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Research Article

Some Applications of the Spectral Theory for the Integral Transform Involving the Spectral Representation

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In many previous papers, an integral transform $\mathcal{F}_{\gamma,\beta}$ was just considered as a transform on appropriate function spaces. In this paper we deal with the integral transform as an operator on a function space. We then apply various operator theories to $\mathcal{F}_{\gamma,\beta}$. Finally we give an application for the spectral representation of a self-adjoint operator which plays a key role in quantum mechanics.

1. Introduction and Definitions

It is a well-known fact that the spectral theory is one of the main subjects of modern functional analysis and applications. It arises quite naturally in connection with the problems of solving equations. In particular, the spectrum of bounded linear operators on a normed and Banach space is the most important concept to understand the spectral theories. Furthermore, the spectral representation is used widely to apply theories in many fields.

Let $C_0[0,T]$ denote one-parameter Wiener space, that is, the space of continuous real-valued functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T]0,\mathcal{M},m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0,T]} F(x)dm(x). \tag{1.1}$$

A subset B of $C_0[0,T]$ is said to be scale-invariant measurable provided ρB is $\mathcal{M}$ measurable for all $\rho \geq 0$, and a scale-invariant measurable set N is said to be a scale-invariant null set provided $m(\rho N) = 0$ for all $\rho \geq 0$. A property that holds except on a scale-invariant null set is said to hold scale invariant almost everywhere (s-a.e.) [1].

In [2], for each pair of nonzero complex numbers γ and β, Lee introduced an integral transform $\mathcal{F}_{\gamma,\beta}(F)$ of a functional F on abstract Wiener space. For certain values of the parameters γ and β and for certain classes of functionals, the Fourier-Wiener transform [3], the modified Fourier-Wiener transform [4], the Fourier-Feynman transform, and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\gamma,\beta}$. These transforms play an important role in studying stochastic processes and functional integrals on infinite dimensional spaces.

In many papers [5–9], the authors studied the integral transform with related topics of functionals in several classes. Recently, in [10, 11], the authors established the existence of a generalized integral transform $\mathcal{F}_{\gamma,\beta,h}$ via a Gaussian process of functionals in a class $\mathcal{S}_\alpha$ and then obtained various relationships involving the convolution product and the first variation of them.

However, in all previous works, the authors have considered $\mathcal{F}_{\gamma,\beta}$ just as a transform of a functional F on function space and then they obtained various relationships as seen in Remark 2.5 below. In this paper, we consider the integral transform $\mathcal{F}_{\gamma,\beta}$ as an operator on a Banach space. We then apply various operator theories to $\mathcal{F}_{\gamma,\beta}$. Furthermore, we obtain various theorems of the spectral theory for $\mathcal{F}_{\gamma,\beta}$ involving the spectral representation.

Now we are ready to define the integral transform (IT) of a functional on $K \equiv K_0[0,T]$, the space of all complex-valued continuous functions defined on $[0,T]$ which vanish at $t = 0$, used in [5–9, 12].

Definition 1.1. Let F be a functional defined on K. For each pair of nonzero complex numbers γ and β, the IT $\mathcal{F}_{\gamma,\beta}(F)$ of F is defined by

$$\mathcal{F}_{\gamma,\beta}(F)(y) \equiv \int_{C_0[0,T]} F(\gamma x + \beta y)\,dm(x), \quad y \in K, \tag{1.2}$$

if it exists.

Remark 1.2. When $\gamma = 1$ and $\beta = i$, $\mathcal{F}_{1,i}$ is the Fourier-Wiener transform introduced by Cameron and Martin in [3], and when $\gamma = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\sqrt{2},i}$ is the modified Fourier-Wiener transform used by Cameron and Martin in [4].

For $v \in L_2[0,T]$ and $x \in C_0[0,T]$, let $\langle v, x\rangle$ denote the Paley-Wiener-Zygmund (PWZ) stochastic integral. Then we have the following assertions.

(1) For each $v \in L_2[0,T]$, $\langle v, x\rangle$ exists for a.e. $x \in C_0[0,T]$.

(2) If $v \in L_2[0,T]$ is a function of bounded variation on $[0,T]$, $\langle v, x\rangle$ equals the Riemann-Stieltjes integral $\int_0^T v(t)\,dx(t)$ for s-a.e. $x \in C_0[0,T]$.

(3) The PWZ stochastic integral $\langle v, x\rangle$ has the expected linearity property.

(4) The PWZ stochastic integral $\langle v, x\rangle$ is a Gaussian process with mean 0 and variance $\|v\|_2^2$.

For a more detailed study of the PWZ stochastic integral, see [5, 10, 12–14].

Let $\{\alpha_1, \alpha_2, \ldots\}$ be an orthonormal set in $L_2[0,T]$. The following formula is a well-known Wiener integration formula. Let $h : \mathbb{R}^n \to \mathbb{R}$ be Borel measurable and let $H(x) = h(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle)$. Then

$$\int_{C_0[0,T]} H(x)dm(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} h(u_1, \ldots, u_n) \cdot \exp\left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2} \right\} du_1 \cdots du_n \tag{1.3}$$

in the sense that if either side of (1.3) exists, both sides exist and equality holds.

2. Some Results as a Transform

In this section, we establish the existence of the IT of functionals in a class $\mathcal{A}_n^{(2)}$, as seen in Theorem 2.4 below. We then give the inverse IT of our IT. Finally we state possible relationships for the IT with related topics.

We start this section by describing the class of functionals that we work with in this paper. Let $\{\alpha_1, \alpha_2, \ldots\}$ be an orthonormal set in $L_2[0,T]$. Let $\mathcal{A}_n^{(2)}$ be the space of all functionals $F : K \to \mathbb{C}$ of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle) \tag{2.1}$$

for some positive integer n (throughout this paper, n is fixed), where $f(u_1, \ldots, u_n)$ is an entire function of the n complex variables $u_1, \ldots, u_n$ and

$$\int_{\mathbb{R}^n} \left| f(\vec{u}) \right|^2 d\vec{u} < \infty. \tag{2.2}$$

To simplify the expressions, we use the following notation:

$$f(\langle \vec{\alpha}, x\rangle) \equiv f(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle). \tag{2.3}$$

Remark 2.1. For any F and G in $\mathcal{A}_n^{(2)}$, we can always express F by (2.1) and G by

$$G(x) = g(\langle \alpha_1, x\rangle, \ldots, \langle \alpha_n, x\rangle) \equiv g(\langle \vec{\alpha}, x\rangle) \tag{2.4}$$

using the same positive integer n, where g is an entire function and

$$\int_{\mathbb{R}^n} \left| g(\vec{u}) \right|^2 d\vec{u} < \infty. \tag{2.5}$$

Note that $\mathcal{A}_n^{(2)}$ is a very rich class of functionals because $\mathcal{A}_n^{(2)}$ contains the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. These functionals are of interest in Feynman integration theories and quantum mechanics.

