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Research Article

Homogeneous Besov Spaces on Stratified Lie Groups and Their Wavelet Characterization

Hartmut Führ[1] and Azita Mayeli[2]

[1] *Lehrstuhl A für Mathematik, RWTH Aachen University, D-52056 Aachen, Germany*
[2] *Department of Mathematics and Computer Sciences, City University of New York (CUNY), Queensborough College, 222-05 56th Avenue Bayside, NY 11364, USA*

Correspondence should be addressed to Azita Mayeli, amayeli@qcc.cuny.edu

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We establish wavelet characterizations of homogeneous Besov spaces on stratified Lie groups, both in terms of continuous and discrete wavelet systems. We first introduce a notion of homogeneous Besov space $\dot{B}^s_{p,q}$ in terms of a Littlewood-Paley-type decomposition, in analogy to the well-known characterization of the Euclidean case. Such decompositions can be defined via the spectral measure of a suitably chosen sub-Laplacian. We prove that the scale of Besov spaces is independent of the precise choice of Littlewood-Paley decomposition. In particular, different sub-Laplacians yield the same Besov spaces. We then turn to wavelet characterizations, first via continuous wavelet transforms (which can be viewed as continuous-scale Littlewood-Paley decompositions), then via discretely indexed systems. We prove the existence of wavelet frames and associated atomic decomposition formulas for all homogeneous Besov spaces $\dot{B}^s_{p,q}$ with $1 \le p, q < \infty$ and $s \in \mathbb{R}$.

1. Introduction

To a large extent, the success of wavelets in applications can be attributed to the realization that wavelet bases are universal unconditional bases for a large class of smoothness spaces, including all homogeneous Besov spaces. Given a wavelet orthonormal basis $\{\psi_{j,k}\}_{j,k} \subset L^2(\mathbb{R}^n)$ (consisting of sufficiently regular wavelets with vanishing moments) and $f \in L^2(\mathbb{R}^n)$, the expansion

$$f = \sum_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k} \tag{1.1}$$

converges not only in $\|\cdot\|_{L^2}$, but also in any other Besov space norm $\|\cdot\|_{\dot{B}^s_{p,q}}$, as soon as f is contained in that space. Furthermore, the latter condition can be read off the decay behaviour of the wavelet coefficients $\{\langle f, \psi_{j,k}\rangle\}_{j,k}$ associated to f in a straightforward manner.

This observation provided important background and heuristics for many wavelet-based methods in applications such as denoising and data compression, but it was also of considerable theoretical interest, for example, for the study of operators. In this paper we provide similar results for simply connected stratified Lie groups. To our knowledge, studies of Besov spaces in this context have been largely restricted to the inhomogeneous cases. The definition of inhomogeneous Besov spaces on stratified Lie groups was introduced independently by Saka [1], and in a somewhat more general setting by Pesenson [2, 3]. Since then, the study of Besov spaces on Lie groups remained restricted to the inhomogeneous cases [4–8], with the notable exception of [9] which studied homogeneous Besov spaces on the Heisenberg group. A further highly influential source for the study of function spaces associated to the sub-Laplacian is Folland's paper [10].

The first wavelet systems on stratified Lie groups (fulfilling certain technical assumptions) were constructed by Lemarié [11], by suitably adapting concepts from spline theory. Lemarié also indicated that the wavelet systems constructed by his approach were indeed unconditional bases of Saka's inhomogeneous Besov spaces. Note that an adaptation, say, of the arguments in [12] for a proof of such a characterization requires a sampling theory for bandlimited functions on stratified groups, which was established only a few years ago by Pesenson [13]; see also [14].

More recent constructions of both continuous and discrete wavelet systems were based on the spectral theory of the sub-Laplacian [15]. Given the central role of the sub-Laplacian both in [8, 15], and in view of Lemarié's remarks, it seemed quite natural to expect a wavelet characterization of homogeneous Besov spaces, and it is the aim of this paper to work out the necessary details. New results in this direction were recently published in [16–18].

The paper is structured as follows. After reviewing the basic notions concerning stratified Lie groups and their associated sub-Laplacians in Section 2, in Section 3 we introduce a Littlewood-Paley-type decomposition of functions and tempered discributions on G. It is customary to employ the spectral calculus of a suitable sub-Laplacian for the definition of such decompositions, see, for example, [8], and this approach is also used here (Lemma 3.7). However, this raises the issue of consistency: the spaces should reflect properties of the group, not of the sub-Laplacian used for the construction of the decomposition. Using a somewhat more general notion than the ϕ-functions in [12] allows to establish that different choices of sub-Laplacian result in the same scale of Besov spaces (Theorem 3.11). In Section 4, we derive a characterization of Besov spaces in terms of continuous wavelet transform, with a wide variety of wavelets to choose from (Theorem 4.4). As a special case one obtains a characterization of homogeneous Besov spaces in terms of the heat semigroup. (See the remarks before Theorem 4.4.)

In Section 5, we study discrete characterizations of Besov spaces obtained by sampling the Calderón decomposition. For this purpose, we introduce the coefficient space $\dot{b}^s_{p,q}$. The chief result is Theorem 5.4, establishing that the wavelet coefficient sequence of $f \in \dot{B}^s_{p,q}$ lies in $\dot{b}^s_{p,q}$. Section 5 introduces our most important tool to bridge the gap between continuous and discrete decompositions, namely, oscillation estimates.

We then proceed to study wavelet synthesis and frame properties of the wavelet system. Our main result in this respect is that for all sufficiently dense regular sampling sets Γ, the discrete wavelet system $\{\psi_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ obtained by shifts from γ and dilations by powers of 2 is a *universal Banach frame* for all Besov spaces. In other words, the wavelet system allows the decomposition

$$f = \sum_{j,\gamma} r_{j,\gamma} \psi_{j,\gamma} \tag{1.2}$$

converging unconditionally in $\dot{B}^s_{p,q}$ whenever $f \in \dot{B}^s_{p,q}$, with coefficients $\{r_{j,\gamma}\}_{j,\gamma} \in \dot{b}^s_{p,q}$ depending linearly and boundedly on f, and satisfying the norm equivalence

$$\left\| \{r_{j,\gamma}\}_{j,\gamma} \right\|_{\dot{b}^s_{p,q}} \asymp \|f\|_{\dot{B}^s_{p,q}}. \tag{1.3}$$

2. Preliminaries and Notation

Following the terminology in [19], we call a Lie group G stratified if it is connected and simply connected, and its Lie algebra $\mathfrak{g}$ decomposes as a direct sum $\mathfrak{g} = V_1 \oplus \cdots \oplus V_m$, with $[V_1, V_k] = V_{k+1}$ for $1 \le k < m$ and $[V_1, V_m] = \{0\}$. Then $\mathfrak{g}$ is nilpotent of step m and generated as a Lie algebra by V_1. Euclidean spaces $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}^n$ are examples of stratified Lie groups.

If G is stratified, its Lie algebra admits a canonical (natural) family of dilations, namely,

$$\delta_r(X_1 + X_2 + \cdots + X_m) = rX_1 + r^2 X_2 + \cdots + r^m X_m \quad \left(X_j \in V_j\right) \ (r > 0), \tag{2.1}$$

which are Lie algebra automorphisms. We identify G with $\mathfrak{g}$ through the exponential map. Hence G is a Lie group with underlying manifold $\mathbb{R}^n$, for some n, and the group product provided by the Campbell-Baker-Hausdorff formula. The dilations are then also group automorphisms of G. Instead of writing $\delta_a(x)$ for $x \in G$ and $a > 0$, we simply use ax, whenever a confusion with the Lie group product is excluded. After choosing a basis of $\mathfrak{g}$ obtained as a union of bases of the V_i, and a possible change of coordinates, one therefore has for $x \in G$ and $a > 0$ that

$$ax = \left(a^{d_1} x_1, \ldots, a^{d_n} x_n\right), \tag{2.2}$$

for integers $d_1 \le \cdots \le d_n$, according to $x_i \in V_{d_i}$.

Under our identification of G with $\mathfrak{g}$, polynomials on G are polynomials on $\mathfrak{g}$ (with respect to any linear coordinate system on the latter). Polynomials on G are written as

$$p\left(\sum_{i=1}^{\dim(G)} x_i Y_i\right) = \sum_I c_I x^I, \tag{2.3}$$

where $c_I \in \mathbb{C}$ are the coefficients with respect to a suitable basis $Y_1, Y_2, \ldots$, and $x^I = x_1^{I_1} x_2^{I_2}, \ldots, x_n^{I_n}$ the monomials associated to the multi-indices $I \in \mathbb{N}^{\{1,\ldots,n\}}$. For a multi-index I, define

$$d(I) = \sum_{i=1}^{n} I_i n(i), \quad n(i) = j \text{ for } Y_i \in V_j. \tag{2.4}$$

A polynomial of the type (2.3) is called *of homogeneous degree* k if $d(I) \leq k$ holds, for all multiindices I with $c_I \neq 0$. We write $\mathcal{P}_k$ for the space of polynomials of homogeneous degree k.

We let $\mathcal{S}(G)$ denote the space of Schwartz functions on G. By definition, $\mathcal{S}(G) = \mathcal{S}(\mathfrak{g})$. Let $\mathcal{S}'(G)$ and $\mathcal{S}'(G)/\mathcal{P}$ denote the space of distributions and distributions modulo polynomials on G, respectively. The duality between the spaces is denoted by the map $(\cdot,\cdot) : \mathcal{S}'(G) \times \mathcal{S}(G) \to \mathbb{C}$. Most of the time, however, we will work with the sesquilinear version $\langle f, g\rangle = (f, \overline{g})$, for $f \in \mathcal{S}'(G)$ and $g \in \mathcal{S}(G)$.

Left Haar measure on G is induced by Lebesgue measure on its Lie algebra, and it is also right-invariant. The number $Q = \sum_1^m j(\dim V_j)$ will be called the *homogeneous dimension* of G. (For instance, for $G = \mathbb{R}^n$ and $\mathbb{H}^n$ we have $Q = n$ and $Q = 2n+2$, respectively.) For any function ϕ on G and $a > 0$, the L^1-normalized dilation of ϕ is defined by

$$D_a\phi(x) = a^Q \phi(ax). \tag{2.5}$$

Observe that this action preserves the L^1-norm, that is, $\|D_a\phi\|_1 = \|\phi\|$. We fix a homogeneous quasi-norm $|\cdot|$ on G which is smooth away from 0 with, $|ax| = a|x|$ for all $x \in G$, $a \geq 0$, $|x^{-1}| = |x|$ for all $x \in G$, with $|x| > 0$ if $x \neq 0$, and fulfilling a triangle inequality $|xy| \leq C(|x| + |y|)$, with constant $C > 0$. Confer [19] for the construction of homogeneous norms, as well as further properties.

Moreover, by [19, Proposition 1.15], for any $r > 0$, there is a finite $C_r > 0$ such that $\int_{|x|>R} |x|^{-Q-r} dx = C_r R^{-r}$ for all $R > 0$.

Our conventions for left-invariant operators on G are as follows. We let $Y_1, \ldots, Y_n$ denote a basis of $\mathfrak{g}$, obtained as a union of bases of the V_i. In particular, $Y_1, \ldots, Y_l$, for $l = \dim(V_1)$, is a basis of V_1. Elements of the Lie algebra are identified in the usual manner with left-invariant differential operators on G. Given a multi-index $I \in \mathbb{N}_0^n$, we write Y^I for $Y_1^{I_1} \circ \cdots \circ Y_n^{I_n}$. A convenient characterization of Schwartz functions in terms of left-invariant operators states that $f \in \mathcal{S}(G)$ if and only if, for all $N \in \mathbb{N}$, $|f|_N < \infty$, where

$$|f|_N = \sup_{|I|\leq N, x\in G} (1+|x|)^N \left| Y^I f(x) \right|. \tag{2.6}$$

In addition, the norms $|\cdot|_N$ induce the topology of $\mathcal{S}(G)$ (see [19]).

The sub-Laplacian operator on G can be viewed as the analog of the Laplacian operator on $\mathbb{R}^n$ defined by $L = -\sum_{i=1}^n \partial^2/\partial x_k^2$. Using the above conventions for the choice of basis $Y_1, \ldots, Y_n$ and $l = \dim(V_1)$, the sub-Laplacian is defined as $L = -\sum_{i=1}^l Y_i^2$. Note that a less restrictive notion of sub-Laplacians can also be found in the literature (e.g., any sum of squares of Lie algebra generators); we stress that the results in this paper crucially rely on the definition presented here. A linear differential operator T on G is called homogenous of degree l if $T(f \circ \delta_a) = a^l (Tf) \circ \delta_a$ for any f on G. By choice of the Y_i for $i \leq l$, these operators are homogeneous of degree 1; it follows that L is homogenous of degree 2, and L^k is homogenous of degree $2k$. Furthermore, any operator of the form Y^I is homogeneous of degree $d(I)$.

When restricted to C_c^∞, L is formally self-adjoint: for any $f, g \in C_c^\infty(G)$, $\langle Lf, g\rangle = \langle f, Lg\rangle$. (For more see [15].) Its closure has domain $\mathfrak{D} = \{u \in L^2(G) : Lu \in L^2(G)\}$, where we take Lu in the sense of distributions. From this fact it quickly follows that this closure is self-adjoint and is in fact the unique self-adjoint extension of $L|_{C_c^\infty}$; we denote this extension also by the symbol L.

Suppose that L has spectral resolution

$$L = \int_0^\infty \lambda dP_\lambda, \tag{2.7}$$

where dP_λ is the projection measure. For a bounded Borel function $\hat{f}$ on $[0, \infty)$, the operator

$$\hat{f}(L) = \int_0^\infty \hat{f}(\lambda) dP_\lambda \tag{2.8}$$

is a bounded integral operator on $L^2(G)$ with a convolution distribution kernel in $L^2(G)$ denoted by f, and

$$\hat{f}(L)\eta = \eta * f \quad \forall \eta \in \mathcal{S}(G). \tag{2.9}$$

An important fact to be used later on is that for rapidly decaying smooth functions, $f \in \mathcal{S}(\mathbb{R}^+)$, the kernel associated to $\hat{f}(L)$ is a Schwartz function. For a function f on G we define $\tilde{f}(x) = f(x^{-1})$ and $f^* = \overline{\tilde{f}}$. For $f \in L^2(G) \cap L^1(G)$, the adjoint of the convolution operator $g \mapsto g * f$ is provided by $g \mapsto g * f^*$.

3. Homogeneous Besov Spaces on Stratified Lie Groups

In this section we define homogeneous Besov spaces on stratified Lie groups via Littlewood-Paley decompositions of distributions u as

$$u = \sum_{j \in \mathbb{Z}} u * \psi_j^* * \psi_j, \tag{3.1}$$

where ψ_j is a dilated copy of a suitably chosen Schwartz function ψ. In the Euclidean setting, it is customary to construct ψ by picking a dyadic partition of unity on the Fourier transform side and applying Fourier inversion. The standard way of transferring this construction to stratified Lie groups consists in replacing the Fourier transform by the spectral decomposition of a sub-Laplacian L, see Lemma 3.7. However, this approach raises the question to what extent the construction depends on the choice of L. It turns out that the precise choice of sub-Laplacian obtained from a basis of V_1 is irrelevant. In order to prove this, we study Littlewood-Paley decompositions in somewhat different terms. The right setting for the study of such decompositions is the space of tempered distributions modulo polynomials, and the easiest approach to this convergence is via duality to a suitable space of Schwartz functions.

Definition 3.1. Let $N \in \mathbb{N}$. A function $f : G \to \mathbb{C}$ has polynomial decay order N if there exists a constant $C > 0$ such that, for all $x \in G$,

$$\left|f(x)\right| \leq C(1+|x|)^{-N}. \tag{3.2}$$

f has vanishing moments of order N, if one has

$$\forall p \in \mathcal{P}_{N-1} \ : \ \int_G f(x)p(x)dx = 0, \tag{3.3}$$

with absolute convergence of the integral.

