}} d\nu(\mathbf{z}) d\mu(\mathbf{w}) \right| \\
&= c_\alpha^2 \left( \int_{\mathbb{B}_n \setminus r\mathbb{B}_n} + \int_{r\mathbb{B}_n} \right) |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^\alpha \int_{\mathbb{B}_n} \frac{\left| f_j(\mathbf{w}) - f_j(\mathbf{z}) \right| \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+1}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&= L_1 + L_2.
\end{aligned} \tag{3.8}$$

For a fixed $\varepsilon > 0$, since $\mu$ is a vanishing $(A^p(\mathbb{B}_n), p)$-Carleson measure, let $r$ sufficiently close to 1 so that

$$\left( 1 - |\mathbf{z}|^2 \right)^{-1/2} \int_{\mathbb{B}_n \setminus r\mathbb{B}_n} \frac{\left( 1 - |\mathbf{w}|^2 \right)^{\alpha - 1/2}}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha}} d\mu(\mathbf{w}) \ d < \varepsilon. \tag{3.9}$$

Similarly, as in the proof of Theorem 2.2, by Proposition 1.1,

$$\begin{aligned}
L_1 &= c_\alpha^2 \int_{\mathbb{B}_n \setminus r\mathbb{B}_n} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^\alpha \int_{\mathbb{B}_n} \frac{\left| f_j(\mathbf{w}) - f_j(\mathbf{z}) \right| \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+1}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\le C \int_{\mathbb{B}_n \setminus r\mathbb{B}_n} \left\| f_j \right\|_{B_p(\mathbb{B}_n)} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^{\alpha - 1/2} \int_{\mathbb{B}_n} \frac{\left( 1 - |\mathbf{w}|^2 \right)^{\alpha - 1/2}}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\le C\varepsilon \| \left\| f_j \right\|_{B_p(\mathbb{B}_n)} \|G\|_{A_\alpha^1(\mathbb{B}_n)} \le C\varepsilon \left\| f_j \right\|_{B_p(\mathbb{B}_n)} \|g\|_{B_q(\mathbb{B}_n)} \le \varepsilon.
\end{aligned} \tag{3.10}$$

Since $f_j \to 0$ as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, we cane choose $j$ big enough so that

$$|G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^\alpha < \varepsilon. \tag{3.11}$$

Therefore,

$$\begin{aligned}
L_2 &= c_\alpha^2 \int_{r\mathbb{B}_n} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^\alpha \int_{\mathbb{B}_n} \frac{\left| f_j(\mathbf{w}) - f_j(\mathbf{z}) \right| \left( 1 - |\mathbf{w}|^2 \right)^\alpha}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha+1}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\le C \int_{r\mathbb{B}_n} \left\| f_j \right\|_{B_q(\mathbb{B}_n)} |G(\mathbf{z})| \left( 1 - |\mathbf{z}|^2 \right)^{\alpha - 1/2} \int_{\mathbb{B}_n} \frac{\left( 1 - |\mathbf{w}|^2 \right)^{\alpha - (1/2)}}{|1 - \langle \mathbf{z}, \mathbf{w} \rangle|^{n+\alpha}} d\mu(\mathbf{w}) d\nu(\mathbf{z}) \\
&\le C\varepsilon \|G\|_{A_\alpha^1(\mathbb{B}_n)} \le C\varepsilon \|g\|_{B_q(\mathbb{B}_n)}.
\end{aligned} \tag{3.12}$$

Hence $|J_1| < C\varepsilon$, where $C$ does not depend on $g(\mathbf{z})$, and so

$$\lim_{j\to\infty} \sup_{\|g\|_{B_q(\mathbb{B}_n)}\le 1} |J_1| = 0. \tag{3.13}$$

Thus, $T_\mu^\alpha$ is compact on $B_p(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|g\|_{B_q(\mathbb{B}_n)}\le 1} |J_2| = 0. \tag{3.14}$$

Again, as in the proof of Theorem 2.2, we have

$$\begin{aligned}
|J_2| &= c_\alpha \left| \int_{\mathbb{B}_n} P_\alpha\left(f_j\overline{G}\right)(\mathbf{z})\, d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha^2 \left| \iint_{\mathbb{B}_n} \frac{f_j(\mathbf{w})\overline{G(\mathbf{w})}\left(1-|\mathbf{w}|^2\right)^\alpha}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}}\, d\nu(\mathbf{w})\, d\mu_\alpha(\mathbf{z}) \right| \\
&= c_\alpha \int_{\mathbb{B}_n} \left|f_j(\mathbf{w})\right| |G(\mathbf{w})| \left(1-|\mathbf{w}|^2\right)^\alpha c_\alpha \int_{\mathbb{B}_n} \frac{\left(1-|\mathbf{z}|^2\right)^\alpha d\mu(\mathbf{z})}{(1-\langle \mathbf{z},\mathbf{w}\rangle)^{n+\alpha+1}}\, d\nu(\mathbf{w}) \\
&\le C \int_{\mathbb{B}_n} \|G\|_{A_\alpha^1(\mathbb{B}_n)} \left|f_j(\mathbf{w})\right| P_\alpha(\mu)(\mathbf{w})\, d\nu(\mathbf{w}) \\
&\le C \int_{\mathbb{B}_n} \|g\|_{B_q(\mathbb{B}_n)} \left|f_j(\mathbf{w})\right| P_\alpha(\mu)(\mathbf{w})\, d\nu(\mathbf{w}).
\end{aligned} \tag{3.15}$$