Now, we will introduce a notation. It will be convenient to express for the type of limiting integral that occurs in our paper. For appropriate functions f and g on $\mathbb{R}^n$, if

$$\lim_{A\to\infty}\int_{\mathbb{R}^n}\left|\int_{-A}^{A}\overset{(n)}{\cdots}\int_{-A}^{A} f(\vec{u};\vec{v})d\vec{u}-g(\vec{v})\right|^2 d\vec{v}=0, \tag{2.6}$$

then we say that the integral

$$\int_{\mathbb{R}^n} f(\vec{u};\vec{v})d\vec{u}=g(\vec{v}) \tag{2.7}$$

is to be interpreted as an L_2-limiting integral, see [15].

The following lemma is due to Cameron and Storvick in [15, Lemma H].

Lemma 2.2. *Let γ be nonzero complex number with* $\mathrm{Re}(1/\gamma^2)\geq 0$. *For $f\in L_2(\mathbb{R}^n)$, let*

$$g(\vec{v})=\left(2\pi\gamma^2\right)^{-n/2}\int_{\mathbb{R}^n} f(\vec{u})\exp\left\{-\sum_{j=1}^{n}\frac{(u_j-v_j)^2}{2\gamma^2}\right\}d\vec{u}. \tag{2.8}$$

Then $g\in L_2(\mathbb{R}^n)$, and

$$\|g\|_2\leq\|f\|_2. \tag{2.9}$$

If $\mathrm{Re}(1/\gamma^2)=0$, *the integral is to be interpreted as an L_2-limiting integral; moreover, in this case*

$$\|g\|_2=\|f\|_2. \tag{2.10}$$

The following lemma is very useful in establishing the existence of the IT.

Lemma 2.3. *Let f, g, and γ be as in Lemma 2.2 and let β be a nonzero complex number with $|\beta|\geq 1$. Let $h(\vec{u})=g(\beta\vec{u})$. Then $h\in L_2(\mathbb{R}^n)$ and*

$$\|h\|_2\leq\|g\|_2\leq\|f\|_2. \tag{2.11}$$

If $\mathrm{Re}(1/\gamma^2)=0$ *and* $|\beta|=1$, *the integral is to be interpreted as an L_2-limiting integral; moreover, in this case*

$$\|h\|_2=\|g\|_2=\|f\|_2. \tag{2.12}$$

Proof. First note that for all nonzero real numbers β with $|\beta|\geq 1$, it follows that

$$\|h\|_2^2=\int_{\mathbb{R}^n}|h(\vec{u})|^2d\vec{u}=\int_{\mathbb{R}^n}\left|g(\beta\vec{u})\right|^2d\vec{u}=\frac{1}{|\beta|^n}\int_{\mathbb{R}^n}\left|g(\vec{u})\right|^2d\vec{u}. \tag{2.13}$$

But each side of the above expression is an analytic function of β throughout the region $\{\beta \in \mathbb{C} : |\beta| \geq 1\}$. Hence, by the uniqueness theorem for analytic functions, the above equality holds for all β with $\{\beta \in \mathbb{C} : |\beta| \geq 1\}$. Since $|\beta| \geq 1$, using Lemma 2.2 we have

$$\|h\|_2^2 \leq \|g\|_2^2 \leq \|f\|_2^2. \tag{2.14}$$

Furthermore, this means that h is an element of $L_2(\mathbb{R}^n)$ and so we complete the proof of Lemma 2.3 as desired. □

In our first theorem, we establish the existence of the IT of a functional F in $\mathcal{A}_n^{(2)}$.

Theorem 2.4. *Let γ and β be as in Lemma 2.3 and let F be given by* (2.1). *Then the IT $\mathcal{F}_{\gamma,\beta}(F)$ of F exists, belongs to $\mathcal{A}_n^{(2)}$, and is given by the formula*

$$\mathcal{F}_{\gamma,\beta}F(y) = \Gamma_{\mathcal{F}_{\gamma,\beta}F}(\langle \vec{\alpha}, y\rangle) \tag{2.15}$$

for $y \in K$, where

$$\Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{v}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\gamma\vec{u} + \beta\vec{v}) \exp\left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2} \right\} d\vec{u}. \tag{2.16}$$

Proof. We first note that (2.15) follows from (1.2) and (1.3). Clearly the function $\Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{\lambda})$ is an entire function since f is an entire function. What is left to show is that the left-hand side of (2.16) is an element of $\mathcal{A}_n^{(2)}$. Now, we note that for all nonzero real values of γ and β,

$$\begin{aligned} \Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{v}) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\gamma\vec{u} + \beta\vec{v}) \exp\left\{ -\sum_{j=1}^{n} \frac{u_j^2}{2} \right\} d\vec{u} \\ &= \left(2\pi\gamma^2\right)^{-n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{ -\sum_{j=1}^{n} \frac{(u_j - \beta v_j)^2}{2\gamma^2} \right\} d\vec{u}. \end{aligned} \tag{2.17}$$

As mentioned in the proof of Lemma 2.3, each side of the above expression is an analytic function of γ throughout the region $\{\gamma \in \mathbb{C} : \mathrm{Re}(1/\gamma^2) \geq 0\}$ and β throughout the region $\{\beta \in \mathbb{C} : |\beta| \geq 1\}$. Hence, by the uniqueness theorem for analytic functions, the above equality holds for all γ and β with $\{\gamma \in \mathbb{C} : \mathrm{Re}(1/\gamma^2) \geq 0\}$ and $\{\beta \in \mathbb{C} : |\beta| \geq 1\}$. Using Lemma 2.3, the function $\Gamma_{\mathcal{F}_{\gamma,\beta}F}$ is an element of $L_2(\mathbb{R}^n)$. In fact,

$$\left\|\Gamma_{\mathcal{F}_{\gamma,\beta}F}\right\|_2 \leq \|f\|_2. \tag{2.18}$$

Moreover, if $\mathrm{Re}(1/\gamma^2) = 0$ and $|\beta| = 1$,

$$\left\|\Gamma_{\mathcal{F}_{\gamma,\beta}F}\right\|_2 = \|f\|_2, \tag{2.19}$$

which completes the proof of Theorem 2.4 as desired. □

Remark 2.5. (1) Under the appropriate conditions for γ and β, we could establish the existences of the convolution product (CP) and the first variation of functionals in $\mathcal{A}_n^{(2)}$ as in Theorem 2.4. We will state just formulas without proof because the main purpose of this paper is to concern $\mathcal{F}_{\gamma,\beta}$ as an operator on Hilbert space.