Under our identification of G with $\mathfrak{g}$, the inversion map $x \mapsto x^{-1}$ is identical to the additive inversion map. That is, $x^{-1} = -x$, and it follows that $\widetilde{p} \in \mathcal{P}_N$ for all $p \in \mathcal{P}_N$. Thus, if f has vanishing moments of order N, then for all $p \in \mathcal{P}_{N-1}$

$$\int_G \widetilde{f}(x)p(x)dx = \int_G f(x)\widetilde{p}(x)dx = 0, \tag{3.4}$$

that is, $\widetilde{f}$ has vanishing moments of order N as well.

Vanishing moments are central to most estimates in wavelet analysis, by the following principle: in a convolution product of the type $g * D_t f$, vanishing moments of one factor together with smoothness of the other result in decay. Later on, we will apply the lemma to Schwartz functions f, g, where only the vanishing moment assumptions are nontrivial. The more general version given here is included for reference.

Lemma 3.2. *Let $N, k \in \mathbb{N}$ be arbitrary.*

(a) *Let $f \in C^k$, such that $Y^I(f)$ is of decay order N, for all I with $d(I) \leq k$. Let g have vanishing moments of order k and decay order $N+k+Q+1$. Then there exists a constant, depending only on the decay of $Y^I(f)$ and g, such that*

$$\forall x \in G \quad \forall 0 < t < 1 \colon \left|g * \left(D_t f\right)(x)\right| \leq Ct^{k+Q}(1+|tx|)^{-N}. \tag{3.5}$$

In particular, if $p > Q/N$,

$$\forall x \in G \quad \forall 0 < t < 1 \colon \left\|g * \left(D_t f\right)\right\|_p \leq C' t^{k+Q(1-1/p)}. \tag{3.6}$$

(b) *Now suppose that $g \in C^k$, with $Y^I(\widetilde{g})$ of decay order N for all I with $d(I) \leq k$. Let f have vanishing moments of order k and decay order $N+k+Q+1$. Then there exists a constant, depending only on the decay of f and $Y^I(\widetilde{g})$, such that*

$$\forall x \in G \quad \forall 1 < t < \infty \colon \left|g * \left(D_t f\right)(x)\right| \leq Ct^{-k}(1+|x|)^{-N}. \tag{3.7}$$

In particular, if $p > Q/N$,

$$\forall x \in G \quad \forall 1 < t < \infty : \left\| g * (D_t f) \right\|_p \le C' t^{-k}. \tag{3.8}$$

Proof. First, let us prove (a). Let $0 < t < 1$. For $x \in G$, let $P^k_{x, D_t \widetilde{f}}$ denote the left Taylor polynomial of $D_t \widetilde{f}$ with homogeneous degree $k-1$, see [19, Definition 1.44]. By that result,

$$\left| D_t f\left(y^{-1} x\right) - P^k_{x, D_t \widetilde{f}}(y) \right| \le C_k |y|^k \sup_{|z| \le b^k |y|, d(I) = k} \left| Y^I \left(D_t \widetilde{f} \right)(xz) \right|, \tag{3.9}$$

with suitable positive constants C_k and b. We next use the homogeneity properties of the partial derivatives [19, page 21], together with the decay condition on $Y^I f$ to estimate for I with $d(I) = k$

$$\begin{aligned}
\sup_{|z| \le b^k |y|} \left| Y^I \left(D_t \widetilde{f} \right)(xz) \right| &= t^k \sup_{|z| \le b^k |y|} \left| D_t \left(Y^I \widetilde{f} \right)(xz) \right| \\
&= t^{k+Q} \sup_{|z| \le b^k |y|} \left| \left(Y^I \widetilde{f} \right)(t(x \cdot z)) \right| \\
&\le t^{k+Q} \sup_{|z| \le b^k |y|} C_f (1 + |t(x \cdot z)|)^{-N} \\
&\le t^{k+Q} \sup_{|z| \le b^k |y|} C_f (1 + |tx|)^{-N} (1 + |tz|)^N \\
&\le t^{k+Q} (1+b)^{kN} C_f (1 + |tx|)^{-N} (1 + |y|)^N,
\end{aligned} \tag{3.10}$$

where the penultimate inequality used [19, 1.10], and the final estimate used $|ty| = t|y| \le |y|$. Thus,

$$\left| D_t f\left(y^{-1} x\right) - P^k_{x, D_t \widetilde{f}}(y) \right| \le \widetilde{C}_k t^{k+Q} (1 + |y|)^{N+k} (1 + |tx|)^{-N}. \tag{3.11}$$

Next, using vanishing moments of g,

$$\begin{aligned}
|(g * D_t f)(x)| &\le \int_G |g(y)| \left| D_t f\left(y^{-1} x\right) - P^k_{x, D_t \widetilde{f}}(y) \right| dy \\
&\le \widetilde{C}_k (1 + |tx|)^{-N} t^{k+Q} \int_G |g(y)| (1 + |y|)^{N+k} dy \\
&\le \widetilde{C}_k (1 + |tx|)^{-N} t^{k+Q} \int_G C_g (1 + |y|)^{-Q-1} dy,
\end{aligned} \tag{3.12}$$

and the integral is finite by [19, 1.15]. This proves (3.5), and (3.6) follows by

$$\|g * D_t f\|_p \leq C' t^{k+Q} \left(\int_G (1 + |tx|)^{-Np} dx \right)^{1/p} \leq C'' t^{k+Q-Q/p}, \tag{3.13}$$

using $Np > Q$.

For part (b), we first observe that

$$(g * D_t f)(x) = t^Q \left(\widetilde{f} * D_{t^{-1}} \widetilde{g} \right)(t \cdot x). \tag{3.14}$$

Our assumptions on f, g allow to invoke part (a) with $\widetilde{g}, \widetilde{f}$ replacing f, g, and (3.7) follows immediately. (3.8) is obtained from this by straightforward integration. □

We let $\mathcal{Z}(G)$ denote the space of Schwartz functions with all moments vanishing. We next consider properties of $\mathcal{Z}(G)$ as a subspace of $\mathcal{S}(G)$ with the relative topology.

Lemma 3.3. *$\mathcal{Z}(G)$ is a closed subspace (in particular complete) of $\mathcal{S}(G)$, with $\mathcal{S}(G) * \mathcal{Z}(G) \subset \mathcal{Z}(G)$, as well as $\widetilde{f} \in \mathcal{Z}(G)$ for all $f \in \mathcal{Z}(G)$. The topological dual of $\mathcal{Z}(G)$, $\mathcal{Z}'(G)$, can be canonically identified with the factor space $\mathcal{S}'(G)/\mathcal{P}$.*

Proof. By definition, $\mathcal{Z}(G)$ is the intersection of kernels of a family of tempered distributions, hence a closed subspace. For $p \in \mathcal{P}$ and $f \in \mathcal{Z}(G)$, one has by unimodularity of G that $\langle p, \widetilde{f} \rangle = \langle \widetilde{p}, f \rangle = 0$, since $\widetilde{p}$ is a polynomial. But then, for any $g \in \mathcal{S}(G)$ and $f \in \mathcal{Z}(G)$, one has for all polynomials p on G that

$$\langle g * f, p \rangle = \left\langle g, p * \widetilde{f} \right\rangle = \langle g, 0 \rangle = 0, \tag{3.15}$$

since $f \in \mathcal{Z}(G)$ implies $p * \widetilde{f} = 0$ (translation on G is polynomial). Thus $g * f \in \mathcal{Z}(G)$. All further properties of $\mathcal{Z}(G)$ follow from the corresponding statements concerning $\mathcal{Z}(\mathbb{R}^n)$. For identification of $\mathcal{Z}'(\mathbb{R}^n)$ with the quotient space $\mathcal{S}(\mathbb{R}^n)'/\mathcal{P}$, we first observe that a tempered distribution φ vanishes on $\mathcal{Z}(\mathbb{R}^n)$ if and only if its (Euclidean) Fourier transform is supported in $\{0\}$, which is well known to be the case if and only if φ is a polynomial. Using this observation, we map $u \in \mathcal{Z}'(\mathbb{R}^n)$ to $\widetilde{u}+\mathcal{P}$, where $\widetilde{u}$ is a continuous extension of u to all of $\mathcal{S}(\mathbb{R}^n)$; such an extension exists by the Hahn-Banach theorem. The map is well defined because the difference between two extensions of u annihilates $\mathcal{Z}(\mathbb{R}^n)$ and hence is a polynomial. Linearity follows from well-definedness. Furthermore, the inverse of the mapping is clearly obtained by assigning $w + \mathcal{P}$ to the restriction $w|_{\mathcal{Z}(G)}$. □

In the following, we will usually not explicitly distinguish between $u \in \mathcal{S}'(G)$ and its equivalence class modulo polynomials, and we will occasionally write $u \in \mathcal{S}'(G)/\mathcal{P}$. The topology of $\mathcal{S}'(G)/\mathcal{P}$ is just the topology of pointwise convergence on the elements of $\mathcal{Z}(G)$. For any net $(u_j + \mathcal{P})_{j \in I}$, $u_j + \mathcal{P} \to u + \mathcal{P}$ holds if and only if $\langle u_j, \varphi \rangle \to \langle u, \varphi \rangle$, for all $\varphi \in \mathcal{Z}(G)$. We next study convolution on $\mathcal{S}'(G)/\mathcal{P}$.

Lemma 3.4. *For every $\psi \in \mathcal{S}(G)$, the map $u \mapsto u * \psi$ is a well-defined and continuous operator $\mathcal{S}'(G)/\mathcal{P} \to \mathcal{S}'(G)/\mathcal{P}$. If $\psi \in \mathcal{Z}(G)$, the associated convolution operator is a well-defined and continuous operator $\mathcal{S}'(G)/\mathcal{P} \to \mathcal{S}'(G)$.*

Proof. Note that $\mathcal{P} * \mathcal{S}(G) \subset \mathcal{P}$. Hence $u \mapsto u * \psi$ induces a well-defined canonical map $\mathcal{S}'(G)/\mathcal{P} \to \mathcal{S}'(G)/\mathcal{P}$. Furthermore, $u \mapsto u * \psi$ is continuous on $\mathcal{S}'(G)$, as a consequence of [19, Proposition 1.47]. Therefore, for any net $u_j \to u$ and any $\varphi \in \mathcal{Z}(G)$, the fact that $\varphi * \psi^* \in \mathcal{Z}(G)$ allows to write

$$\langle u_j * \psi, \varphi \rangle = \langle u_j, \varphi * \psi^* \rangle \longrightarrow \langle u, \varphi * \psi^* \rangle = \langle u * \psi, \varphi \rangle, \tag{3.16}$$

showing $u_j * \psi \to u * \psi$ in $\mathcal{S}'(G)/\mathcal{P}$.

For $\psi \in \mathcal{Z}(G)$, the fact that $\mathcal{P} * \psi = \{0\}$ makes the mapping $u \mapsto u * \psi \in \mathcal{S}'(G)$ well-defined modulo polynomials. The continuity statement is proved by (3.16), with assumptions on ψ and φ switched. □

The definition of homogeneous Besov spaces requires taking L^p-norms of elements of $\mathcal{S}'(G)/\mathcal{P}$. The following remark clarifies this.

Remark 3.5. Throughout this paper, we use the canonical embedding $L^p(G) \subset \mathcal{S}'(G)$. For $p < \infty$, this gives rise to an embedding $L^p(G) \subset \mathcal{S}'(G)/\mathcal{P}$, using that $\mathcal{P} \cap L^p(G) = \{0\}$. Consequently, given $u \in \mathcal{S}'(G)/\mathcal{P}$, we let

$$\|u\|_p = \|u + q\|_p \quad \text{whenever } u + q \in L^p(G), \quad \text{for suitable } q \in \mathcal{P} \tag{3.17}$$

assigning the value ∞ otherwise. Here the fact that $\mathcal{P} \cap L^p(G) = \{0\}$ guarantees that the decomposition is unique, and thus (3.17) well-defined.

By contrast, $\|\cdot\|_\infty$ can only be defined on $\mathcal{S}'(G)$, if we assign the value ∞ to $u \in \mathcal{S}'(G) \setminus L^\infty(G)$.

Note that with these definitions, the Hausdorff-Young inequality $\|u * f\|_p \leq \|u\|_p \|f\|_1$ remains valid for all $f \in \mathcal{S}(G)$, and all $u \in \mathcal{S}'(G)/\mathcal{P}$ (for $p < \infty$), respectively, $u \in \mathcal{S}'(G)$ (for $p = \infty$). For $p = \infty$, this is clear. For $p < \infty$, note that if $u + q \in L^p(G)$, then $(u + q) * \psi = u * \psi + q * \psi \in L^p(G)$ with $q * \psi \in \mathcal{P}$.

We now introduce a general Littlewood-Paley-type decomposition. For this purpose we define for $\psi \in \mathcal{S}(G)$,

$$\psi_j = D_{2^j} \psi. \tag{3.18}$$

Definition 3.6. A function $\psi \in \mathcal{S}(G)$ is called LP-admissible if for all $g \in \mathcal{Z}(G)$,

$$g = \lim_{N \to \infty} \sum_{|j| \leq N} g * \psi_j^* * \psi_j \tag{3.19}$$

holds, with convergence in the Schwartz space topology. Duality entails the convergence

$$u = \lim_{N\to\infty} \sum_{|j|\le N} u * \psi_j^* * \psi_j \tag{3.20}$$

for all $u \in \mathcal{S}'(G)/\mathcal{P}$.

The following lemma yields the chief construction of LP-admissible functions.

Lemma 3.7. *Let $\widehat{\phi}$ be a function in C^∞ with support in $[0,4]$ such that $0 \le \widehat{\phi} \le 1$ and $\widehat{\phi} \equiv 1$ on $[0,1/4]$. Take $\widehat{\psi}(\xi) = \sqrt{\widehat{\phi}(2^{-2}\xi) - \widehat{\phi}(\xi)}$. Thus, $\widehat{\psi} \in C_c^\infty(\mathbb{R}^+)$, with support in the interval $[1/4,4]$, and*

$$1 = \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}\left(2^{2j}\xi\right)\right|^2 \ a.e. \tag{3.21}$$

Pick a sub-Laplacian L, and let ψ denote the convolution kernel associated to the bounded left-invariant operator $\widehat{\psi}(L)$. Then ψ is LP-admissible, with $\psi \in \mathcal{Z}(G)$.

Proof. Let us first comment on the properties of ψ that are immediate from the construction via spectral calculus: $\psi \in \mathcal{S}(G)$ follows from [20] and vanishing moments by [15, Proposition 1].

Now let $g \in \mathcal{Z}(G)$. First note that 2-homogeneity of L implies that the convolution kernel associated to $\widehat{\psi}(2^{-2j}\cdot)(L)$ coincides with ψ_j. Then, by the spectral theorem and (3.21),

$$g = \sum_{j\in\mathbb{Z}} \left[\widehat{\psi}\left(2^{-2j}\cdot\right)(L)\right]^* \circ \left[\widehat{\psi}\left(2^{-2j}\cdot\right)(L)\right] g = \sum_{j\in\mathbb{Z}} g * \psi_j^* * \psi_j \tag{3.22}$$

holds in L^2-norm.

For any positive integer N,

$$\sum_{|j|\le N} g * \psi_j^* * \psi_j = g * D_{2^{N+1}}\phi - g * D_{2^{-N}}\phi, \tag{3.23}$$

where $\phi \in \mathcal{S}(G)$ is the convolution kernel of $\widehat{\phi}(L)$. Since ϕ is a Schwartz function, it follows by [19, Proposition (1.49)] that $g * D_{2^{N+1}}\phi \to c_\phi g$, for $N \to \infty$, for all $g \in \mathcal{S}(G)$, with convergence in $\mathcal{S}(G)$ and a suitable constant c_ϕ.