Therefore, $T_\mu^\alpha$ is compact on $B_p(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \int_{\mathbb{B}_n} \left|f_j(\mathbf{w})\right| P_\alpha(\mu)(\mathbf{w})\, d\nu(\mathbf{w}) = 0, \tag{3.16}$$

which is equivalent to say that $P_\alpha(\mu)(\mathbf{w})$ is a vanishing $(B_p(\mathbb{B}_n), p)$-Carleson measure. □

**Theorem 3.3.** *Let $\mu$ be a positive Borel measure on $\mathbb{B}_n$. If $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, then the Toeplitz operator $T_\mu^\alpha$ is compact on $B_1(\mathbb{B}_n)$ spaces if and only if*

$$\sum_{|m|=n+1} \left| \frac{\partial^m P_\alpha(\mu)}{\partial \mathbf{w}^m}(\mathbf{w}) \right| d\nu(\mathbf{w}) \tag{3.17}$$

*is a vanishing $(B_1(\mathbb{B}_n), 1)$-Carleson measure.*

*Proof.* Let $\{f_j\}$ be a sequence in $B_p(\mathbb{B}_n)$ satisfying $\|f_j\|_{B_1(\mathbb{B}_n)} \le 1$ and such that $f_j$ converges to 0 uniformly as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, and let $g \in \mathcal{B}(\mathbb{B}_n)$. By duality, we have that $T_\mu^\alpha$ is compact on $B_1(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} \left|\left\langle T_\mu^\alpha(f_j), g\right\rangle\right| = 0. \tag{3.18}$$

Thus, $T_\mu^\alpha$ is compact on $B_1(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} \left|\int_{\mathbb{B}_n} f_j(\mathbf{w})\overline{G(\mathbf{w})}\,d\mu_\alpha(\mathbf{w})\right| = 0. \tag{3.19}$$

Using the operator $P_\alpha$, we have that

$$\begin{aligned}\int_{\mathbb{B}_n} f_j(\mathbf{w})\overline{G(\mathbf{w})}\,d\mu_\alpha(\mathbf{w}) &= c_\alpha \int_{\mathbb{B}_n} \left[(I - P_\alpha)\left(f_j\overline{G}\right)\right](\mathbf{z})\,d\mu_\alpha(\mathbf{z}) + c_\alpha \int_{\mathbb{B}_n} P_\alpha\left(f_j\overline{h}\right)(\mathbf{z})\,d\mu_\alpha(\mathbf{z}). \\ &= J_1 + J_2.\end{aligned} \tag{3.20}$$

As in the proof of Theorem 2.3, we have

$$|J_1| \le C\int_{\mathbb{B}_n} \|f_j\|_{B_1(\mathbb{B}_n)}\|G\|_{A^1(\mathbb{B})}\left(1 - |\mathbf{z}|^2\right)^{-1/2} \int_{\mathbb{B}_n} \frac{\left(1 - |\mathbf{w}|^2\right)^{\alpha-1/2}}{|1 - \langle \mathbf{z}, \mathbf{w}\rangle|^{n+\alpha+1}}\,d\mu(\mathbf{w})\,d\nu(\mathbf{z}). \tag{3.21}$$

Notice that $\|f_j\|_{B_p(\mathbb{B}_n)}$ implies that $\|f_j\|_{B_1(\mathbb{B}_n)} \le C$. Since $f_j$ converges to 0 uniformly as $j \to \infty$ on compact subsets of $\mathbb{B}_n$, and $\mu$ is a $(A^p(\mathbb{B}_n), p)$-Carleson measure, we get that $G \in \mathcal{B}^{\alpha+2}(\mathbb{B}_n)$ and $\|G\|_{\mathcal{B}^{\alpha+2}(\mathbb{B}_n)} \le C\|g\|_{\mathcal{B}(\mathbb{B}_n)}$. Thus

$$\begin{aligned}|J_1| &\le C\|f_j\|_{B_1(\mathbb{B}_n)}\|g\|_{\mathcal{B}(\mathbb{B}_n)}, \\ \lim_{j\to\infty} &\sup_{\|g\|_{\mathcal{B}}\le 1} |J_1| = 0.\end{aligned} \tag{3.22}$$

Therefore, $T_\mu^\alpha$ is compact on $B_1(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty} \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} |J_2| = 0. \tag{3.23}$$

We have shown in the proof of Theorem 2.3

$$\sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} |J_2| \le C \sup_{\|g\|_{\mathcal{B}(\mathbb{B}_n)}\le 1} \int_{\mathbb{B}_n} |f_j(\mathbf{w})|\Re P_\alpha(\mu)(\mathbf{w})\,d\nu(\mathbf{w}). \tag{3.24}$$

Therefore, $T_\mu^\alpha$ is compact on $B_1(\mathbb{B}_n)$ if and only if

$$\lim_{j\to\infty}\int_{\mathbb{B}_n}\left|f_j(\mathbf{w})\right|\sum_{|m|=n+1}\left|\frac{\partial^m}{\partial\mathbf{w}^m}P_\alpha(\mu)(\mathbf{w})\right|d\nu(\mathbf{w})=0, \tag{3.25}$$

which is equivalent to saying that the measure $\sum_{|m|=n+1}|(\partial^m/\partial\mathbf{w}^m)P_\alpha(\mu)(\mathbf{w})|d\nu(\mathbf{w})$ is a vanishing $(B_1(\mathbb{B}_n),1)$-Carleson measure. □

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