(2) In [5–8, 10], the authors established various basic formulas for the IT involving the CP and the first variation of functionals in various classes. Like these, we can obtain various basic relationships for the IT with related topic of functionals in $\mathcal{A}_n^{(2)}$ under appropriate conditions for γ and β. We list some relationships as follows.

(i) The IT of a CP is the product of ITs,

$$\mathcal{F}_{\gamma,\beta}(F * G)_\gamma(y) = \mathcal{F}_{\gamma,\beta} F\left(\frac{y}{\sqrt{2}}\right)\mathcal{F}_{\gamma,\beta} G\left(\frac{y}{\sqrt{2}}\right). \tag{2.20}$$

(ii) A relationship among the CP, the IT, and the Inverse IT,

$$(F * G)_\gamma(y) = \mathcal{F}_{i(\gamma/\beta),1/\beta}\left(\mathcal{F}_{\gamma,\beta} F\left(\frac{\cdot}{\sqrt{2}}\right)\mathcal{F}_{\gamma,\beta} G\left(\frac{\cdot}{\sqrt{2}}\right)\right)(y). \tag{2.21}$$

(iii) A relationship between the IT, and the first variation,

$$\beta\mathcal{F}_{\gamma,\beta}\delta F(\cdot \mid w)(y) = \delta\mathcal{F}_{\gamma,\beta} F(y \mid w). \tag{2.22}$$

(iv) A relationship among the CP, the IT, and the first variation,

$$\beta^2\mathcal{F}_{\gamma,\beta}(\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_\gamma(z) = \delta\mathcal{F}_{\gamma,\beta} F\left(\frac{y}{\sqrt{2}} \mid w\right)\delta\mathcal{F}_{\gamma,\beta} G\left(\frac{y}{\sqrt{2}} \mid w\right). \tag{2.23}$$

(v) A relationship among the CP, the inverse IT, and the first variation,

$$\mathcal{F}_{i(\gamma/\beta),1/\beta}(\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_\gamma(y) = \beta^2\delta\mathcal{F}_{i(\gamma/\beta),1/\beta} F\left(\frac{y}{\sqrt{2}} \mid w\right)\delta\mathcal{F}_{i(\gamma/\beta),1/\beta} G\left(\frac{y}{\sqrt{2}} \mid w\right) \tag{2.24}$$

for $y, w \in K$, where the CP $(F * G)_\gamma$ of F and G is defined by

$$(F * G)_\gamma(y) = \int_{C_0[0,T]} F\left(\frac{y + \gamma x}{\sqrt{2}}\right)G\left(\frac{y - x}{\sqrt{2}}\right)dm(x), \quad y \in K, \tag{2.25}$$

and the first variation is defined by formula

$$\delta F(x \mid w) = \frac{\partial}{\partial k}F(x + kw)\bigg|_{k=0}, \quad x, w \in K, \tag{2.26}$$

if they exist.

In our next theorem, we establish the inverse IT of our IT of functionals in $\mathcal{A}_n^{(2)}$.

Theorem 2.6. *Let γ and β be as in Theorem 2.4 with $\operatorname{Re}(\beta^2/\gamma^2) \leq 0$ and $|\beta| = 1$, and let $F \in \mathcal{A}_n^{(2)}$ be given by* (2.1)*. Then*

$$\mathcal{F}_{i(\gamma/\beta),1/\beta}(\mathcal{F}_{\gamma,\beta}F)(y) = F(y) = \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i(\gamma/\beta),1/\beta}F)(y) \tag{2.27}$$

for all $y \in K$. That is to say, $\mathcal{F}_{i(\gamma/\beta),1/\beta}$ is the inverse IT of the IT.

Proof. Since $\operatorname{Re}(1/\gamma^2) \geq 0$ and $|\beta| = 1$, $\mathcal{F}_{\gamma,\beta} \in \mathcal{A}_n^{(2)}$ for all $F \in \mathcal{A}_n^{(2)}$ by Theorem 2.4. Also, since $\operatorname{Re}(1/(i\gamma^2/\beta)) = \operatorname{Re}(-\beta^2/\gamma^2) \geq 0$ and $|1/\beta| = 1$, $\mathcal{F}_{i(\gamma/\beta),1/\beta}(\mathcal{F}_{\gamma,\beta}F) \in \mathcal{A}_n^{(2)}$ for all $F \in \mathcal{A}_n^{(2)}$. By using the similar method, we can show that $\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i(\gamma/\beta),1/\beta}F) \in \mathcal{A}_n^{(2)}$ for all $F \in \mathcal{A}_n^{(2)}$. In [2], the author showed that for a integrable functional F,

$$\int_{C_0[0,T]}\int_{C_0[0,T]} F(px_1 + qx_2 + y)\,dm(x_1)\,dm(x_2) = \int_{C_0[0,T]} F\left(\sqrt{p^2+q^2}z + y\right)dm(z) \tag{2.28}$$

for all nonzero complex numbers p and q. Using this formula and (1.2), we have

$$\begin{aligned}\mathcal{F}_{i(\gamma/\beta),1/\beta}(\mathcal{F}_{\gamma,\beta}F)(y) &= \int_{C_0[0,T]}\int_{C_0[0,T]} F(\gamma x_2 + i\gamma x_1 + y)\,dm(x_1)\,dm(x_2) = F(y),\\ \mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i(\gamma/\beta),1/\beta}F)(y) &= \int_{C_0[0,T]}\int_{C_0[0,T]} F\left(i\frac{\gamma}{\beta}x_2 + \frac{\gamma}{\beta}x_1 + y\right)dm(x_1)\,dm(x_2) = F(y).\end{aligned} \tag{2.29}$$

Hence we complete the proof of Theorem 2.6. □

3. A Bounded Linear Operator $\mathcal{F}_{\gamma,\beta}$

In previous Section 2, we have considered $\mathcal{F}_{\gamma,\beta}$ as a transform of functionals in $\mathcal{A}_n^{(2)}$. From now on we will consider $\mathcal{F}_{\gamma,\beta}$ as an operator from $\mathcal{A}_n^{(2)}$ into $\mathcal{A}_n^{(2)}$ and then apply various operator theories to the IT. In particular, we obtain various spectral theorems for an operator $\mathcal{F}_{\gamma,\beta}$.

For F and G in $\mathcal{A}_n^{(2)}$, let

$$(F,G)_{\mathcal{A}_n^{(2)}} = \int_{\mathbb{R}^n} f(\vec{u})\overline{g(\vec{u})}\,d\vec{u} \tag{3.1}$$

denote the inner product on $\mathcal{A}_n^{(2)}$ and $\|F\|_{\mathcal{A}_n^{(2)}} = (F,F)_{\mathcal{A}_n^{(2)}}^{1/2}$.