We next show that $g * D_t f \to 0$ in $\mathcal{S}(G)$, as $t \to 0$, for any $f \in \mathcal{S}(G)$. Fix a multi-index I and $N, k \in \mathbb{N}$ with $k \ge N$. Then left-invariance and homogeneity of Y^I yield

$$\begin{aligned} \left|Y^I(g * D_t f)(x)\right| &= t^{d(I)} \left|g * D_t\left(Y^I f\right)(x)\right| \\ &\le C_{f,g} t^{k+Q+d(I)} (1+|tx|)^{-N} \\ &\le C_{f,g} t^{k+Q+d(I)-N} (1+|x|)^{-N}. \end{aligned} \tag{3.24}$$

Here the first inequality is an application of (3.5); the constant $C_{f,g}$ can be estimated in terms of $|f|_M, |g|_M$, for M sufficiently large. But this proves $g * D_t f \to 0$ in the Schwartz topology.

Summarizing, $\sum_{|j|\le N} g * \psi_j^* * \psi_j \to c_\phi g$ in $\mathcal{S}(G)$, and in addition by (3.22), $\sum_{|j|\le N} g * \psi_j^* * \psi_j \to g$ in L^2, whence $c_\phi = 1$ follows. □

Note that an LP-admissible function ψ as constructed in Lemma 3.7 fulfills the convenient relation

$$\forall j, l \in \mathbb{Z}: \ \left|j-l\right| > 1 \Longrightarrow \psi_j^* * \psi_l = 0, \tag{3.25}$$

which follows from $[\widehat{\psi}(2^{-2j}\cdot)(L)] \circ [\widehat{\psi}(2^{-2l}\cdot)(L)] = 0$.

Remark 3.8. By spectral calculus, we find that $\psi = L^k g_k$, with $g_k \in \mathcal{Z}(G)$. In particular, the decomposition

$$\begin{aligned} f &= \lim_{N\to\infty} \sum_{|j|\le N} f * \psi_j^* * D_{2^j} L^k(g_k) \\ &= \lim_{N\to\infty} L^k \left(\sum_{|j|\le N} f * \psi_j^* * 2^{-kj} D_{2^j} g_k \right) \end{aligned} \tag{3.26}$$

shows that $L^k(\mathcal{Z}(G)) \subset \mathcal{Z}(G)$ is dense.

We now associate a scale of homogeneous Besov spaces to the function ψ.

Definition 3.9. Let $\psi \in \mathcal{Z}(G)$ be LP-admissible, let $1 \le p \le \infty$, $1 \le q \le \infty$, and $s \in \mathbb{R}$. *The homogeneous Besov space associated to ψ is defined as*

$$\dot{B}^{s,\psi}_{p,q} = \left\{ u \in \mathcal{S}'(G)/\mathcal{P}: \ \left\{ 2^{js} \left\| u * \psi_j^* \right\|_p \right\}_{j\in\mathbb{Z}} \in \ell^q(\mathbb{Z}) \right\}, \tag{3.27}$$

with associated norm

$$\|u\|_{\dot{B}^{s,\psi}_{p,q}} = \left\| \left\{ 2^{js} \|u * \psi_j^*\|_p \right\}_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}. \tag{3.28}$$

Remark 3.10. The definition relies on the conventions regarding L^p-norms of distributions (modulo polynomials), as outlined in Remark 3.5. Definiteness of the Besov norm holds because of (3.20).

The combination of Lemma 3.7 with Definition 3.9 shows that we cover the homogeneous Besov spaces defined in the usual manner via the spectral calculus of sub-Laplacians. Hence the following theorem implies in particular that different sub-Laplacians yield the same homogeneous Besov spaces (at least within the range of sub-Laplacians that we consider).

Theorem 3.11. *Let $\psi^1, \psi^2 \in \mathcal{Z}(G)$ be LP-admissible. Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Then, $\dot{B}_{p,q}^{s,\psi^1} = \dot{B}_{p,q}^{s,\psi^2}$, with equivalent norms.*

Proof. It is sufficient to prove the norm equivalence, and here symmetry with respect to ψ^1 and ψ^2 immediately reduces the proof to showing, for a suitable constant $C > 0$,

$$\forall u \in \mathcal{S}'(G)/\mathcal{P}\colon \ \|u\|_{\dot{B}_{p,q}^{s,\psi^1}} \le C\|u\|_{\dot{B}_{p,q}^{s,\psi^2}}, \tag{3.29}$$

in the extended sense that the left-hand side is finite whenever the right-hand side is. Hence assume that $u \in \dot{B}_{p,q}^{s,\psi^2}$; otherwise, there is nothing to show. In the following, let $\psi_{i,j} = D_{2^j}\psi^i$ $(i = 1,\ 2)$.

By LP-admissibility of ψ^2,

$$u = \lim_{N\to\infty} \sum_{|j|\le N} u * \psi_{2,j}^* * \psi_{2,j}, \tag{3.30}$$

with convergence in $\mathcal{S}'(G)/\mathcal{P}$. Accordingly,

$$u * \psi_{1,\ell}^* = \lim_{N\to\infty} \sum_{|j|\le N} u * \psi_{2,j}^* * \psi_{2,j} * \psi_{1,\ell}^*, \tag{3.31}$$

where the convergence on the right-hand side holds in $\mathcal{S}'(G)$, by Lemma 3.4. We next show that the right-hand side also converges in L^p. For this purpose, we observe that

$$\left\|\psi_{2,j} * \psi_{1,\ell}^*\right\|_1 = \left\|D_{2^j}\left(\psi^2 * D_{2^{\ell-j}}\psi_1^{1*}\right)\right\|_1 = \left\|\psi^2 * D_{2^{\ell-j}}\psi^{1*}\right\|_1 \le C2^{-|\ell-j|k}, \tag{3.32}$$

where $k > s$ is a fixed integer. For $\ell - j \ge 0$, this follows directly from (3.8), using $\psi^1, \psi^2 \in \mathcal{S}(G)$, and vanishing moments of ψ^1, whereas for $\ell - j < 0$, the vanishing moments of ψ^2 allow to apply (3.6).

Using Young's inequality, we estimate with C from above that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \left\|u * \psi_{2,j}^* * \psi_{2,j} * \psi_{1,\ell}^*\right\|_p &\le \sum_{j\in\mathbb{Z}} \left\|u * \psi_{2,j}^*\right\|_p \left\|\psi_{2,j} * \psi_{1,\ell}^*\right\|_1 \\ &\le C\left\|u * \psi_{2,j}^*\right\|_p 2^{-|j-\ell|k} \qquad (3.33) \\ &\le C\sum_{j\in\mathbb{Z}} 2^{js}\left\|u * \psi_{2,j}^*\right\|_p 2^{-|j-\ell|k-js}. \qquad (3.34)\end{aligned}$$

Next observe that

$$2^{-|j-\ell|k-js} = 2^{-\ell s} \cdot \begin{cases} 2^{-|j-\ell|(k+s)} & j \ge \ell \\ 2^{-|j-\ell|(k-s)} & j < l \end{cases} \le 2^{-\ell s} 2^{-|j-\ell|(k-|s|)}. \tag{3.35}$$

By assumption, the sequence $(2^{js}\|u * \psi_{j,2}^*\|_p)_{j\in\mathbb{Z}}$ is in ℓ^q, in particular, bounded. Therefore, $k - |s| > 0$ yields that (3.34) converges. But then the right-hand side of (3.31) converges unconditionally with respect to $\|\cdot\|_p$. This limit coincides with the $\mathcal{S}'(G)/\mathcal{P}$-limit $u * \psi_{1,\ell}^*$ (which because of $\psi_{1,\ell}^* \in \mathcal{Z}(G)$ is even a $\mathcal{S}'(G)$-limit), yielding $u * \psi_{1,\ell}^* \in L^p(G)$, with

$$\begin{aligned} 2^{\ell s}\left\|u * \psi_{1,\ell}^*\right\|_p &\leq 2^{\ell s}\sum_{j\in\mathbb{Z}}\left\|u * \psi_{2,j}^* * \psi_{2,j} * \psi_{1,\ell}^*\right\|_p \\ &\leq C_3 2^{\ell s}\sum_{j\in\mathbb{Z}} 2^{js}\left\|u * \psi_{2,j}^*\right\|_p 2^{-|j-\ell|(k-|s|)}. \end{aligned} \tag{3.36}$$

Now an application of Young's inequality for convolution over $\mathbb{Z}$, again using $k - |s| > 0$, provides (3.29). □

As a consequence, we write $\dot{B}_{p,q}^s = \dot{B}_{p,q}^{s,\psi}$, for any LP-admissible $\psi \in \mathcal{Z}(G)$. These spaces coincide with the homogeneous Besov spaces for the Heisenberg group in [9], and with the usual definitions in the case $G = \mathbb{R}^n$.

In the remainder of the section we note some functional-analytic properties of Besov spaces and Littlewood-Paley-decompositions for later use.

Lemma 3.12. *For all $1 \leq p, q \leq \infty$ and all $s \in \mathbb{R}$, one has continuous inclusion maps $\mathcal{Z}(G) \hookrightarrow \dot{B}_{p,q}^s \hookrightarrow S'(G)/\mathcal{P}$, as well as $\mathcal{Z}(G) \hookrightarrow \dot{B}_{p,q}^{s*}$, where the latter denotes the dual of $\dot{B}_{p,q}^s$. For $p, q < \infty$, $\mathcal{Z}(G) \subset \dot{B}_{p,q}^s$ is dense.*

Proof. We pick ψ as in Lemma 3.7 and define $\Delta_j g = g * \psi_j^*$ for $g \in \mathcal{S}'(G)$. For the inclusion $\mathcal{Z}(G) \subset \dot{B}_{p,q}^s$, note that (3.6) and (3.8) allow to estimate for all $g \in \mathcal{Z}(G)$ and $k \in \mathbb{N}$ that

$$\left\|\Delta_j g\right\|_p \leq C_k 2^{-|j|k}. \tag{3.37}$$

Here the constant C_k is a suitable multiple of $|g|_M$, for $M = M(k)$ sufficiently large. But this implies that $\mathcal{Z}(G) \subset \dot{B}_{p,q}^s$ continuously.

For the other embedding, repeated applications of Hölder's inequality yield the estimate

$$\begin{aligned} |\langle f, g\rangle| &= \left|\sum_{j\in\mathbb{Z}}\left\langle f, g * \psi_j^* * \psi_j\right\rangle\right| \\ &\leq \sum_{j\in\mathbb{Z}}\left|\left\langle f * \psi_j^*, g * \psi_j^*\right\rangle\right| \\ &\leq \sum_{j\in\mathbb{Z}}\left\|f * \psi_j^*\right\|_{p'}\left\|g * \psi_j^*\right\|_p \\ &= \sum_{j\in\mathbb{Z}}\left(2^{-js}\left\|f * \psi_j^*\right\|_{p'}\right)\left(2^{js}\left\|f * \psi_j^*\right\|_p\right) \\ &\leq \|f\|_{p',q'}^{-s}\|g\|_{p,q}^s \end{aligned} \tag{3.38}$$

valid for all $f \in \mathcal{Z}(G) \subset \dot{B}^{-s}_{p',q'}$ and $g \in \dot{B}^{s}_{p,q}$. Here p', q' are the conjugate exponents of p, q, respectively. But this estimate implies continuity of the embeddings $\dot{B}^{s}_{p,q} \subset \mathcal{S}'(G)/\mathcal{P}$ and $\mathcal{Z}(G) \subset \dot{B}^{s*}_{p,q}$.

For the density statement, let $u \in \dot{B}^{s}_{p,q}$, and $\epsilon > 0$. For convenience, we pick ψ according to Lemma 3.7. Since $q < \infty$, there exists $N \in \mathbb{N}$ such that

$$\sum_{|j|>N-1} 2^{jsq} \left\| \Delta_j u \right\|_p^q < \epsilon. \tag{3.39}$$

Next define

$$K_N = \sum_{|j|\le N} \psi_j^* * \psi_j = D_{2^{N+1}}\phi - D_{2^{-N}}\phi. \tag{3.40}$$

Let $w = u * K_N$. By assumption on u and Young's inequality, $w \in L^p(G)$, and since $p < \infty$, there exists $g \in \mathcal{S}(G)$ with $\|w - g\|_p < \epsilon^{1/q}$. Let $f = g * K_N$, then $f \in \mathcal{Z}(G)$, and for $j \in \mathbb{Z}$,

$$\begin{aligned} \left\| \Delta_j (u - f) \right\|_p &= \left\| (u - f) * \psi_j^* \right\|_p \\ &\le \left\| u * \psi_j^* - u * K_N * \psi_j^* \right\|_p + \left\| w * \psi_j^* - g * K_N * \psi_j^* \right\|_p. \end{aligned} \tag{3.41}$$

For $|j| \le N-1$, the construction of ψ_j and K_N implies that $K_N * \psi_j^* = \psi_j^*$, whereas for $|j| > N+1$, one has $K_N * \psi_j^* = 0$. As a consequence, one finds for $|j| < N - 1$

$$\left\| \Delta_j (u - f) \right\|_p \le \|w - g\|_p \left\| \psi_j^* \right\|_1 = \|w - g\|_p \|\psi\|_1 < \epsilon^{1/q} \|\psi\|_1, \tag{3.42}$$

and for $|j| > N + 1$

$$\left\| \Delta_j (u - f) \right\|_p \le \left\| u * \psi_j^* \right\|_p < \epsilon^{1/q}. \tag{3.43}$$

For $||j| - N| \le 1$, one finds

$$\left\| \Delta_j (u - f) \right\|_p \le C \epsilon^{1/q} \tag{3.44}$$

with some constant $C > 0$ depending only on ψ. For instance, for $j = N$,

$$\begin{aligned} \left\| \Delta_j (u - f) \right\|_p \le{} & \left\| u * \psi_N^* - u * (\psi_{N-1}^* * \psi_{N-1} + \psi_N^* * \psi_N) * \psi_N^* \right\|_p \\ &+ \left\| w * \psi_N^* - g * (\psi_{N-1}^* * \psi_{N-1} + \psi_N^* * \psi_N) * \psi_N^* \right\|_p. \end{aligned} \tag{3.45}$$

A straight forward application of triangle and Young's inequality yields

$$\begin{aligned}
&\left\|u * \psi_N^* - u * \left(\psi_{N-1}^* * \psi_{N-1} + \psi_N^* * \psi_N\right) * \psi_N^*\right\|_p \le \left\|u * \psi_N^*\right\|_p \left(1 + 2\left\|\psi^* * \psi\right\|_1\right) \\
&\quad < \epsilon^{1/q}\left(1 + 2\left\|\psi^* * \psi\right\|_1\right).
\end{aligned} \tag{3.46}$$

Similar considerations applied to $w = u * K_N$ yield

$$\begin{aligned}
&\left\|w * \psi_N^* - g * \left(\psi_{N-1}^* * \psi_{N-1} + \psi_N^* * \psi_N\right) * \psi_N^*\right\|_p \\
&\quad \le 2\left\|u * \psi_N^*\right\|_p \left\|\psi^* * \psi\right\|_1 + 2\left\|g * \psi_N^*\right\|_p \left\|\psi^* * \psi\right\|_1 \\
&\quad \le 2\epsilon^{1/q}\left\|\psi^* * \psi\right\|_1 + 2\Big(\left\|w * \psi_N^*\right\|_p + \left\|(w - g) * \psi_N^*\right\|_p\Big)\left\|\psi^* * \psi\right\|_1 \\
&\quad \le \left(4\left\|\psi^* * \psi\right\|_1 + \left\|\psi^* * \psi\right\|_1 \left\|\psi\right\|_1\right)\epsilon^{1/q}.
\end{aligned} \tag{3.47}$$

Now summation over j yields

$$\left\|u - f\right\|_{\dot{B}^s_{p,q}} \le C'\epsilon, \tag{3.48}$$

as desired. □

Remark 3.13. Let ψ be as in Lemma 3.7. As a byproduct of the proof, we note that the space

$$\mathfrak{D} = \left\{f * K_N : f \in \mathcal{S}(G),\ N \in \mathbb{N}\right\} \tag{3.49}$$

is dense in $\mathcal{Z}(G)$ as well as $\dot{B}^s_{p,q}$, if $p, q < \infty$. In $\mathfrak{D}$, the decomposition

$$g = \sum_{j \in \mathbb{Z}} g * \psi_j^* * \psi_j \tag{3.50}$$

holds with finitely many nonzero terms.