Remark 3.1. One can show that $(\mathcal{A}_n^{(2)}, \|\cdot\|_{\mathcal{A}_n^{(2)}})$ is a complex normed linear space. Also, from the fact that $L_2(\mathbb{R}^n)$ is complete, one can easily show that the space $(\mathcal{A}_n^{(2)}, \|\cdot\|_{\mathcal{A}_n^{(2)}})$ is also complete.

In our first theorem in this section, we show that the operator $\mathcal{F}_{\gamma,\beta}$ is well defined on $\mathcal{A}_n^{(2)}$.

Theorem 3.2. *Let γ and β be as in Lemma 2.3. Then the $\mathcal{F}_{\gamma,\beta}$ is a well defined operator from $\mathcal{A}_n^{(2)}$ into $\mathcal{A}_n^{(2)}$ and*

$$\left\|\mathcal{F}_{\gamma,\beta}(F)\right\|_{\mathcal{A}_n^{(2)}} \le \|F\|_{\mathcal{A}_n^{(2)}}. \tag{3.2}$$

Moreover, if $\mathrm{Re}(1/\gamma^2) = 0$ *and* $|\beta| = 1$, *then the operator preserves the norm, namely,*

$$\left\|\mathcal{F}_{\gamma,\beta}(F)\right\|_{\mathcal{A}_n^{(2)}} = \|F\|_{\mathcal{A}_n^{(2)}}. \tag{3.3}$$

Proof. In Section 2, we showed that for each $F \in \mathcal{A}_n^{(2)}$, $\mathcal{F}_{\gamma,\beta}(F)$ exists, belongs to $\mathcal{A}_n^{(2)}$ and

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \Gamma_{\mathcal{F}_{\gamma,\beta}F}(\langle\vec{\alpha}, y\rangle), \tag{3.4}$$

where

$$\Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{v}) = \left(2\pi\gamma^2\right)^{-n/2}\int_{\mathbb{R}^n} f(\vec{u})\exp\left\{-\sum_{j=1}^{n}\frac{\left(u_j-\beta v_j\right)^2}{2\gamma^2}\right\}d\vec{u}. \tag{3.5}$$

Furthermore, we obtained that

$$\left\|\Gamma_{\mathcal{F}_{\gamma,\beta}F}\right\|_2 \le \|f\|_2. \tag{3.6}$$

This tells us that

$$\left\|\mathcal{F}_{\gamma,\beta}(F)\right\|_{\mathcal{A}_n^{(2)}} \le \|F\|_{\mathcal{A}_n^{(2)}} \tag{3.7}$$

for all γ and β satisfy the conditions in Lemma 2.3. Moreover, if $\mathrm{Re}(1/\gamma^2) = 0$ and $|\beta| = 1$,

$$\left\|\mathcal{F}_{\gamma,\beta}(F)\right\|_{\mathcal{A}_n^{(2)}} = \|F\|_{\mathcal{A}_n^{(2)}}, \tag{3.8}$$

which completes the proof of Theorem 3.2 as desired. □

Next, we give a simple example to illustrate our results and formulas in Theorem 3.2.

Example 3.3. Let γ and β be are nonzero real numbers with $|\beta| \ge 1$. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of infinitely differentiable functions $f(\vec{u})$ decaying at infinity together with all its derivatives faster than any polynomial of $|\vec{u}|^{-1}$. For nonzero real values of γ, let

$$f(\vec{u}) = \exp\left\{-\sum_{j=1}^{n}\frac{u_j^2}{2\gamma^2}\right\}. \tag{3.9}$$

Then $f \in \mathcal{S}(\mathbb{R}^n) \subset L_2(\mathbb{R}^n)$ and hence $F(x) = f(\langle \vec{\alpha}, x\rangle)$ is an element of $\mathcal{A}_n^{(2)}$. Also, using (2.15), (2.16), and (1.3), we have

$$\mathcal{F}_{\gamma,\beta}(F)(y) = 2^{-n/2} \exp\left\{ -\sum_{j=1}^{n} \frac{\beta^2}{4\gamma^2} \langle \alpha_j, y\rangle^2 \right\}. \tag{3.10}$$

Furthermore, we note that

$$\|F\|^2_{\mathcal{A}_n^{(2)}} = |\gamma|^n \pi^{n/2}, \qquad \|\mathcal{F}_{\gamma,\beta}(F)\|^2_{\mathcal{A}_n^{(2)}} = 2^{-n/2} \frac{|\gamma|^n \pi^{n/2}}{|\beta|^n}. \tag{3.11}$$

In fact,

$$\|\mathcal{F}_{\gamma,\beta}(F)\|^2_{\mathcal{A}_n^{(2)}} = 2^{-n/2} \frac{|\gamma|^n (\pi)^{n/2}}{|\beta|^n} \leq |\gamma|^n (2\pi)^n = \|F\|^2_{\mathcal{A}_n^{(2)}}. \tag{3.12}$$

All expressions in Example 3.3 are valid for nonzero real values of γ and β. But $f \in \mathcal{S}(\mathbb{R}^n)$ and hence they are still valid for nonzero complex values of γ and β which satisfy the conditions in Lemma 2.3.

Now, we establish some basic operator theories for the operator $\mathcal{F}_{\gamma,\beta}$. First, in our next theorem, we show that the operator $\mathcal{F}_{\gamma,\beta}$ is a bounded linear operator on $\mathcal{A}_n^{(2)}$.

Theorem 3.4. *Let γ and β be as in Lemma 2.3. Then $\mathcal{F}_{\gamma,\beta}$ is a bounded operator on $\mathcal{A}_n^{(2)}$ and hence it is continuous on $\mathcal{A}_n^{(2)}$. Furthermore, if* $\mathrm{Re}(1/\gamma^2) = 0$ *and* $|\beta| = 1$, *then $\mathcal{F}_{\gamma,\beta}$ is injective from $\mathcal{A}_n^{(2)}$ into $\mathcal{A}_n^{(2)}$.*

Proof. We first note that

$$\|\mathcal{F}_{\gamma,\beta}\|_0 = \sup_{\|F\|_{\mathcal{A}_n^{(2)}}=1} \|\mathcal{F}_{\gamma,\beta}(F)\|_{\mathcal{A}_n^{(2)}} \leq \sup_{\|F\|_{\mathcal{A}_n^{(2)}}=1} \|F\|_{\mathcal{A}_n^{(2)}} = 1, \tag{3.13}$$

where $\|T\|_0$ is the operator norm of an operator T. Hence $\mathcal{F}_{\gamma,\beta}$ is bounded and so it is continuous. Furthermore if $\mathrm{Re}(1/\gamma^2) = 0$ and $|\beta| = 1$, then (3.3) tells us that $\mathcal{F}_{\gamma,\beta}$ preserves the norm and hence it is injective from $\mathcal{A}_n^{(2)}$ into $\mathcal{A}_n^{(2)}$. So we complete the proof of Theorem 3.4. □

The following corollary follows from Theorem 3.4 and some basic properties for bounded linear operators on Hilbert space.

Corollary 3.5. *Let γ and β be as in Theorem 3.4. Then we have the following assertions.*

(1) *The null space $\mathcal{N}(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is closed.*

(2) *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{A}_n^{(2)}$ with $F_n \to F$ as $n \to \infty$ for some $F \in \mathcal{A}_n^{(2)}$. Then $\mathcal{F}_{\gamma,\beta}(F_n) \to \mathcal{F}_{\gamma,\beta}(F)$ as $n \to \infty$. That is to say, it is closed.*

(3) *For all F and G in $\mathcal{A}_n^{(2)}$, we have the Cauchy-Schwartz inequality*

$$\left|\left(\mathcal{F}_{\gamma,\beta}F,\mathcal{F}_{\gamma,\beta}G\right)_{\mathcal{A}_n^{(2)}}\right| \le \left\|\mathcal{F}_{\gamma,\beta}F\right\|_{\mathcal{A}_n^{(2)}}\left\|\mathcal{F}_{\gamma,\beta}G\right\|_{\mathcal{A}_n^{(2)}} \le \|F\|_{\mathcal{A}_n^{(2)}}\|G\|_{\mathcal{A}_n^{(2)}}. \tag{3.14}$$

Remark 3.6. As mentioned in Section 1, the Fourier-Wiener transform, the modified Fourier-Wiener transform, the Fourier-Feynman transform, and the Gauss transform are also well-defined operators on $\mathcal{A}_n^{(2)}$. In particular, from the definition of analytic Fourier-Feynman transform, it is an injective operator on $\mathcal{A}_n^{(2)}$. Hence all those transforms can be applied to our main results and formulas in this paper. In particular, the authors studied that for $|\beta| = 1$ and $F \in L_2(C_0[0,T])$, the IT $\mathcal{F}_{\gamma,\beta}F$ is an element of $L_2(C_0[0,T])$ and

$$\left\|\mathcal{F}_{\gamma,\beta}F\right\|_2 = \|F\|_2. \tag{3.15}$$

That is to say, the IT is injective [7, 9]. This result is a special case of our result in this paper. In addition, in [3, 4], the authors showed that the Fourier Wiener transform acts as a unitary operator on $L_2(C_0[0,T])$.

We finish this section by stating that the operator $\mathcal{F}_{\gamma,\beta}$ is invertible.

Theorem 3.7. *Let γ and β be as in Theorem 3.4 with $\mathrm{Re}(\beta^2/\gamma^2) \le 0$ and $|\beta| = 1$. Then the inverse operator of the IT $\mathcal{F}_{\gamma,\beta}$ exists and is given by*

$$\mathcal{F}_{\gamma,\beta}^{-1} = \mathcal{F}_{i(\gamma/\beta),1/\beta}. \tag{3.16}$$

Furthermore, the null space $\mathcal{N}(\mathcal{F}_{\gamma,\beta})$ consists of the zero vector only.

Proof. From Theorem 3.4, the operator is continuous from $\mathcal{A}_n^{(2)}$ into $\mathcal{A}_n^{(2)}$. Since $\mathrm{Re}(\beta^2/\gamma^2) \le 0$ and $\mathcal{F}_{\gamma,\beta}(F)$ is in $\mathcal{A}_n^{(2)}$, $\mathcal{F}_{i(\gamma/\beta),1/\beta}(\mathcal{F}_{\gamma,\beta}(F))$ exists and is in $\mathcal{A}_n^{(2)}$. Also, since $\mathrm{Re}(1/\gamma^2) \ge 0$ and $\mathcal{F}_{i(\gamma/\beta),1/\beta}(F)$ is in $\mathcal{A}_n^{(2)}$, $\mathcal{F}_{\gamma,\beta}(\mathcal{F}_{i(\gamma/\beta),1/\beta}F)$ exists and is in $\mathcal{A}_n^{(2)}$. Now, using (2.27), for $F \in \mathcal{A}_n^{(2)}$,

$$\mathcal{F}_{i(\gamma/\beta),1/\beta}\left(\mathcal{F}_{\gamma,\beta}(F)\right)(x) = F(x) = \mathcal{F}_{\gamma,\beta}\left(\mathcal{F}_{i(\gamma/\beta),1/\beta}F\right)(x), \tag{3.17}$$

which completes the proof of Theorem 3.7 as desired. □

We have some observations for the inverse operator $\mathcal{F}_{i(\gamma/\beta),1/\beta}$ of $\mathcal{F}_{\gamma,\beta}$.

Remark 3.8. (1) If $\gamma = \sqrt{2}$ and $\beta = i$, then γ and β always satisfy the hypotheses of Theorems 3.2 and 3.4. In fact, there are many pairs (γ,β) satisfying the hypotheses of Theorems 3.2 and 3.4.

(2) The operator $\mathcal{F}_{\gamma,\beta}$ might not be bijective. Hence we should consider that the domain of the inverse operator $\mathcal{F}_{i(\gamma/\beta),1/\beta}$ is the range of $\mathcal{F}_{\gamma,\beta}$.

(3) The operator $\mathcal{F}_{\gamma,\beta}$ is an homeomorphism from $\mathcal{A}_n^{(2)}$ into $\mathcal{R}(\mathcal{F}_{\gamma,\beta})$, where $\mathcal{R}(\mathcal{F}_{\gamma,\beta})$ is the range of $\mathcal{F}_{\gamma,\beta}$.

4. Some Spectral Theorems for the Bounded Linear Operator $\mathcal{F}_{\gamma,\beta}$

In this section we will apply some spectral theories to the $\mathcal{F}_{\gamma,\beta}$. To do this, we need some concepts related to the spectral theory on a Banach space.

With $\mathcal{F}_{\gamma,\beta}$ we associate the operator

$$\mathcal{F}_{\gamma,\beta}^{\lambda} = \mathcal{F}_{\gamma,\beta} - \lambda I, \tag{4.1}$$

where λ is a complex number and I is the identity operator on $\mathcal{A}_n^{(2)}$. If $\mathcal{F}_{\gamma,\beta}^{\lambda}$ has an inverse, we denote it by $R_\lambda(\mathcal{F}_{\gamma,\beta})$; that is,

$$R_\lambda(\mathcal{F}_{\gamma,\beta}) = {\mathcal{F}_{\gamma,\beta}^{\lambda}}^{-1} = (\mathcal{F}_{\gamma,\beta} - \lambda I)^{-1}, \tag{4.2}$$

and call it the resolvent operator of $\mathcal{F}_{\gamma,\beta}$ or, simply, the resolvent of $\mathcal{F}_{\gamma,\beta}$.