We next extend the Littlewood-Paley decomposition to the elements of the Besov space. For simplicity, we prove the result only for certain LP-admissible functions.

Proposition 3.14. *Let $1 \le p, q < \infty$, and let $\psi \in \mathcal{Z}(G)$ be an LP-admissible vector constructed via Lemma 3.7. Then the decomposition* (3.19) *converges for all $g \in \dot{B}^s_{p,q}$ in the Besov space norm.*

Proof. Consider the operators $\Sigma_N : \dot{B}^s_{p,q} \to \dot{B}^s_{p,q}$,

$$\Sigma_N g = \sum_{|j| \le N} g * \psi_j^* * \psi_j. \tag{3.51}$$

By suitably adapting the arguments proving the density statement of Lemma 3.12, it is easy to see that the family of operators $(\Sigma_N)_{N \in \mathbb{N}}$ is bounded in the operator norm. As noted in

Remark 3.13, the Σ_N strongly converges to the identity operator on a dense subspace. But then boundedness of the family implies strong convergence everywhere. □

A further class of spaces for which the decomposition converges is L^p.

Proposition 3.15. *Let $1 < p < \infty$, and let $\psi \in \mathcal{Z}(G)$ be an LP-admissible vector constructed via Lemma 3.7. Then the decomposition* (3.19) *converges with respect to $\|\cdot\|_p$, for all $g \in L^p(G)$.*

Proof. Let the operator family $(\Sigma_N)_{N\in\mathbb{N}}$ be defined as in the previous proof. Then, $\Sigma_N f = g * D_{2^{N+1}}\phi - g * D_{2^{-N}}\phi$, and Young's inequality implies that the sequence of operators is norm-bounded. It therefore suffices to prove the desired convergence on the dense subspace $\mathcal{S}(G)$. By [19, Proposition 1.49], $g * D_{2^{N+1}}\phi \to c_\phi g$. Furthermore, for $N \in \mathbb{N}$,

$$
\begin{aligned}
(g * D_{2^{-N}}\phi)(x) &= 2^{-NQ}\int_G g(y)\phi\left(2^{-N}\left(y^{-1}x\right)\right)dy \\
&= \int_G g\left(2^N y\right)\phi\left(y^{-1}\cdot 2^{-N}x\right)dy \\
&= 2^{-NQ}(D_{2^N}g * \phi)\left(2^{-N}x\right),
\end{aligned}
\tag{3.52}
$$

and thus

$$
\begin{aligned}
\|g * D_{2^{-NQ}}\phi\|_p &= 2^{-NQ}\left(\int_G \left|(D_{2^N}g * \phi)\left(2^{-N}x\right)\right|^p dx\right)^{1/p} \\
&= 2^{-NQ+NQ/p}\|D_{2^N}g * \phi\|_p.
\end{aligned}
\tag{3.53}
$$

Again by [19, Proposition 1.49], $(D_{2^N}g * \phi) \to c_g\phi$, in particular,

$$
2^{-NQ+NQ/p}\|D_{2^N}g * \phi\|_p \longrightarrow 0 \quad \text{as } N \longrightarrow \infty. \tag{3.54}
$$

Hence, $\Sigma_N g \to c_\phi g$, and the case $p = 2$ yields $c_\phi = 1$. □

Theorem 3.16. *$\dot{B}^s_{p,q}$ is a Banach space.*

Proof. Completeness is the only issue here. Again, we pick $\psi \in \mathcal{Z}(G)$ as an LP-admissible vector via Lemma 3.7. Suppose that $\{u_n\}_{n\in\mathbb{N}} \subset \dot{B}^s_{p,q}$ is a Cauchy sequence. As a consequence, one has in particular, for all $j \in \mathbb{Z}$, that $\{u_n * \psi_j^*\}_{n\in\mathbb{N}} \subset L^p(G)$ is a Cauchy sequence, hence $u_n * \psi_j^* \to v_j$, for a suitable $v_j \in L^p(G)$. Furthermore, the Cauchy property of $\{u_n\}_{n\in\mathbb{N}} \subset \dot{B}^s_{p,q}$ implies that

$$
\left\{\left\{2^{js}\left\|u_n * \psi_j^*\right\|_p\right\}_{j\in\mathbb{Z}}\right\}_{n\in\mathbb{N}} \subset \ell^q(\mathbb{Z}) \tag{3.55}
$$

is a Cauchy sequence. On the other hand, the sequence converges pointwise to $\{2^{js}\|v_j\|_p\}_j$, whence

$$\sum_{j\in\mathbb{Z}} 2^{jsq}\|v_j\|_p^q < \infty. \tag{3.56}$$

We define

$$u = \lim_{M\to\infty} \sum_{|j|\le M} v_j * \psi_j. \tag{3.57}$$

Now, using (3.56) and $\mathcal{Z}(G) \subset \dot{B}^{-s}_{p',q'}$, where p', q' are the conjugate exponents of p, q, respectively, a straightforward calculation as in the proof of Lemma 3.12 shows that the sum defining u converges in $\mathcal{S}'(G)/\mathcal{P}$. Furthermore, (3.56) and (3.25) easily imply that $u \in \dot{B}^s_{p,q}$. Finally, for the proof of $u_n \to u$, we employ (3.25) together with the equality $\psi_j^* = \sum_{|l-j|\le 1} \psi_l^* * \psi_l * \psi_j^*$, to show that

$$\begin{aligned}
\|(u_n - u) * \psi_j\|_p &= \left\| u_n * \psi_j - \sum_{|l-j|\le 1} v_l * \psi_l * \psi_j^* \right\|_p \\
&\le \sum_{|l-j|\le 1} \left\| \left(u_n * \psi_l^* - v_l\right) * \psi_l * \psi_j^* \right\|_p \\
&\le \sum_{|l-j|\le 1} \|u_n * \psi_l^* - v_l\|_p \left\|\psi_l * \psi_j^*\right\|_1 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.58}$$

Summarizing, the sequence $\{\{2^{js}\|(u_n - u) * \psi_j^*\|_p\}_{j\in\mathbb{Z}}\}_{n\in\mathbb{N}} \in \ell^q(\mathbb{N})$ is a Cauchy sequence, converging pointwise to 0. But then $\|u_n - u\|_{\dot{B}^s_{p,q}} \to 0$ follows. □

4. Characterization via Continuous Wavelet Transform

The following definition can be viewed as a continuous-scale analog of LP-admissibility.

Definition 4.1. $\psi \in \mathcal{S}(G)$ is called *$\mathcal{Z}$-admissible*, if for all $f \in \mathcal{Z}(G)$,

$$f = \lim_{\epsilon\to 0, A\to\infty} \int_\epsilon^A f * D_a\left(\psi^* * \psi\right) \frac{da}{a} \tag{4.1}$$

holds with convergence in the Schwartz topology.

The next theorem reveals a large class of $\mathcal{Z}$-admissible wavelets. In fact, all the wavelets studied in [15] are also $\mathcal{Z}$-admissible in the sense considered here. Its proof is an adaptation of the argument showing [15, Theorem 1].

We let

$$S(\mathbb{R}^+) = \left\{ f \in C^\infty(0,\infty) : \forall k \in \mathbb{N}_0, f^{(k)}\text{decreases rapidly, } \lim_{\xi} f^{(k)}(\xi) \text{ exists} \right\}. \tag{4.2}$$

Theorem 4.2. *Let $\widehat{h} \in \mathcal{S}(\mathbb{R}^+)$, and let ψ be the distribution kernel associated to the operator $L\widehat{h}(L)$. Then ψ is $\mathcal{Z}$-admissible up to normalization.*

Proof. The main idea of the proof is to write, for $f \in \mathcal{Z}(G)$,

$$\begin{aligned} \int_\epsilon^A f * D_a(\psi^* * \psi)\frac{da}{a} &= f * \int_\epsilon^A D_a(\psi^* * \psi)\frac{da}{a} \\ &= f * D_A g - f * D_\epsilon g, \end{aligned} \tag{4.3}$$

with suitable $g \in \mathcal{S}(G)$. Once this is established, $f * D_A g \to c_g f$ for $A \to \infty$ follows by [19, Proposition (1.49)], with convergence in the Schwartz topology. Moreover, $f \in \mathcal{Z}(G)$ entails that $f * D_\epsilon g \to 0$ in the Schwartz topology: given any $N > 0$ and $I \in \mathbb{N}_0^n$ with associated left-invariant differential operator Y^I, we can employ (3.5) to estimate

$$\begin{aligned} \sup_{x\in G}(1+|x|)^N \left|\left(Y^I f * D_\epsilon g\right)(x)\right| &= \sup_{x\in G}(1+|x|)^N \epsilon^{Q+d(I)}\left|f * D_\epsilon\left(Y^I g\right)(x)\right| \\ &\le C\sup_{x\in G}(1+|x|)^N \epsilon^{Q+d(I)+k}(1+|\epsilon x|)^{-M} \\ &\le C\sup_{x\in G}(1+|x|)^{N-M}\epsilon^{Q+d(I)+k-M}, \end{aligned} \tag{4.4}$$

which converges to zero for $\epsilon \to 0$, as soon as $M \ge N$ and $k > M - Q - d(I)$. But this implies $f * D_\epsilon g \to 0$ in $\mathcal{S}(G)$, by [19].

Thus it remains to construct g. To this end, define

$$\widehat{g}(\xi) = -\frac{1}{2}\int_\xi^\infty a\left|\widehat{h}\left(a^2\right)\right|^2 da, \tag{4.5}$$

which is clearly in $\mathcal{S}(\mathbb{R}^+)$, and let g denote the associated convolution kernel of $\widehat{g}(L)$. By the definition, $g \in \mathcal{S}(G)$. Let φ_1, φ_2 be in $\mathcal{S}(G)$, and let $d\lambda_{\varphi_1,\varphi_2}$ denote the scalar-valued Borel measure associated to φ_1, φ_2 by the spectral measure. Then, by spectral calculus and the invariance properties of da/a,

$$\begin{aligned}\left\langle \int_\epsilon^A \varphi_1 * D_a(\psi^* * \psi) f \frac{da}{a}, \varphi_2 \right\rangle &= \int_0^\infty \int_\epsilon^A \left(a^2\xi\right)^2 \left|\widehat{h}\left(a^2\xi\right)\right|^2 \frac{da}{a} d\lambda_{\varphi_1,\varphi_2}(\xi) \\ &= \frac{1}{2}\int_0^\infty \int_{\epsilon^2\xi}^{A^2\xi} a\left|\widehat{h}\left(a^2\xi\right)\right|^2 da\, d\lambda_{\varphi_1,\varphi_2}(\xi) \\ &= \int_0^\infty \widehat{g}\left(A^2\xi\right) - \widehat{g}\left(\epsilon^2\xi\right) d\lambda_{\varphi_1,\varphi_2}(\xi) \\ &= \langle \varphi_1 * (D_A g - D_\epsilon g), \varphi_2 \rangle,\end{aligned} \tag{4.6}$$

as desired. □

Hence, by [15, Corollary 1] we have the following.

Corollary 4.3. (a) *There exist* $\mathcal{Z}$*-admissible* $\psi \in \mathcal{Z}(G)$.

(b) *There exist* $\mathcal{Z}$*-admissible* $\psi \in C_c^\infty(G)$ *with vanishing moments of arbitrary finite order.*

Given a tempered distribution $u \in \mathcal{S}'(G)/\mathcal{P}$ and a $\mathcal{Z}(G)$-admissible function ψ, the continuous wavelet transform of u is the family $(u * D_a\psi^*)_{a>0}$ of convolution products. We will now prove a characterization of Besov spaces in terms of the continuous wavelet transform.

Another popular candidate for defining scales of Besov spaces is the heat semigroup; see for example, [1] for the inhomogeneous case on stratified groups, or rather [21] for the general treatment. In our setting, the heat semigroup associated to the sub-Laplacian is given by right convolution with $h_t(x) = D_t h(x)$, where h is the kernel of $\widehat{h}(L)$ with $\widehat{h}(\xi) = e^{-\xi}$. Theorem 4.2 implies that $\psi = L^k h$ is $\mathcal{Z}$-admissible; it can be viewed as an analog of the well-known *Mexican Hat* wavelet. (For general stratified Lie groups, this class of wavelets was studied for the first time in [15].) The wavelet transform of $f \in \mathcal{S}'(G)$ associated to ψ is then very closely related to the k-fold time derivative of the solution to the heat equation with initial condition f. By choice of h,

$$u(x,t) = \left(f * D_t h\right)(x) \tag{4.7}$$

denotes the solution of the heat equation associated to L, with initial condition f. A formal calculation using left invariance of L then yields

$$\partial_t^k u = L^k\left(f * D_t h\right) = f * L^k(D_t h) = t^{2k} f * D_t \psi^*. \tag{4.8}$$

Thus the following theorem also implies a characterization of Besov spaces in terms of the heat semigroup.

Theorem 4.4. *Let $\psi \in \mathcal{S}(G)$ be $\mathcal{Z}$-admissible, with vanishing moments of order k. Then, for all $s \in \mathbb{R}$ with $|s| < k$, and all $1 \le p < \infty$, $1 \le q \le \infty$, the following norm equivalence holds:*

$$\forall u \in S'(G)/p \ \|u\|_{\dot{B}^s_{p,q}} \asymp \left\| a \longmapsto a^s \left\| u * D_a \psi^* \right\|_p \right\|_{L^q(\mathbb{R}^+; da/a)}. \tag{4.9}$$

Here the norm equivalence is understood in the extended sense that one side is finite if and only if the other side is. If $\psi \in \mathcal{Z}(G)$, the equivalence is also valid for the case $p = \infty$.

Proof. The strategy consists in adapting the proof of Theorem 3.11 to the setting where one summation over scales is replaced by integration. This time, however, we have to deal with both directions of the norm equivalence. In the following estimates, the symbol C denotes a constant that may change from line to line, but in a way that is independent of $u \in S'(G)$.

Let us first assume that

$$\int_{\mathbb{R}} a^{sq} \left\| u * D_a \psi^* \right\|_p^q \frac{da}{a} < \infty, \tag{4.10}$$

for $u \in \mathcal{S}'(G)/\mathcal{P}$, $1 \le p, q \le \infty$, for a $\mathcal{Z}$-admissible function $\psi \in S(G)$ with $k_\psi > |s|$ vanishing moments ($\psi \in \mathcal{Z}(G)$, if $p = \infty$). Let $\varphi \in \mathcal{Z}(G)$ be LP-admissible. Then, for all $j \in \mathbb{Z}$,

$$u * \varphi_j^* = \lim_{\epsilon \to 0, A \to \infty} \int_\epsilon^A u * D_a \psi^* * D_a \psi * \varphi_j^* \frac{da}{a} \tag{4.11}$$

holds in $\mathcal{S}'(G)$, by Lemma 3.4.