Definition 4.1. A regular value λ of $\mathcal{F}_{\gamma,\beta}$ is a complex number such that

(1) $R_\lambda(\mathcal{F}_{\gamma,\beta})$ exists,

(2) $R_\lambda(\mathcal{F}_{\gamma,\beta})$ is bounded,

(3) $R_\lambda(\mathcal{F}_{\gamma,\beta})$ is defined on a set which is dense in $\mathcal{A}_n^{(2)}$.

The resolvent set $\rho(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is the set of all regular values λ of $\mathcal{F}_{\gamma,\beta}$. Its complement $\sigma(\mathcal{F}_{\gamma,\beta}) = \mathbb{C} - \rho(\mathcal{F}_{\gamma,\beta})$ is called the spectrum of $\mathcal{F}_{\gamma,\beta}$, and a $\lambda \in \sigma(\mathcal{F}_{\gamma,\beta})$ is called a spectral value of $\mathcal{F}_{\gamma,\beta}$. For more details, see [16].

Remark 4.2. (1) The spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ is partitioned into three disjoint sets as follows.

(i) The point spectrum or discrete spectrum $\sigma_p(\mathcal{F}_{\gamma,\beta})$ is the set such that $R_\lambda(\mathcal{F}_{\gamma,\beta})$ does not exist. A $\lambda \in \sigma_p(\mathcal{F}_{\gamma,\beta})$ is called an eigenvalue of $\mathcal{F}_{\gamma,\beta}$.

(ii) The continuous spectrum $\sigma_c(\mathcal{F}_{\gamma,\beta})$ is the set such that $R_\lambda(\mathcal{F}_{\gamma,\beta})$ exists and satisfies (3) but not (2) in Definition 4.1; that is to say, $R_\lambda(\mathcal{F}_{\gamma,\beta})$ is unbounded.

(iii) The residual spectrum $\sigma_r(\mathcal{F}_{\gamma,\beta})$ is the set such that $R_\lambda(\mathcal{F}_{\gamma,\beta})$ exists (and it may be bounded or not) but does not satisfy (3) in Definition 4.1. That is to say, the domain of $R_\lambda(\mathcal{F}_{\gamma,\beta})$ is not dense in $\mathcal{A}_n^{(2)}$.

(2) We know that $\mathbb{C} = \rho(\mathcal{F}_{\gamma,\beta}) \cup \sigma(\mathcal{F}_{\gamma,\beta}) = \rho(\mathcal{F}_{\gamma,\beta}) \cup \sigma_p(\mathcal{F}_{\gamma,\beta}) \cup \sigma_c(\mathcal{F}_{\gamma,\beta}) \cup \sigma_r(\mathcal{F}_{\gamma,\beta})$.

From now on, if what operator $\mathcal{F}_{\gamma,\beta}$ refers to is clear, we will write R_λ instead of $R_\lambda(\mathcal{F}_{\gamma,\beta})$.

In our next theorem, we apply the spectral theory to the operator $\mathcal{F}_{\gamma,\beta}$.

Theorem 4.3. *Let γ and β be as in Theorem 3.4. Then the resolvent set $\rho(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is open and hence the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ is closed. Furthermore, for every $\lambda_0 \in \rho(\mathcal{F}_{\gamma,\beta})$, the resolvent R_λ has the representation*

$$R_\lambda = \sum_{j=1}^{\infty} (\lambda - \lambda_0)^j R_{\lambda_0}^{j+1}, \tag{4.3}$$

where the series is absolutely convergent for every λ in the open disk given by

$$|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|_0} \tag{4.4}$$

in the complex plane. This disk is a subset of $\rho(\mathcal{F}_{\gamma,\beta})$.

Proof. Theorem 4.3 immediately follows the fact that the $\mathcal{F}_{\gamma,\beta}$ is a bounded linear operator on $\mathcal{A}_n^{(2)}$. □

Next, we note that the spectral radius $r_\sigma(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is the radius

$$r_\sigma(\mathcal{F}_{\gamma,\beta}) = \sup_{\lambda\in\sigma(\mathcal{F}_{\gamma,\beta})} |\lambda| \tag{4.5}$$

of the smallest closed disk centered at the origin of the complex λ-plane and containing $\sigma(\mathcal{F}_{\gamma,\beta})$.

Theorem 4.4. *Let γ and β be as in Theorem 3.4. Then the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is compact and lies in the disk given by*

$$|\lambda| \leq \|\mathcal{F}_{\gamma,\beta}\|_0. \tag{4.6}$$

Hence the resolvent set $\rho(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is not empty. Furthermore, the spectral radius $r_\sigma(\mathcal{F}_{\gamma,\beta}) \leq \|\mathcal{F}_{\gamma,\beta}\|_0$ and

$$r_\sigma(\mathcal{F}_{\gamma,\beta}) = \lim_{n\to\infty}\left(\|\mathcal{F}_{\gamma,\beta}\|_0^n\right)^{1/n}. \tag{4.7}$$

Proof. From Theorem 3.4, the $\mathcal{F}_{\gamma,\beta}$ is a bounded linear operator on $\mathcal{A}_n^{(2)}$. Using a basic property for the spectrum, we establish (4.6), and hence the resolvent set is not empty. Furthermore, using (4.6), it is obvious that for the spectral radius of a bounded linear operator $\mathcal{F}_{\gamma,\beta}$ we have

$$r_\sigma(\mathcal{F}_{\gamma,\beta}) \leq \|\mathcal{F}_{\gamma,\beta}\|_0. \tag{4.8}$$

Also, we can easily obtain (4.7) as desired. □

In our next theorem, we give a spectral mapping theorem for polynomials of $\mathcal{F}_{\gamma,\beta}$. The proof of Theorem 4.5 is omitted because it immediately follows the spectral mapping theorem for polynomial on a Banach space.

Theorem 4.5. *Let γ and β be as in Theorem 3.4. Let*

$$P(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0, \quad a_n \neq 0,\ n = 1,2,\ldots. \tag{4.9}$$

Then

$$\sigma(P(\mathcal{F}_{\gamma,\beta})) = P(\sigma(\mathcal{F}_{\gamma,\beta})). \tag{4.10}$$

This implies that the spectrum $\sigma(P(\mathcal{F}_{\gamma,\beta}))$ of the operator

$$P(\lambda) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_0 I \tag{4.11}$$

consists precisely of all those values which the polynomial P assumes on the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$.

Next, we will explain that our study is meaningful to obtain the solution to a differential equation.

Let H be a real separable infinite-dimensional Hilbert space with the inner product $\langle \cdot,\cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot,\cdot \rangle}$. Let $\|\cdot\|_0$ be a measurable norm on H with respect to the Gaussian cylinder set measure ν_0 on H. Let B denote the completion of H with respect to $\|\cdot\|_0$. Let i denote the natural injection from H to B. The adjoint operator i^* of i is one to one and maps B^* continuously onto a dense subset H^*, where B^* and H^* are topological duals of B and H, respectively. By identifying H^* with H and B^* with i^*B^*, we have a triple $B^* \subset H^* \approx H \subset B$ with $\langle x, y \rangle = (x, y)^\sim$ for all x in H and y in B^*, where $(\cdot,\cdot)^\sim$ denotes the natural dual pairing between B and B^*. By a well-known result of Gross [17], $\nu_0 \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (B, H, ν) is called an abstract Wiener space. The classical Wiener space $C_0[0,T]$ is one of the examples of abstract Wiener space.

For an appropriate functional $u(x)$ on B, let N_c be an operator defined by the formula

$$N_c u(x) = -\mathrm{Tr}_H D^2 u(x) + c(x, Du(x))^\sim, \quad x \in B,\ c \in \mathbb{C}/\{0\}, \tag{4.12}$$

where D denotes the second Fréchet derivative and Tr_H denotes the trace of an operator. In [2], Lee showed that the integral transform $\mathcal{F}_{1/c,i}$, $c \in \mathbb{C}/\{0\}$ forms the solution of a differential equation which is called a Cauchy problem

$$\begin{aligned} u_t(x,t) &= \mathcal{P}(N_c) u(x,t), \quad x \in B,\ t > 0 \\ u(x,0) &= F(x), \end{aligned} \tag{4.13}$$

where $\mathcal{P}(\eta) = a_m \eta^m + \cdots + a_1 \eta + a_0$ is an m-dimensional polynomial function with respect to η. In addition, let $\mathcal{P} = -\eta$ and $c = 1$ in (4.13). Then the solution of the Cauchy problem is given by formula

$$u(x,t) = \int_B F\left(e^{-t} x + \sqrt{1 - e^{-2t}}\, y\right) d\nu(y), \tag{4.14}$$

or, equivalently,

$$u(x,t) = \int_B F(y) o_t(x, dy), \tag{4.15}$$

where $o_t(x,dy) = \nu_{1-e^{-2t}}(e^{-t}x,dy)$. This showed that the family of measures $\{o_t(x,dy)\}$ serves as the "fundamental solution" of the operator $\partial/\partial t + N_1$. For more details see [2, 18]. Hence our discussions as a bounded linear operator in this paper have some meaningful subjects. That is to say, from Theorems 3.2 through 4.5, if we take a complex number c such that $\mathrm{Re}(c^2) \geq 0$, then the transform $\mathcal{F}_{1/c,i}$ is a well-defined bounded linear operator on $\mathcal{A}_n^{(2)}$ and could be applied to all the results and formulas in this paper.

5. Applications for the Spectral Theory

In Sections 3 and 4, we treated the IT $\mathcal{F}_{\gamma,\beta}$ as a bounded linear operator on $\mathcal{A}_n^{(2)}$. Also, we applied spectral theorems to the IT to obtain various useful formulas and results. In this section we will show that the operator $\mathcal{F}_{\gamma,\beta}$ is self-adjoint under appropriate parameters γ and β. We then apply the spectral theory to a self-adjoint operator on a Banach space. In particular, we obtain the spectral representation for IT $\mathcal{F}_{\gamma,\beta}$.

In our next theorem, we show that the operator $\mathcal{F}_{\gamma,\beta}$ is a self-adjoint operator on $\mathcal{A}_n^{(2)}$ under an appropriate condition for γ and β.

Theorem 5.1. *Let γ be as in Theorem 3.4 with $\overline{\gamma^2} = \gamma^2$ and let $\beta = 1$. Then the operator $\mathcal{F}_{\gamma,\beta}$ is a self-adjoint operator on $\mathcal{A}_n^{(2)}$.*

Proof. Let T^* denote the adjoint operator on an operator T. For all F and G in $\mathcal{A}_n^{(2)}$, we note that

$$
\begin{aligned}
\left(F, \mathcal{F}^*_{\gamma,\beta}G\right)_{\mathcal{A}_n^{(2)}} &= \left(\mathcal{F}_{\gamma,\beta}F, G\right)_{\mathcal{A}_n^{(2)}} \\
&= \int_{\mathbb{R}^n} \Gamma_{\mathcal{F}_{\gamma,\beta}F}(\vec{v})\overline{g(\vec{v})}d\vec{v} \\
&= \left(\prod_{j=1}^{n} 2\pi\gamma^2\right)^{-1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\sum_{j=1}^{n}\frac{\left(u_j - v_j\right)^2}{2\gamma^2}\right\} d\vec{u}\overline{g(\vec{v})}d\vec{v} \\
&= \left(\prod_{j=1}^{n} 2\pi\gamma^2\right)^{-1} \int_{\mathbb{R}^n} f(\vec{u}) \int_{\mathbb{R}^n} \overline{g(\vec{v})} \exp\left\{-\sum_{j=1}^{n}\frac{\left(v_j - u_j\right)^2}{2\gamma^2}\right\} d\vec{v}\, d\vec{u} \\
&= \overline{\left(\prod_{j=1}^{n} 2\pi\gamma^2\right)^{-1}} \int_{\mathbb{R}^n} f(\vec{u}) \overline{\int_{\mathbb{R}^n} g(\vec{v}) \exp\left\{-\sum_{j=1}^{n}\frac{\left(v_j - u_j\right)^2}{2\gamma^2}\right\} d\vec{v}}\, d\vec{u} \\
&= \left(F, \mathcal{F}_{\gamma,\beta}G\right)_{\mathcal{A}_n^{(2)}},
\end{aligned}
\tag{5.1}
$$

which completes the proof of Theorem 5.1 as desired. □

Remark 5.2. We gave the conditions for γ and β in Sections 2 and 3, and Theorem 5.1. We note that these conditions imply that γ is real and $\beta = 1$ only. But, a self-adjoint operator may not have eigenvalues. So if it has eigenvalues, then it must be real. Hence these are very natural conditions.

Throughout the next corollary, we give some results of the spectral theories for self-adjoint operator $\mathcal{F}_{\gamma,\beta}$.