We next prove that the right-hand side of (4.11) converges in L^p. For this purpose, introduce

$$c_j = \int_0^\infty \left\| u * D_a \psi^* * D_a \psi * \varphi_j^* \right\|_p \frac{da}{a}. \tag{4.12}$$

We estimate

$$\begin{aligned} c_j &\le \int_0^\infty \left\| u * D_a \psi^* \right\|_p \left\| D_a \psi * \varphi_j^* \right\|_1 \frac{da}{a} \\ &= \int_1^2 \sum_{\ell \in \mathbb{Z}} \left\| u * D_{a2^\ell} \psi^* \right\|_p \left\| D_{a2^\ell} \psi * \varphi_j^* \right\|_1 \frac{da}{a} \end{aligned} \tag{4.13}$$

$$\le \left(\int_1^2 \left(\sum_{\ell \in \mathbb{Z}} \left\| u * D_{a2^\ell} \psi^* \right\|_p \left\| D_{a2^\ell} \psi * \varphi_j^* \right\|_1 \right)^q \frac{da}{a} \right)^{1/q} \log(2)^{1/q'}, \tag{4.14}$$

where we used that da/a is scaling invariant. Note that the last inequality is Hölder's inequality for $q < \infty$. In this case, taking qth powers and summing over j yields

$$\sum_{j \in \mathbb{Z}} 2^{jsq} c_j^q \le C \int_1^2 \sum_{j \in \mathbb{Z}} 2^{jsq} \left(\sum_{\ell \in \mathbb{Z}} \left\| u * D_{a2^\ell} \psi^* \right\|_p \left\| D_{a2^\ell} \psi * \varphi_j^* \right\|_1 \right)^q \frac{da}{a}. \tag{4.15}$$

Using vanishing moments and Schwartz properties of ψ and φ, we can now employ (3.6) and (3.8) to obtain

$$\left\| D_{a2^\ell}\psi * \varphi_j^* \right\|_1 \le C 2^{-|j-\ell|k}, \tag{4.16}$$

with a constant independent of $a \in [1,2]$. But then, since $k > |s|$, we may proceed just as in the proof of Theorem 3.11 to estimate the integrand in (4.15) via

$$\sum_{j\in\mathbb{Z}} 2^{jsq} \left(\sum_{\ell\in\mathbb{Z}} \| u * D_{a2^\ell}\psi^* \|_p \left\| D_{a2^\ell}\psi * \varphi_j^* \right\|_1 \right)^q \le C \sum_{\ell\in\mathbb{Z}} 2^{\ell sq} \| u * D_{a2^\ell}\psi^* \|_p^q. \tag{4.17}$$

Summarizing, we obtain

$$\begin{aligned} \sum_j 2^{jsq} c_j^q &\le C \int_1^2 \sum_{\ell\in\mathbb{Z}} 2^{\ell sq} \| u * D_{a2^\ell}\psi^* \|_p^q \frac{da}{a} \\ &\le C \int_0^\infty a^{sq} \| u * D_{a2^\ell}\psi^* \|_p^q \frac{da}{a} < \infty. \end{aligned} \tag{4.18}$$

In particular, $c_j < \infty$. But then the right-hand side of (4.11) converges to $u * \varphi_j^*$ in L^p. The Minkowski inequality for integrals yields $\| u * \varphi_j^* \|_p \le c_j$, and thus

$$\| u \|_{\dot{B}^s_{p,q}}^q \le C \int_0^\infty a^{sq} \| u * D_{a2^\ell}\psi^* \|_p^q \frac{da}{a}, \tag{4.19}$$

as desired. In the case $q = \infty$, (4.16) yields that

$$\sup_j 2^{js} \left(\sum_{\ell\in\mathbb{Z}} \| u * D_{a2^\ell}\psi^* \|_p \left\| D_{a2^\ell}\psi * \varphi_j^* \right\|_1 \right) \le C \sup_\ell 2^{\ell s} \| u * D_{a2^\ell}\psi^* \|_p^q. \tag{4.20}$$

Thus, by (4.13),

$$\begin{aligned} \sup_j 2^{js} c_j &\le C \int_1^2 \sup_\ell 2^{\ell s} \| u * D_{a2^\ell}\psi^* \|_p \frac{da}{a} \\ &\le C \operatorname{ess\,sup}_a a^s \| u * D_a\psi^* \|_p. \end{aligned} \tag{4.21}$$

The remainder of the argument is the same as for the case $q < \infty$.

Next assume $u \in \dot{B}^s_{p,q}$. Then, for all $a \in [1,2]$ and $\ell \in \mathbb{Z}$,

$$u * D_{a2^\ell}\psi^* = \sum_{j\in\mathbb{Z}} u * \varphi_j^* * \varphi_j * D_{a2^\ell}\psi^*, \tag{4.22}$$

with convergence in $\mathcal{S}'(G)/\mathcal{P}$; for $\psi \in \mathcal{Z}(G)$, convergence holds even in $\mathcal{S}'(G)$. As before,

$$\left\| \sum_{j\in\mathbb{Z}} u * \varphi_j^* * \varphi_j * D_{a2^\ell}\psi^* \right\|_p \leq \sum_{j\in\mathbb{Z}} \left\| u * \varphi_j^* \right\|_p \|\varphi_j * D_{a2^\ell}\psi^*\|_1. \tag{4.23}$$

Again, we have $\|\varphi_j * D_{a2^\ell}\psi^*\|_1 \preceq 2^{-|j-\ell|k}$ with a constant independent of a. Hence, one concludes in the same fashion as in the proof of Theorem 3.11 that, for all $a \in [1,2]$,

$$\left\| \left(2^{\ell s} \|u * D_{a2^\ell}\psi^*\|_p \right)_{\ell\in\mathbb{Z}} \right\|_q \leq C \left\| \left(2^{js} \left\| u * \varphi_j^* \right\|_p \right)_{j\in\mathbb{Z}} \right\|_q, \tag{4.24}$$

again with a constant independent of a. In the case $q = \infty$, this finishes the proof immediately, and for $q < \infty$, we integrate the qth power over $a \in [1,2]$ and sum over ℓ to obtain the desired inequality. □

Remark 4.5. Clearly, the proof of Theorem 4.4 can be adapted to consider discrete Littlewood-Paley decompositions based on integer powers of any $a > 1$ instead of $a = 2$. Thus consistently replacing powers of 2 in Definitions 3.6 and 3.9 by powers of $a > 1$ results in the same scale of Besov spaces.

As an application of the characterization via continuous wavelet transforms, we exhibit certain of the homogeneous Besov spaces as homogeneous Sobolev spaces, and we investigate the mapping properties of sub-Laplacians between Besov spaces of different smoothness exponents.

Lemma 4.6. $\dot{B}^0_{2,2} = L^2(G)$, *with equivalent norms.*

Proof. Pick ψ by Lemma 3.7. Then spectral calculus implies that for all $f \in \mathcal{Z}(G)$

$$\|f\|^2_{\dot{B}^0_{2,2}} = \sum_{j\in\mathbb{Z}} \left\| f * \varphi_j^* \right\|_2^2 = \|f\|_2^2. \tag{4.25}$$

Since $\mathcal{Z}(G)$ is dense in both spaces, and both spaces are complete, it follows that $\dot{B}^0_{2,2} = L^2(G)$. □

The next lemma investigates the mapping properties of sub-Laplacians between Besov spaces of different smoothness exponents. Its proof is greatly facilitated by the characterization via continuous wavelet transforms.

Lemma 4.7. *Let L denote a sub-Laplacian. For all $u \in \mathcal{S}'(G)/\mathcal{P}$, $1 \leq p, q < \infty$, $s \in \mathbb{R}$ and $k \geq 0$,*

$$\left\| L^k u \right\|_{\dot{B}^{s-2k}_{p,q}} \asymp \|u\|_{\dot{B}^s_{p,q}}, \tag{4.26}$$

in the extended sense that one side is infinite if and only if the other side is. In particular, $L^k : \dot{B}^s_{p,q} \to \dot{B}^{s-2k}_{p,q}$ *is a bijection, and it makes sense to extend the definition to negative* k*. Thus, for all* $k \in \mathbb{Z}$,

$$L^k : \dot{B}^s_{p,q} \longrightarrow \dot{B}^{s-2k}_{p,q} \tag{4.27}$$

is a topological isomorphism of Banach spaces.

Proof. Pick a nonzero real-valued $h \in \mathcal{S}(\mathbb{R}^+)$, an integer $m > |s|$ and let ψ denote the distribution kernel of $L^m \widehat{h}(L)$. Hence ψ is admissible by Theorem 4.2, with vanishing moments of order $2m$ and $\psi^* = \psi$. On $L^2(G)$, the convolution operator $u \mapsto u * D_a\psi^*$ can be written as $\widehat{\Psi}_a(L)$ with a suitable function $\widehat{\Psi}_a$. For $u \in \mathcal{Z}(G) \subset L^2(G)$, spectral calculus implies

$$\begin{aligned} \left\| \left(L^k u \right) * D_a \psi^* \right\|_p &= \left\| \left(\widehat{\Psi}_a(L) \circ L^k \right)(u) \right\|_p \\ &= \left\| \left(L^k \circ \widehat{\Psi}_a(L) \right)(u) \right\| \\ &= \left\| L^k \left(u * D_a \psi^* \right) \right\|_p \\ &= \left\| u * L^k \left(D_a \psi^* \right) \right\|_p \\ &= a^{2k} \left\| u * D_a \left(L^k \psi \right)^* \right\|_p , \end{aligned} \tag{4.28}$$

where we employed left invariance to pull L^k past u in the convolution. Note that up to normalization, $L^k\psi$ is admissible with vanishing moments of order $2m + 2k > |s - 2k|$. Thus, applying Theorem 4.4, we obtain

$$\begin{aligned} \left\| L^k u \right\|_{\dot{B}^{s-2k}_{p,q}} &\asymp \left\| a \longmapsto a^{s-2k} \left\| \left(L^k u \right) * D_a \psi^* \right\|_p \right\|_{L^q(\mathbb{R}^+; da/a)} \\ &= \left\| a \longmapsto a^{s} \left\| u * D_a \left(L^k \psi \right)^* \right\|_p \right\|_{L^q(\mathbb{R}^+; da/a)} \\ &\asymp \|u\|_{\dot{B}^s_{p,q}} . \end{aligned} \tag{4.29}$$

Now assume that $L^k u \in \dot{B}^{s-2k}_{p,q}$. Then, combining the density statements from Lemma 3.12 and Remark 3.8, we obtain a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{Z}(G)$ with $L^k u_n \to L^k u$ in $\dot{B}^{s-2k}_{p,q}$; thus also with convergence in $\mathcal{S}'(G)/\mathcal{P}$. The norm equivalence and completeness of $\dot{B}^s_{p,q}$ yield that $u_n \to v \in \dot{B}^s_{p,q}$, for suitable $v \in \dot{B}^s_{p,q}$. Again, this implies convergence in $\mathcal{S}'(G)/\mathcal{P}$. Since L^k is continuous on that space, it follows that $L^k u_n \to L^k v$, establishing that $L^k v = L^k u$. Since any distribution annihilated by L^k is a polynomial, this finally yields $u = v \in \dot{B}^s_{p,q}$, and $\|u\|_{\dot{B}^s_{p,q}} \asymp \|L^k u\|_{\dot{B}^{s-2k}_{p,q}}$ follows by taking limits. A similar but simpler argument establishes the norm equivalence under the assumption that $u \in \dot{B}^s_{p,q}$. □

This observation shows that we can regard certain Besov spaces as homogeneous Sobolev spaces, or, more generally, as generalizations of Riesz potential spaces.

Corollary 4.8. *For all* $k \in \mathbb{N}$: $B^{2k}_{2,2} = \{f \in \mathcal{S}'(G)/\mathcal{P} : L^k f \in L^2(G)\}$.

As a further corollary, we obtain the following interesting result relating two sub-Laplacians L_1 and L_2. For all $k \in \mathbb{Z}$, the operator

$$L_1^k \circ L_2^{-k} : L^2(G) \longrightarrow L^2(G) \tag{4.30}$$

is densely defined and has a bounded extension with bounded inverse. More general analogues involving more than two sub-Laplacians are also easily formulated. For the Euclidean case, this is easily derived using the Fourier transform, which can be viewed as a joint spectral decomposition of commuting operators. In the general, nonabelian case however, this tool is not readily available, and we are not aware of a direct proof of this observation, nor of a previous source containing it.

5. Characterization of Besov Spaces by Discrete Wavelet Systems

We next show that the Littlewood-Paley characterization of $\dot{B}^s_{p,q}$ can be discretized by sampling the convolution products $f * \psi_j^*$ over a given discrete set $\Gamma \subset G$. This is equivalent to the study of the analysis operator associated to a discrete wavelet system $\{\psi_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}$, defined by

$$\psi_{j,\gamma}(x) = D_{2^j} T_\gamma \psi(x) = 2^{jQ} \psi\left(\gamma^{-1} \cdot 2^j x\right). \tag{5.1}$$

Throughout the rest of the paper, we assume that the wavelet $\psi \in \mathcal{Z}(G)$ has been chosen according to Lemma 3.7 and $\psi^* = \psi$.

We first define the discrete coefficient spaces which will be instrumental in the characterization of the Besov spaces.

Definition 5.1. Fix a discrete set $\Gamma \subset G$. For a family $\{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ of complex numbers, we define

$$\left\| \{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma} \right\|_{\dot{b}^s_{p,q}} = \left(\sum_j \left(\sum_{\gamma\in\Gamma} \left(2^{j(s-Q/p)} |c_{j,\gamma}| \right)^p \right)^{q/p} \right)^{1/q}. \tag{5.2}$$

The coefficient space $\dot{b}^s_{p,q}(\Gamma)$ associated to $\dot{B}^s_{p,q}$ and Γ is then defined as

$$\dot{b}^s_{p,q}(\Gamma) := \left\{ \{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma} : \left\| \{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma} \right\|_{\dot{b}^s_{p,q}} < \infty \right\}. \tag{5.3}$$

We simply write $\dot{b}^s_{p,q}$ if Γ is understood from the context.

We define the analysis operator A_ψ associated to the function ψ and Γ, assigning each $u \in \mathcal{S}'(G)/P$ the family of coefficients $A_\psi(u) = \{\langle u, \psi_{j,\gamma}\rangle\}_{j,\gamma}$. Note that the analysis operator is implicitly assumed to refer to the same set Γ that is used in the definition of $\dot{b}^s_{p,q}$.

We next formulate properties of the sampling sets we intend to use in the following. We will focus on *regular sampling*, as specified in the next definition. Most of the results are obtainable for less regular sampling sets, at the cost of more intricate notation.

Definition 5.2. A subset $\Gamma \subset G$ is called *regular sampling set*, if there exists a relatively compact Borel neighborhood $W \subset G$ of the identity element of G satisfying $\bigcup_{\gamma\in\Gamma} \gamma W = G$ (up to a set of measure zero) as well as $|\gamma W \cap \alpha W| = 0$, for all distinct $\gamma, \alpha \in \Gamma$. Such a set W is called a Γ-*tile*. A regular sampling set Γ is called U-dense, for $U \subset G$, if there exists a Γ-tile $W \subset U$.

Note that the definition of U-dense used here is somewhat more restrictive than, for example, in [14]. A particular class of regular sampling sets is provided by *lattices*, that is, cocompact discrete subgroups $\Gamma \subset G$. Here, Γ-tiles are systems of representatives mod Γ. However, not every stratified Lie group admits a lattice. By contrast, there always exist sufficiently dense regular sampling sets, as the following result shows.