Corollary 5.3. *Let γ and β be as in Theorem 5.1. Then we have the following assertions.*

(1) *For $\lambda \in \sigma(\mathcal{F}_{\gamma,\beta})$, there exists a positive real number k_0 such that*

$$\left\| (\mathcal{F}_{\gamma,\beta} - \lambda I)F \right\|_{\mathcal{A}_n^{(2)}} \geq k_0 \|F\|_{\mathcal{A}_n^{(2)}} \tag{5.2}$$

and so there exists a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{A}_n^{(2)}$ with $\|F_n\|_{\mathcal{A}_n^{(2)}} = 1$, $n = 1, 2, \ldots$ such that

$$\lim_{n\to\infty} \left\| (\mathcal{F}_{\gamma,\beta} - \lambda I)F_n \right\|_{\mathcal{A}_n^{(2)}} = 0. \tag{5.3}$$

(2) *For all $F \in \mathcal{A}_n^{(2)}$ and $\lambda \in \mathbb{C}$,*

$$\left\| (\mathcal{F}_{\gamma,\beta} - \lambda I)F \right\|_{\mathcal{A}_n^{(2)}} \geq |\mathrm{Im}\,\lambda| \|F\|_{\mathcal{A}_n^{(2)}}. \tag{5.4}$$

In our next theorem, we apply the spectral theory to the operator $\mathcal{F}_{\gamma,\beta}$ as a self-adjoint operator.

Theorem 5.4. *Let γ and β be as in Theorem 5.1. Then the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ is real and it lies in the closed interval $[k_1, k_2]$ where*

$$k_1 = \inf_{\|F\|_{\mathcal{A}_n^{(2)}}=1} (\mathcal{F}_{\gamma,\beta}(F), F)_{\mathcal{A}_n^{(2)}}, \tag{5.5}$$

$$k_2 = \sup_{\|F\|_{\mathcal{A}_n^{(2)}}=1} (\mathcal{F}_{\gamma,\beta}(F), F)_{\mathcal{A}_n^{(2)}}. \tag{5.6}$$

Furthermore, k_1 and k_2 are spectral values of $\mathcal{F}_{\gamma,\beta}$ and $\max\{|k_1|, |k_2|\} = 1$.

Proof. First, since $\mathcal{F}_{\gamma,\beta}$ is a self-adjoint operator on $\mathcal{A}_n^{(2)}$, the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ of $\mathcal{F}_{\gamma,\beta}$ must be real. Next we recall that for each bounded self-adjoint operator $T : H \to H$ on a complex Hilbert space H, $\sigma(T) \subset [m, M]$ on the real axis and $\|T\|_0 = \max\{|m|, |M|\}$, where

$$m = \inf_{\|h\|_H=1} (Th, h)_H, \qquad M = \sup_{\|h\|_H=1} (Th, h)_H, \tag{5.7}$$

and $(\cdot,\cdot)_H$ is an inner product on H. Hence the spectrum $\sigma(\mathcal{F}_{\gamma,\beta})$ lies in the closed interval $[k_1, k_2]$. Furthermore, k_1 and k_2 are spectral values of $\mathcal{F}_{\gamma,\beta}$ and $\max\{|k_1|, |k_2|\} = \|\mathcal{F}_{\gamma,\beta}\|_0 = 1$. Hence we complete the proof of Theorem 5.4. □

We finish this paper by giving an application for the spectral representation of the self-adjoint operator which is one of very important subjects in the fields of the quantum mechanics and physical theories.

In our last theorem, we give the spectral representation for the self-adjoint operator. To do this, we need some concepts for the spectral theory.

Let $(\lambda_n)_{n=1}^{\infty}$ be the set of eigenvalues of $\mathcal{F}_{\gamma,\beta}$ with $\lambda_n < \lambda_m$ for $m < n$ and let $(e_n)_{n=1}^{\infty}$ be the set of eigenfunctions corresponding to $(\lambda_n)_{n=1}^{\infty}$. Then we note that

$$\mathcal{F}_{\gamma,\beta}F(x) = \sum_{n=1}^{\infty} a_n \lambda_n e_n(x), \tag{5.8}$$

where $a_n = (F, e_n)_{\mathcal{A}_n^{(2)}}$. For each $j = 1, 2, \ldots$, define an (orthogonal) projection P_j on $\mathcal{A}_n^{(2)}$ by $P_j F(x) = a_j e_j(x)$. Then we also note that

$$\mathcal{F}_{\gamma,\beta}F(x) = \sum_{n=1}^{\infty} \lambda_n P_n F(x). \tag{5.9}$$

Now, for $\lambda \in \mathbb{R}$, define an operator E_λ on $\mathcal{A}_n^{(2)}$ by $E_\lambda F(x) = \sum_{\lambda_j \leq \lambda} P_j F(x)$. In this case, $\mathcal{E} \equiv (E_\lambda)$ is called the spectral family of $\mathcal{F}_{\gamma,\beta}$.

Theorem 5.5. *Let γ and β be as in Theorem 5.1. Then we have the spectral representation for $\mathcal{F}_{\gamma,\beta}$ as follows:*

$$\mathcal{F}_{\gamma,\beta} = \int_{k_1}^{k_2} \lambda \, dE_\lambda, \tag{5.10}$$

where k_1 and k_2 are given by (5.5) *and* (5.6), *and $\{E_\lambda\}$ is the spectral family of $\mathcal{F}_{\gamma,\beta}$.*

Proof. First, we note that for γ and β in Theorem 5.1, $\mathcal{F}_{\gamma,\beta}$ is a self-adjoint operator on $\mathcal{A}_n^{(2)}$. Furthermore, $\sigma(\mathcal{F}_{\gamma,\beta}) = [k_1, k_2] \subset [-1, 1]$. Using the spectral representation of the self-adjoint operator, we establish (5.10). □

Remark 5.6. (1) In view of Theorem 5.5, for all real-valued continuous functions f on $[k_1, k_2]$ and for F and G in $\mathcal{A}_n^{(2)}$,

$$\begin{aligned} f(\mathcal{F}_{\gamma,\beta}) &= \int_{k_1}^{k_2} f(\lambda) dE_\lambda, \\ (\mathcal{F}_{\gamma,\beta}F, G)_{\mathcal{A}_n^{(2)}} &= \int_{k_1}^{k_2} \lambda \, dw(\lambda), \end{aligned} \tag{5.11}$$

where $w(\lambda) = (E_\lambda F, G)_{\mathcal{A}_n^{(2)}}$ and the integral is an ordinary Riemann-Stieltjes integral.

(2) An alternative formulation of the spectral theorem expresses the operator $\mathcal{F}_{\gamma,\beta}$ as an integral of the coordinate function over the operator's spectrum with respect to a projection-valued measure $\mathcal{F}_{\gamma,\beta} = \int_{\sigma(\mathcal{F}_{\gamma,\beta})} \lambda \, dE_\lambda$. When the normal operator in question is compact, this version of the spectral theorem reduces to the finite-dimensional spectral theorem, except that the operator is expressed as a linear combination of possibly infinitely many projections.

(3) In Sections 3, 4, and 5, we considered the IT as an operator. Like this, we expect that the convolution product could be dealt with as an operator. Furthermore, we could obtain various relations between the IT and the convolution product as a composition of operators.

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