Lemma 5.3. *For every neighborhood U of the identity, there exists a U-dense regular sampling set.*

Proof. By [14, Lemma 5.10], there exists $\Gamma \subset G$ and a relatively compact W with nonempty open interior, such that $\bigcup \gamma W$ tiles G (up to sets of measure zero). Then $V = W x_0^{-1}$ is a Γ-tile, for some point x_0 in the interior of W. Finally, choosing $b > 0$ sufficiently small ensures that $bV \subset U$, and bV is a $b\Gamma$-tile. □

The chief result of this section is the following theorem which shows that the Besov norms can be expressed in terms of discrete coefficients. Note that the constants arising in the following norm equivalences may depend on the space, but the same sampling set is used simultaneously for all spaces.

Theorem 5.4. *There exists a neighborhood U of the identity, such that for all U-dense regular sampling sets Γ, and for all $u \in \mathcal{S}'(G)/\mathcal{P}$ and all $1 \le p, q \le \infty$, the following implication holds:*

$$u \in \dot{B}^s_{p,q} \Longrightarrow \left\{ \langle u, \psi_{j,\gamma} \rangle \right\}_{j\in\mathbb{Z},\gamma\in\Gamma} \in \dot{b}^s_{p,q}(\Gamma). \tag{5.4}$$

Furthermore, the induced coefficient operator $A_\psi : \dot{B}^s_{p,q} \to \dot{b}^s_{p,q}$ is a topological embedding. In other words, on $\dot{B}^s_{p,q}$ one has the norm equivalence

$$\|u\|_{\dot{B}^s_{p,q}} \asymp \left(\sum_j \left(\sum_\gamma \left(2^{j(s-Q/p)} \left| \langle u, \psi_{j,\gamma} \rangle \right| \right)^p \right)^{q/p} \right)^{1/q}, \tag{5.5}$$

with constants depending on p, q, s, and Γ.

Remark 5.5. As a byproduct of the discussion in this section, we will obtain that the tightness of the frame estimates approaches 1, as the density of the sampling set increases. That is, the wavelet frames are asymptotically tight.

For the proof of Theorem 5.4, we need to introduce some notations. In the following, we write

$$X_j = \left\{ u * \psi_j^* : u \in \mathcal{S}'(G) \right\}, \tag{5.6}$$

which is a space of smooth functions, as well as $X_j^p = X_j \cap L^p(G)$. Furthermore, let $\Gamma_j = 2^j\Gamma$, and denote by $R_{\Gamma_j} : X_j \ni g \mapsto g|_{\Gamma_j}$ the restriction operator.

In order to prove Theorem 5.4, it is enough to prove the following sampling result for the spaces X_j; the rest of the argument consists in summing over j. In particular, note that the sampling set Γ is independent of p and j, and the associated constants are independent of j.

Lemma 5.6. *There exists a neighborhood U of the identity, such that for all U-dense regular sampling sets Γ, the implication*

$$g \in X_j^p \Longrightarrow R_{\Gamma_j} g \in \ell^p(\Gamma_j), \tag{5.7}$$

holds. Furthermore, with suitable constants $0 < c(p) \le C(p) < \infty$ (for $1 \le p \le \infty$), the inequalities

$$c(p) \left\| u * \psi_j^* \right\|_p \le \left(\sum_{\gamma \in \Gamma} 2^{-jQ} \left| \langle u, \psi_{j,\gamma} \rangle \right|^p \right)^{1/p} \le C(p) \left\| u * \psi_j^* \right\|_p \tag{5.8}$$

hold for all $j \in \mathbb{Z}$ and all $u \in X_j$.

Proof. Here we only show that the case $j = 0$ implies the other cases; the rest will be established below. Hence assume (5.8) is known for $j = 0$. Let $g = u * \psi_j^* \in X_j$. For arbitrary j, we have that $\psi_j^* = 2^{jQ}\psi_0^* \circ \delta_{2^j}$, and thus

$$u * \psi_j^* = 2^{jQ} u * \left(\psi^* \circ \delta_{2^j} \right) = \left(v^j * \psi^* \right) \circ \delta_{2^j}. \tag{5.9}$$

Here $v^j = u \circ \delta_{2^{-j}}$, where the dilation action on distributions is defined in the usual manner by duality. The last equality follows from the fact that δ_{2^j} is a group homomorphism. Recall that for any j and γ, $\psi_{j,\gamma}(x) = 2^{jQ}\psi(\gamma^{-1} \cdot 2^j x)$, applying the case $j = 0$, we obtain for $p < \infty$ that

$$\begin{aligned}
\left(\sum_{\gamma \in \Gamma} \left| \langle u, \psi_{j,\gamma} \rangle \right|^p \right)^{1/p} &= \left(\sum_{\gamma \in \Gamma} \left| \left\langle v^j, \psi_{0,\gamma} \right\rangle \right|^p \right)^{1/p} \\
&\le C(p) \left\| v^j * \psi_0^* \right\|_p \\
&= C(p) \left\| \left(v^j * \psi_0^* \right) \circ \delta_{2^j} \circ \delta_{2^{-j}} \right\|_p \\
&= C(p) \left\| \left(u * \psi_j^* \right) \circ \delta_{2^{-j}} \right\|_p \\
&= C(p) 2^{jQ/p} \left\| u * \psi_j^* \right\|_p,
\end{aligned} \tag{5.10}$$

which is the upper estimate for arbitrary j. The lower estimate and the case $p = \infty$ follow by similar calculations. □

For the remainder of this section, we will therefore be concerned with the case $j = 0$, which will be treated using ideas similar to the ones in [14], relying mainly on oscillation estimates. Given any function f on G and a set $U \subset G$, we define the oscillation

$$\mathrm{osc}_U(f)(x) = \sup_{y \in U} \left| f(x) - f\left(xy^{-1}\right) \right|. \tag{5.11}$$

We can then formulate the following result.

Proposition 5.7. *Let $X_0 \subset \mathcal{S}'(G)$ be a space of continuous functions. Suppose that there exists $K \in \mathcal{S}(G)$ such that, for all $f \in X_0$, $f = f * K$ holds pointwise. Define $X_0^p = X_0 \cap L^p(G)$, for $1 \leq p \leq \infty$. Let $\epsilon < 1$, and, U be a neighborhood of the unit element fulfilling $\|osc_U(K)\|_1 \leq \epsilon$. Then, for all U-dense regular sampling sets Γ, the following implication holds:*

$$\forall f \in X_0 : \; f \in X_0^p \Longrightarrow f|_\Gamma \in \ell^p(\Gamma). \tag{5.12}$$

The restriction map $R_\Gamma : f \to f|_\Gamma$ induces a topological embedding $(X_0^p, \|\cdot\|_p) \to l^p(\Gamma)$.

More precisely, for $p < \infty$,

$$\frac{1}{|W|^{1/p}}(1-\epsilon)\|f\|_p \leq \|R_\Gamma f\|_p \leq \frac{1}{|W|^{1/p}}(1+\epsilon)\|f\|_p, \quad \forall f \in X_0^p, \tag{5.13}$$

where W denotes a Γ-tile, and

$$(1-\epsilon)\|f\|_\infty \leq \|R_\Gamma f\|_\infty \leq (1+\epsilon)\|f\|_\infty, \quad \forall f \in X_0^\infty. \tag{5.14}$$

Proof. We introduce the auxiliary operator $T : \ell^p(\Gamma) \to L^p(G)$ defined by

$$T(c) = \sum_{\gamma \in \Gamma} c_\gamma L_\gamma \chi_W, \tag{5.15}$$

with $c = (c_\gamma)_{\gamma \in \Gamma}$. Since the sets γW are pairwise disjoint, T is a multiple of an isometry, $\|Tc\|_p = |W|^{1/p}\|c\|_p$. In particular, T has a bounded inverse on its range, and $Tc \in L^p(G)$ implies $c \in \ell^p(\Gamma)$ for any sequence $c \in \mathbb{C}^\Gamma$.

The equation $f = f * K$ implies the pointwise inequality

$$\mathrm{osc}_U(f) \leq |f| * \mathrm{osc}_U(K) \tag{5.16}$$

(see [14, page 185]). Now Young's inequality provides for $f \in X^p$:

$$\|\mathrm{osc}_U(f)\|_p \leq \|f\|_p \|\mathrm{osc}_U(K)\|_1 \leq \epsilon \|f\|_p. \tag{5.17}$$

Since the γW's are disjoint, we may then estimate, for all $f \in X^p$,

$$\begin{aligned} \|f - TR_\Gamma f\|_p^p &= \sum_{\gamma\in\Gamma} \int_{\gamma W} |f(x) - f(\gamma)|^p dx \\ &\leq \sum_{\gamma\in\Gamma} \int_{\gamma W} |\mathrm{osc}_U(f)(x)|^p dx \\ &= \|\mathrm{osc}_U(f)\|_p^p \\ &\leq \epsilon^p \|f\|_p^p. \end{aligned} \tag{5.18}$$

In particular, $TR_\Gamma f \in L^p(G)$, whence $R_\Gamma f \in \ell^p(\Gamma)$. In addition, we obtain the upper bound of the sampling inequality for $f \in X^p$

$$\begin{aligned} \|R_\Gamma f\|_p &= \left\|T^{-1} TR_\Gamma f\right\|_p \\ &\leq \left\|T^{-1}\right\|_\infty \|TR_\Gamma f\|_p \\ &\leq \left\|T^{-1}\right\|_\infty \left(\|f\|_p + \|f - TR_\Gamma f\|_p\right) \\ &\leq \left\|T^{-1}\right\|_\infty (1+\epsilon)\|f\|_p \\ &\leq \frac{1}{|W|^{1/p}} (1+\epsilon)\|f\|_p. \end{aligned} \tag{5.19}$$

The lower bound follows similarly by

$$\begin{aligned} \|R_\Gamma f\|_p &\geq \|T\|_\infty^{-1} \|TR_\Gamma f\|_p \\ &\geq \|T\|_\infty^{-1} \left(\|f\|_p - \|f - TR_\Gamma f\|_p\right) \\ &\geq \|T\|_\infty^{-1} (1-\epsilon)\|f\|_p \\ &\geq \frac{1}{|U|^{1/p}} (1-\epsilon)\|f\|_p. \end{aligned} \tag{5.20}$$

Thus (5.13) and (5.12) are shown, for $1 \leq p < \infty$. For $p = \infty$, we note that $\|T\|_\infty = \|T^{-1}\|_\infty = 1$. Furthermore,

$$\begin{aligned} \|f - TR_\gamma f\|_\infty &\leq \sup_\gamma \ \mathrm{ess} \sup_{x\in\gamma W} |f(x) - f(\gamma)| \\ &\leq \|\mathrm{osc}_U(f)\|_\infty. \end{aligned} \tag{5.21}$$

Now the remainder of the proof is easily adapted from the case $p < \infty$. □

It remains to check the conditions of the proposition for

$$X_0 = \left\{ f = u * \psi_0^* : u \in \frac{\mathcal{S}'(G)}{P} \right\}. \tag{5.22}$$

Lemma 5.8. *There exists a Schwartz function K acting as a reproducing kernel for X_0, that is, $f = f * K$ holds for all $f \in X_0^p$.*

Proof. We pick a real-valued C_c^∞-function k on $\mathbb{R}^+$ that is identically 1 on the support of $\widehat{\varphi}_0$, and let K be the associated distribution kernel to $k(L)$. Then $\psi_0^* = \psi_0^* * K$, whence $f = f * K$ follows, for all $f \in X_0$. □

Lemma 5.9. *Let K be a Schwartz function. For every $\epsilon > 0$, there exists a compact neighborhood U of the unit element such that $\|osc_U(K)\|_1 < \epsilon$.*

Proof. First observe that, by continuity, $\mathrm{osc}_U(K) \to 0$ pointwise, as U runs through a neighborhood base at the identity element. Thus by dominated convergence it suffices to prove $\|\mathrm{osc}_V(K)\|_1 < \infty$, for some neighborhood V.

Let $V = \{x \in G : |x| < 1\}$. A straightforward application of the mean value theorem [19, Theorem 1.33] yields

$$\mathrm{osc}_V(K)(x) \leq C \sup_{|z|\leq\beta, 1\leq i\leq n} |Y_i K(xz)|. \tag{5.23}$$

Here C and β are constants depending on G. The Sobolev estimate [22, (5.13)] for $p = 1$ yields that for all z with $|z| < \beta$

$$|Y_i K(xz)| \leq C' \sum_Y \int_{xW} |YK(y)| dy, \tag{5.24}$$

where Y runs through all possible Y^I with $d(I) \leq Q + 1$, including the identity operator corresponding to $I = (0, \ldots, 0)$. Furthermore, $W = \{x \in G : |x| < \beta\}$, and $C' > 0$ is a constant. Now integrating against Haar-measure (which is two-sided invariant) yields

$$\begin{aligned}\int_G \mathrm{osc}_V(K)(x) dx &\leq C \sum_Y \int_G \int_{xW} |YK(y)| dy\, dx \\ &= CC' \sum_Y \int_G \int_W |YK(xy)| dy\, dx \\ &= |W| CC' \sum_Y \int_G |YK(x)| dx,\end{aligned} \tag{5.25}$$

and the last integral is finite because K is a Schwartz function. □

Now Lemma 5.6 is a direct consequence of Proposition 5.7 and Lemmas 5.8 and 5.9. Note that the tightness in Proposition 5.7 converges to 1, as U runs through a neighborhood of the identity. This property is then inherited by the norm estimates in Theorem 5.4.

6. Banach Wavelet Frames for Besov Spaces

In Hilbert spaces a norm equivalence such as (5.5) would suffice to imply that the wavelet system is a frame, thus entailing a bounded reconstruction from the discrete coefficients. For

Banach spaces one needs to use the extended definition of frames [23], that is, to show the invertibility of associated frame operator. In this section we will establish these statements for wavelet systems in Besov space. We retain the assumption that the wavelet ψ was chosen according to Lemma 3.7.

We first prove that any linear combination of wavelet systems with coefficients in $\dot{b}^s_{p,q}$ converges unconditionally in $\dot{B}^s_{p,q}$, compare [12, Theorem 3.1]. We then show that for all sufficiently dense choices of the sampling set Γ, the wavelet system $\{2^{-jQ}\psi_{j,\gamma}\}$ constitutes a Banach frame for $\dot{B}^s_{p,q}$.

Recall that the sampled convolution products studied in the previous sections can be read as scalar products

$$f * \psi_j^*\left(2^j\gamma\right) = \langle f, \psi_{j,\gamma}\rangle, \tag{6.1}$$

where $\psi_{j,\gamma}(x) = 2^{jQ}\psi(\gamma^{-1}\cdot 2^j x)$ denotes the wavelet of scale 2^{-j} at position $2^{-j}\gamma$. In the following, the wavelet system is used for synthesis purposes, that is, we consider linear combinations of discrete wavelets. The next result can be viewed in parallel to synthesis results, for example, in [7]. It establishes synthesis for a large class of systems. Note in particular that the functions $g_{j,\gamma}$ need not be obtained by dilation and shifts from a single function g.

Theorem 6.1. *Let $\Gamma \subset G$ be a regular sampling set. Let $1 \le p, q < \infty$.*

(a) *Suppose that one is given tempered distributions $(g_{j,\gamma})_{j\in\mathbb{Z},\gamma\in\Gamma}$ satisfying the following decay conditions: for all $N, \theta \in \mathbb{N}$, there exist constants c_1, c_2 such that for all $j, l \in \mathbb{Z}$, $\gamma \in \Gamma$, $x \in G$:*

$$\left|g_{j,\gamma} * \psi_l^*(x)\right| \le \begin{cases} c_1 2^{jQ}2^{-(j-l)N}\left(1 + 2^l\left|2^{-j}\gamma^{-1}\cdot x\right|\right)^{-(Q+1)} & \text{for } l \le j, \\ c_2 2^{jQ}2^{-(l-j)\theta}\left(1 + 2^j\left|2^{-j}\gamma^{-1}\cdot x\right|\right)^{-(Q+1)} & \text{for } l \ge j, \end{cases} \tag{6.2}$$

Then for all $\{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma} \in \dot{b}^s_{p,q}(\Gamma)$, the sum

$$f = \sum_{j,\gamma} c_{j,\gamma} g_{j,\gamma} \tag{6.3}$$

converges unconditionally in the Besov norm, with

$$\|f\|_{\dot{B}^s_{p,q}} \le c\left(\sum_j\left(\sum_\gamma\left(2^{j(s-Q/p)}\left|c_{j,\gamma}\right|\right)^p\right)^{q/p}\right)^{1/q} \tag{6.4}$$

for some constant c independent of $\{c_{j,\gamma}\}_{j\in\mathbb{Z},\gamma\in\Gamma}$. In other words, the synthesis operator $\dot{b}^s_{p,q}(\Gamma) \to \dot{B}^s_{p,q}$ associated to the system $(g_{j,\gamma})_{j,\gamma}$ is bounded.

(b) *The synthesis result in (a) holds in particular for*

$$g_{j,\gamma}(x) = \psi_{j,\gamma}(x) = 2^{jQ}\psi_j\left(\gamma^{-1}\cdot\left(2^j x\right)\right). \tag{6.5}$$

In order to motivate the following somewhat technical lemmas, let us give a short sketch of the proof strategy for the theorem. It suffices to show (6.4) for all finitely supported sequences; the rest follows by density arguments, using that $\dot{B}^s_{p,q}$ is a Banach space. Hence, given a finitely supported coefficient sequence $\{c_{j,\gamma}\}$ and $f = \sum_{j,\gamma} c_{j,\gamma} g_{j,\gamma}$, we need estimates for the L^p-norms of

$$f * \psi_l = \sum_{j,\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l. \tag{6.6}$$

These estimates are obtained by first looking at the summation over γ, with j fixed, and then summing over j. In both steps, we use the decay condition (6.2).

The following lemma shows that (6.2) is fulfilled for $g_{j,\gamma} = \psi_{j,\gamma}$ and thus allows to conclude part (b) of Theorem 6.1.

Lemma 6.2. *There exists a constant $C > 0$ such that for any $j, l \in \mathbb{Z}$, $\gamma \in \Gamma$, $x \in G$, the following estimate holds:*

$$\left|\psi_{j,\gamma} * \psi_l^*(x)\right| \le \begin{cases} C 2^{jQ}\left(1 + 2^j \left|\left(2^{-j}\gamma^{-1}\right) \cdot x\right|\right)^{-(Q+1)} & |l-j| \le 1 \\ 0 & \text{otherwise}. \end{cases} \tag{6.7}$$

Proof. We first compute

$$\begin{aligned} \left(\psi_{j,\gamma} * \psi_l^*\right)(x) &= \int_G 2^{jQ}\psi\left(\gamma^{-1} \cdot 2^j y\right) 2^{lQ} \overline{\psi\left(2^l\left(x^{-1} \cdot y\right)\right)} dy \\ &= \int_G \psi\left(\gamma^{-1} \cdot y\right) 2^{lQ} \overline{\psi\left(2^l\left(x^{-1} \cdot 2^{-j} y\right)\right)} dy \\ &= \int \psi(y) 2^{jQ} 2^{lQ} \overline{\left(\psi_{2^{l-j}}\left(\left(\gamma^{-1} \cdot 2^j x\right)^{-1} \cdot y\right)\right)} dy \\ &= 2^{jQ}\left(\psi * \psi_{l-j}^*\right)\left(\gamma^{-1} \cdot 2^j x\right). \end{aligned} \tag{6.8}$$

In particular, (3.25) implies that the convolution vanishes if $|j - l| > 1$. For the other case, we observe that the convolution products $\psi * \psi_l$, for $l \in \{-1, 0, 1\}$ are Schwartz functions, hence

$$\left|\psi_{j,\gamma} * \psi_l^*(x)\right| \le C 2^{jQ}\left(1 + 2^j \left|2^{-j}\gamma^{-1} \cdot x\right|\right)^{-Q-1}. \tag{6.9}$$

□

For the convergence of the sums over Γ, we will need the Schur test for boundedness of infinite matrices on ℓ^p-spaces.

Lemma 6.3. *Let $1 \le p \le \infty$. Let Γ be some countable set, and let $A = \left(a_{\lambda,\gamma}\right)_{\lambda,\gamma \in \Gamma}$ denote a matrix of complex numbers. Assume that, for some finite constant M,*

$$\sup_{\gamma} \sum_{\lambda \in \Gamma} \left|a_{\lambda,\gamma}\right| \le M, \qquad \sup_{\lambda} \sum_{\gamma \in \Gamma} \left|a_{\lambda,\gamma}\right| \le M. \tag{6.10}$$

Then the operator

$$T_A\colon (x_\gamma)_{\gamma\in\Gamma} \longmapsto \left(\sum_{\gamma\in\Gamma} a_{\lambda,\gamma}x_\gamma\right)_{\lambda\in\Gamma} \tag{6.11}$$

is bounded on $\ell^p(\Gamma)$, with operator norm $\le M$.

Lemma 6.4. *Let $\eta, j \in \mathbb{Z}$, with $\eta \le j$ and $N \ge Q+1$. Let $\Gamma \subset G$ be separated. Then for any $x \in G$, one has*

$$\sum_\gamma 2^{-jQ}\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)\cdot x\right|\right)^{-N} \le C2^{-\eta Q}, \tag{6.12}$$

where the constant C depends only on N and Γ.

Proof. By assumption, there exists an open set W such that $\gamma W \cap \gamma' W = \emptyset$, for $\gamma, \gamma' \in \Gamma$ with $\gamma \ne \gamma'$. In addition, we may assume W is relatively compact. Then,

$$\sum_\gamma 2^{-jQ}\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)\cdot x\right|\right)^{-N} \le \sum_\gamma \frac{1}{|W|}\int_{2^{-j}(\gamma W)}\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)\cdot x\right|\right)^{-N}dy. \tag{6.13}$$

For $y \in 2^{-j}(\gamma W)$, the triangle inequality of the quasi-norm yields

$$\begin{aligned} 1+2^\eta\left|y^{-1}x\right| &\le 1+2^\eta C\left(\left|y^{-1}\left(2^{-j}\gamma\right)\right| + \left|\left(2^{-j}\gamma^{-1}\right)x\right|\right) \\ &\le 1+C2^\eta\left(2^{-j}\operatorname{diam}(W) + \left|\left(2^{-j}\gamma^{-1}\right)x\right|\right) \\ &\le C'\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)x\right|\right), \end{aligned} \tag{6.14}$$

with the last inequality due to $\eta \le j$. Accordingly,

$$\begin{aligned} \sum_\gamma \frac{1}{|W|}\int_{2^{-j}(\gamma W)}\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)\cdot x\right|\right)^{-N}dy &\le C''\sum_\gamma\int_{2^{-j}(\gamma W)}\left(1+2^\eta\left|y^{-1}\cdot x\right|\right)^{-N}dy \\ &= C''2^{-\eta Q}\int_G (1+|y|)^{-N}dy, \end{aligned} \tag{6.15}$$

where the inequality used disjointness of the γW. For $N \ge Q+1$, the integral is finite. □

The next lemma is an analog of [12, Lemma 3.4], which we will need for the proof of Theorem 6.1.

Lemma 6.5. *Let $1 \le p \le \infty$ and $j, \eta \in \mathbb{Z}$ be fixed with $\eta \le j$. Suppose that $\Gamma \subset G$ is a regular sampling set. For any $\gamma \in \Gamma$, let $f_{j,\gamma}$ be a function on G. Assume that the $f_{j,\gamma}$ fulfill the decay estimate*

$$\forall x \in G, \quad \forall \eta, j \in \mathbb{Z}, \forall \gamma \in \Gamma\colon \left|f_{j,\gamma}(x)\right| \le C_1\left(1+2^\eta\left|\left(2^{-j}\gamma^{-1}\right)\cdot x\right|\right)^{-(Q+1)}, \tag{6.16}$$

with a constant $C_1 > 0$. *Define* $F = \sum_{\gamma\in\Gamma} c_{j,\gamma} f_{j,\gamma}$, *where* $\{c_{j,\gamma}\}_\gamma \in l^p(\Gamma)$. *Then the series converges unconditionally in* L^p, *with*

$$\|F\|_p \le C_2 2^{(j-\eta)Q} 2^{-jQ/p} \left\| \{c_{j,\gamma}\} \right\|_{\ell^p(\Gamma)}, \tag{6.17}$$

with a constant C_2 *independent of* j, γ, η, *and of the coefficient sequence.*

Proof. To prove the assertion, let W be a Γ-tile. Then,

$$\begin{aligned} \|F\|_p^p &= \sum_{\alpha\in\Gamma} \int_{2^{-j}(\alpha W)} \left| \sum_\gamma c_{j,\gamma} f_{j,\gamma}(x) \right|^p dx \\ &\le C_1^p \sum_{\alpha\in\Gamma} \int_{2^{-j}(\alpha W)} \left| \sum_\gamma |c_{j,\gamma}| \left(1 + 2^\eta \left| \left(2^{-j}\gamma^{-1}\right) \cdot x \right|\right)^{-(Q+1)} \right|^p dx. \end{aligned} \tag{6.18}$$

On each integration patch $2^{-j}(\alpha W)$, the triangle inequality of the quasi-norm yields the estimate

$$1 + 2^\eta \left| 2^{-j}\left(\gamma^{-1}\alpha\right) \right| \le C' \left(1 + 2^\eta \left| \left(2^{-j}\gamma^{-1}\right) x \right|\right), \tag{6.19}$$

compare the proof of Lemma 6.4, and thus the integrand can be estimated from above by the constant

$$\left| \sum_\gamma |c_{j,\gamma}| \left(1 + 2^\eta \left| 2^{-j}\left(\gamma^{-1}\alpha\right) \right|\right)^{-(Q+1)} \right|^p \tag{6.20}$$

whence

$$\begin{aligned} &\sum_{\alpha\in\Gamma} \int_{2^{-j}(\alpha W)} \left| \sum_\gamma |c_{j,\gamma}| \left(1 + 2^\eta \left| \left(2^{-j}\gamma^{-1}\right) \cdot x \right|\right)^{-(Q+1)} \right|^p dx \\ &\quad\le C' \sum_{\alpha\in\Gamma} 2^{-jQ} \left(\sum_\gamma |c_{j,\gamma}| \left(1 + 2^\eta \left| 2^{-j}\left(\gamma^{-1}\alpha\right) \right|\right)^{-(Q+1)} \right)^p \\ &\quad= C' \sum_{\alpha\in\Gamma} 2^{-jQ} \left(\sum_\gamma |c_{j,\gamma}| a_{\alpha,\gamma} \right)^p. \end{aligned} \tag{6.21}$$

Here $a_{\alpha,\gamma} = (1 + 2^{\eta}|2^{-j}(\gamma^{-1}\alpha)|)^{-(Q+1)}$. Now Lemma 6.4 yields that the Schur test is fulfilled for the coefficients $\{a_{\alpha,\gamma}\}$ with $M = 2^{Q(j-\eta)}$ (observe in particular that the right-hand side of the estimate above is independent of x), thus Lemma 6.3 yields

$$\|F\|_p \le C'' 2^{-jQ/p} \left(\sum_{\alpha\in\Gamma} \left(\sum_{\gamma} |c_{j,\gamma}| a_{\alpha,\gamma} \right)^p \right)^{1/p} \le C_2 2^{-jQ/p} 2^{(j-\eta)Q} \left\| \{c_{j,\gamma}\}_{\gamma} \right\|_{\ell^p}, \tag{6.22}$$

as desired. □

Proof of Theorem 6.1. We still need to prove part (a) of the theorem, and here it is sufficient to show the norm estimate for all finitely supported coefficient sequences $\{c_{j,\gamma}\}_{j,\gamma}$. The full statement then follows by completeness of $\dot{B}^s_{p,q}$ and from the fact that the Kronecker-δs are an unconditional basis of $\dot{b}^s_{p,q}$ (here we need $p, q < \infty$).

Repeated applications of the triangle inequality yield

$$\begin{aligned} \|f\|_{\dot{B}^s_{p,q}} &= \left\| \left\{ 2^{ls} \left\| \sum_{j,\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \right\}_l \right\|_{\ell^q(\mathbb{Z})} \\ &\le \left\| \left\{ 2^{ls} \sum_{j=-\infty}^{l-1} \left\| \sum_{\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \right\}_l \right\|_{\ell^q(\mathbb{Z})} + \left\| \left\{ 2^{ls} \sum_{j=l}^{\infty} \left\| \sum_{\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \right\}_l \right\|_{\ell^q(\mathbb{Z})}. \end{aligned} \tag{6.23}$$

Pick $N, \theta \in \mathbb{N}$ such that $N > Q - s + 1$ and $\theta > s + 1$. Define

$$d_j := 2^{j(s-Q/p)} \|\{c_{j,\gamma}\}\|_{\ell^p}. \tag{6.24}$$

For $j < l$, assumption (6.2) yields

$$\left|(g_{j,\gamma} * \psi_l^*)(x)\right| \le C 2^{jQ} 2^{-(l-j)\theta} \left(1 + 2^j \left| \left(2^{-j}\gamma^{-1}\right) x \right| \right)^{-Q-1}, \tag{6.25}$$

and thus, by Lemma 6.5,

$$\left\| \sum_{\gamma\in\Gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \le C' 2^{jQ} 2^{-(l-j)\theta} 2^{-jQ/p} \left\| \{c_{j,\gamma}\}_{\gamma} \right\|_{\ell^p}. \tag{6.26}$$

But then

$$\begin{aligned} 2^{ls} \sum_{j=-\infty}^{l-1} \left\| \sum_{\gamma\in\Gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p &\le C' \sum_{j=-\infty}^{l-1} 2^{-(l-j)(\theta-s)} 2^{j(s-Q/p)} \left\| \{c_{j,\gamma}\}_{\gamma} \right\|_{\ell^p} \\ &= C' \left(b *_{\mathbb{Z}} \{d_j\}_j \right)(l) \quad , \end{aligned} \tag{6.27}$$

where $*_{\mathbb{Z}}$ denotes convolution over $\mathbb{Z}$, and

$$b(j) = 2^{-j(\theta-s)}\chi_{\mathbb{N}}(j). \tag{6.28}$$

By choice of θ, $b \in \ell^1(\mathbb{Z})$, and Young's inequality allows to conclude that

$$\left\| \left\{ 2^{ls} \sum_{j=-\infty}^{l-1} \left\| \sum_{\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \right\}_l \right\|_{\ell^q(\mathbb{Z})} \leq C'' \left\| \{d_j\}_j \right\|_{\ell^q} = C'' \left\| \{c_{j,\gamma}\}_{j,\gamma} \right\|_{\dot{b}^s_{p,q}}. \tag{6.29}$$

For $j \geq l$, assumption (6.2) provides the estimate

$$\left| (g_{j,\gamma} * \psi_l^*)(x) \right| \leq C 2^{jQ} 2^{-(l-j)\theta} \left(1 + 2^j \left| \left(2^{-j}\gamma^{-1} \right) x \right| \right)^{-Q-1}. \tag{6.30}$$

Here, Lemma 6.5 and straightforward calculation allow to conclude that

$$2^{ls} \sum_{j=-\infty}^{l-1} \left\| \sum_{\gamma\in\Gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \leq C' \left(\widetilde{b} *_{\mathbb{Z}} \{d_j\}_j \right)(l), \tag{6.31}$$

with

$$\widetilde{b}(j) = 2^{-j(s+N-Q)}\chi_{\mathbb{N}}(j). \tag{6.32}$$

Hence, Young's theorem applies again and yields

$$\left\| \left\{ 2^{ls} \sum_{j=-\infty}^{l-1} \left\| \sum_{\gamma} c_{j,\gamma} g_{j,\gamma} * \psi_l^* \right\|_p \right\}_l \right\|_{\ell^q(\mathbb{Z})} \leq C'' \left\| \{c_{j,\gamma}\}_{j,\gamma} \right\|_{\dot{b}^s_{p,q}}, \tag{6.33}$$

and we are done. □

We conclude this section by showing that wavelets provide a simultaneous Banach frame for $\dot{B}^s_{p,q}$, for all $1 \leq p, q < \infty$ and $s \in \mathbb{R}$; see [24] for an introduction to Banach frames. In the following, we consider the frame operator associated to a regular sampling set Γ, given by

$$S_{\psi,\Gamma}(f) = \sum_{j\in\mathbb{Z},\gamma\in\Gamma} 2^{-jQ} \langle f, \psi_{j,\gamma} \rangle \psi_{j,\gamma}. \tag{6.34}$$

By Theorems 6.1 and 5.4, $S_{\psi,\Gamma} : \dot{B}^s_{p,q}(G) \to \dot{B}^s_{p,q}(G)$ is bounded, at least for sufficiently dense sampling sets Γ. Our aim is to show that, for all sufficiently dense regular sampling sets, the operator $S_{\psi,\Gamma}$ is in fact invertible, showing that the wavelet system is a Banach frame for $\dot{B}^s_{p,q}(G)$. The following lemma contains the main technical ingredient for the proof. Once again, we will rely on oscillation estimates.

Lemma 6.6. *Let $f = u * \psi_j^*$, with $u \in \mathcal{S}'(G)/\mathcal{P}$, such that $f \in L^p(G)$, for some $1 \leq p < \infty$. For $\epsilon > 0$, there exists a neighborhood U of the identity such that, for all U-dense regular sampling sets $\Gamma \subset G$ and all Γ-tiles $W \subset G$, one has*

$$\left\| f * \psi_j - \sum_{\gamma\in\Gamma} |W| 2^{-jQ} \langle u, \psi_{j,\gamma} \rangle \psi_{j,\gamma} \right\|_p \leq \epsilon \left\| f * \psi_j \right\|_p. \tag{6.35}$$

Proof. We first consider the case $j = 0$. Let W denote a Γ-tile. We define the auxiliary function

$$h = \sum_{\gamma\in\Gamma} f(\gamma) L_\gamma \chi_{\gamma W} = T R_\Gamma f, \tag{6.36}$$

using the notation of the proof of Proposition 5.7. By the triangle inequality,

$$\left\| f * \psi_0 - \sum_{\gamma\in\Gamma} |W| \langle u, \psi_{0,\gamma} \rangle \psi_{0,\gamma} \right\|_p \leq \left\| (f-h) * \psi_0 \right\|_p + \left\| h * \psi_0 - \sum_{\gamma\in\Gamma} |W| \langle u, \psi_{0,\gamma} \rangle \psi_{0,\gamma} \right\|_p. \tag{6.37}$$

Now Young's inequality, together with the proof of Proposition 5.7, implies that for all sufficiently dense Γ,

$$\left\| (f-h) * \psi_0 \right\|_p \leq \| f - h \|_p \| \psi_0 \|_1 \leq \frac{\epsilon \| f \|_p}{2}. \tag{6.38}$$

For the second term in the right hand side of (6.37), we first observe that $\langle u, \psi_{0,\gamma} \rangle = f(\gamma)$, and thus using the tiling $G = \bigcup_{\gamma\in\Gamma} \gamma W$,

$$\begin{aligned}
\left| \left(\sum_{\gamma\in\Gamma} |W| f(\gamma) \psi_{0,\gamma} - h * \psi_0 \right)(y) \right| &= \left| \sum_{\gamma\in\Gamma} |W| f(\gamma) \psi\left(\gamma^{-1} y\right) - \sum_{\gamma\in\Gamma} \int_{\gamma W} f(\gamma) \psi_0\left(x^{-1} y\right) dx \right| \\
&= \left| \sum_{\gamma\in\Gamma} \int_{\gamma W} f(\gamma) \left(\psi_0\left(\gamma^{-1} y\right) - \psi_0\left(x^{-1} y\right) \right) dx \right| \\
&\leq \sum_{\gamma\in\Gamma} \int_{\gamma W} |f(\gamma)| \left| \psi_0\left(\gamma^{-1} y\right) - \psi_0\left(x^{-1} y\right) \right| dx.
\end{aligned} \tag{6.39}$$

Since $x \in \gamma W$ if and only if $y^{-1}\gamma \in y^{-1} x W^{-1}$, it follows that

$$\left| \psi_0\left(\gamma^{-1} y\right) - \psi_0\left(x^{-1} y\right) \right| \leq \mathrm{osc}_{W^{-1}}(\psi_0)\left(y^{-1} x\right), \tag{6.40}$$

thus we can continue the estimate by

$$\begin{aligned}(6.39) &\leq \sum_{\gamma\in\Gamma}\int_{\gamma W}|f(\gamma)|\mathrm{osc}_{W^{-1}}(\psi_0)\left(y^{-1}x\right)dx \\ &= |h| * (\mathrm{osc}_{W^{-1}}(\psi_0))^{\sim}(y),\end{aligned} \tag{6.41}$$

leading to

$$\left\| h * \psi_0 - \sum_{\gamma\in\Gamma}|W|\langle u, \psi_{0,\gamma}\rangle\psi_{0,\gamma}\right\|_p \leq \|h\|_p\|\mathrm{osc}_{W^{-1}}(\psi_0)\|_1 < \frac{\epsilon\|f\|_p}{2}, \tag{6.42}$$

using $\|h\|_p \leq 2\|f\|_p$ as well as $\|\mathrm{osc}_{W^{-1}}(\psi_0)\|_1 < \epsilon/4$, both valid for sufficiently dense Γ, by the proof of Proposition 5.7, and by Lemma 5.9, respectively.

Thus (6.35) is established for $j = 0$. The statement for general $j \in \mathbb{Z}$ now follows by dilation, similar to the proof of Lemma 5.6. We write $f = u * \psi_j^* = (v^j * \psi_0^*) \circ \delta_{2^j}$, where $v^j = u \circ \delta_{2^{-j}}$. Hence, for

$$g = \sum_{\gamma\in\Gamma}|W|2^{-jQ}\langle u, \psi_{j,\gamma}\rangle\psi_{j,\gamma}, \tag{6.43}$$

we obtain that

$$\begin{aligned}\|f * \psi_j - g\|_p &= \left\|\left(v^j * \psi_0^* * \psi_0\right) \circ \delta_{2^j} - g\right\|_p \\ &= \left\|\left(v^j * \psi_0^* * \psi_0 - g \circ \delta_{2^{-j}}\right) \circ \delta_{2^j}\right\|_p \\ &= 2^{-jQ/p}\left\|v^j * \psi_0^* * \psi_0 - g \circ \delta_{2^{-j}}\right\|_p.\end{aligned} \tag{6.44}$$

Now

$$\begin{aligned}g \circ \delta_{2^{-j}} &= \sum_{\gamma\in\Gamma}|W|2^{-jQ}\langle u, \psi_{j,\gamma}\rangle(\psi_{j,\gamma} \circ \delta_{2^{-j}}) \\ &= \sum_{\gamma\in\Gamma}|W|\left(u * \psi_j^*\right)\left(2^{-j}\gamma\right)\psi_{0,\gamma} \\ &= \sum_{\gamma\in\Gamma}|W|\left(v^j * \psi_0^*\right)(\gamma)\psi_{0,\gamma}.\end{aligned} \tag{6.45}$$

Thus, by the case $j = 0$,

$$\begin{aligned}2^{-jQ/p}\left\| v^j * \psi_0^* * \psi_0 - g \circ \delta_{2^{-j}} \right\|_p &= \epsilon 2^{-jQ/p} \left\| v^j * \psi_0^* * \psi_0 \right\|_p \\ &= \epsilon \left\| u * \psi_j^* \right\|_p,\end{aligned} \tag{6.46}$$

as desired. □

Now, invertibility of the frame operator is easily established. In fact, we can even show the existence of a dual frame and an atomic decomposition for our homogeneous Besov spaces. Note however that the notation of the following theorem is somewhat deceptive. The dual wavelet frame might depend on the space $\dot{B}^s_{p,q}$, whereas the well-known result for wavelet bases in the Euclidean setting allows to take $\widetilde{\psi}_{j,k} = \psi_{j,k}$, regardless of the Besov space under consideration.

Theorem 6.7 (Atomic decomposition). *Let $1 \le p, q < \infty$. There exists a neighborhood U of the identity such that, for all U-dense regular sampling sets $\Gamma \subset G$, the frame operator $S_{\psi,\Gamma}$ is an automorphism of $\dot{B}^s_{p,q}(G)$.*

In this case, there exists a dual wavelet family $\{\widetilde{\psi}_{j,\gamma}\}_{j,\gamma} \subset \dot{B}^{s*}_{p,q}$, such that for all $f \in \dot{B}^s_{p,q}(G)$, one has

$$f = \sum_{j \in \mathbb{Z}, \gamma \in \Gamma} 2^{-jQ} \langle f, \widetilde{\psi}_{j,\gamma} \rangle \psi_{j,\gamma}, \tag{6.47}$$

and in addition

$$\| f \|_{\dot{B}^s_{p,q}} \asymp \left(\sum_{j \in \mathbb{Z}} \left(\sum_{\gamma \in \Gamma} 2^{j(s-Q/p)p} \left| \langle f, \widetilde{\psi}_{j,\gamma} \rangle \right|^p \right)^{q/p} \right)^{1/q}. \tag{6.48}$$

Proof. Fix $0 < \epsilon < 1$, and choose the neighborhood U according to the previous lemma, with ϵ replaced by

$$\epsilon_0 = \frac{\epsilon}{\left(2^{-sq} + 1 + 2^{sq}\right)^{1/q} 3^{(q-1)/q} \left\| \psi_0 \right\|_1}. \tag{6.49}$$

Let Γ be a U-dense regular sampling set, and let W denote a Γ-tile. Let $f \in \mathfrak{D}$, where $\mathfrak{D} \subset \dot{B}^s_{p,q}(G)$ is the dense subspace of functions for which

$$f = \sum_{j \in \mathbb{Z}} f * \psi_j^* * \psi_j \tag{6.50}$$

holds with finitely many nonzero terms; see Remark 3.13. For $l \in \mathbb{Z}$, we then obtain from (3.25) that

$$\begin{aligned}\left\| f - |W| S_{\psi,\Gamma} f \right\|_p &= \left\| \sum_{|j-l|\le 1} \left(f * \psi_j^* * \psi_j - \sum_{\gamma\in\Gamma} 2^{-jQ} |W| \langle f, \psi_{j,\gamma}\rangle \psi_{j,\gamma} \right) * \psi_l^* \right\|_p \\ &\le \sum_{|j-l|\le 1} \epsilon_0 \left\| f * \psi_j^* \right\|_p \left\| \psi_0 \right\|_1,\end{aligned} \tag{6.51}$$

where the inequality used Lemma 6.6 and Young's inequality. But then it follows that

$$\begin{aligned}\left\| f - |W| S_{\psi,\Gamma} f \right\|_{\dot{B}^s_{p,q}}^q &= \sum_{l\in\mathbb{Z}} 2^{lsq} \left\| f - |W| S_{\psi,\Gamma} f \right\|_p^q \\ &\le \sum_{l\in\mathbb{Z}} 2^{lsq} \left(\sum_{|j-l|\le 1} \epsilon_0 \left\| f * \psi_j^* \right\|_p \left\| \psi_0 \right\|_1 \right)^q \\ &\le \sum_{l\in\mathbb{Z}} 2^{lsq} \sum_{|j-l|\le 1} 3^{q-1} \epsilon_0^q \left\| \psi_0 \right\|_1^q \left\| f * \psi_j^* \right\|_p^q \\ &= \sum_{j\in\mathbb{Z}} 2^{jsq} \left\| f * \psi_j^* \right\|_p^q 3^{q-1} \epsilon_0^q \left(2^{-sq} + 1 + 2^{sq} \right) \\ &= \epsilon^q \left\| f \right\|_{\dot{B}^s_{p,q}}^q.\end{aligned} \tag{6.52}$$

Since $S_{\psi,\Gamma}$ is bounded, the estimate extends to all $f \in \dot{B}^s_{p,q}$. Therefore, the operator $S_{\Gamma,\psi}$ is invertible by its Neumann series on $\dot{B}^s_{p,q}$, for all sufficiently dense quasi-lattices Γ. In this case, we define the dual wavelet frame by

$$\left\langle f, \widetilde{\psi}_{j,\gamma} \right\rangle = \left\langle S_{\psi,\Gamma}^{-1}(f), \psi_{j,\gamma} \right\rangle. \tag{6.53}$$

$\widetilde{\psi}_{j,\gamma} \in \dot{B}^{s*}_{p,q}$, since $S_{\psi,\Gamma}^{-1}$ is bounded and $\psi_{j,\gamma} \in \dot{B}^{s*}_{p,q}$.

Let $f \in \dot{B}^s_{p,q}$. By Theorem 5.4, $S_{\psi,\Gamma}^{-1}(f) \in \dot{B}^s_{p,q}$ implies $\{\langle f, \widetilde{\psi}_{j,\gamma}\rangle\}_{j,\gamma} \in \dot{b}^s_{p,q}$. Theorem 6.1 then implies that

$$f = S_{\psi,\Gamma}\left(S_{\psi,\Gamma}^{-1}(f) \right) = \sum_{j\in\mathbb{Z},\gamma\in\Gamma_b} 2^{-jQ} \left\langle S_{\psi,\Gamma}^{-1}(f), \psi_{j,\gamma} \right\rangle \psi_{j,\gamma} = \sum_{j\in\mathbb{Z},\gamma\in\Gamma} 2^{-jQ} \left\langle f, \widetilde{\psi}_{j,\gamma} \right\rangle \psi_{j,\gamma} \tag{6.54}$$

with unconditional convergence in the Besov norm. Furthermore, Theorems 6.1 and 5.4 yield that

$$\|f\|_{\dot{B}^s_{p,q}} \leq \left(\sum_j \left(\sum_\gamma 2^{j(s-Q/p)p} |\langle f, \widetilde{\varphi}_{j,\gamma} \rangle|^p \right)^{q/p} \right)^{1/q} \leq \left\| S^{-1}_{\psi,\Gamma}(f) \right\|_{\dot{B}^s_{p,q}} \leq \|f\|_{\dot{B}^s_{p,q}}, \tag{6.55}$$

up to constants depending on p, q, s, but not on f. This completes the proof. □

Remark 6.8. We wish to stress that an appropriate choice of Γ provides a wavelet frame in $\dot{B}^s_{p,q}$, simultaneously valid for all $1 \leq p, q < \infty$ and all $s \in \mathbb{R}$. As the discussion in Section 5 shows, the tightness of the oscillation estimates converges to one with increasing density of the quasi-lattices. As a consequence, the tightness of the wavelet frame in $\dot{B}^s_{p,q}$ converges to one also, at least when measured with respect to the Besov norm from Definition 3.9, applied to the same window ψ. However, the tightness will depend on p, q, and s.

Remark 6.9. We expect to remove the restriction on p and q in our future work and prove the existence of (quasi)Banach frame for all homogeneous Besov spaces $\dot{B}^s_{p,q}$ with $0 < p, q \leq \infty$ and $s \in \mathbb{R}$.

Remark 6.10. Our treatment of discretization problems via oscillation estimates is heavily influenced by the work of Feichtinger and Gröchenig on atomic decomposition, in particular the papers [23, 25] on coorbit spaces. A direct application of these results to our problem is difficult, since the representations underlying our wavelet transforms are not irreducible if the group G is noncommutative, whereas irreducibility is an underlying assumption in [23, 25]. However, the recent extensions of coorbit theory, most notably [26], provide a unified approach to our results (see [27]).

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