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Research Article

Variable Exponent Spaces of Differential Forms on Riemannian Manifold

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We introduce the Lebesgue space and the exterior Sobolev space for differential forms on Riemannian manifold M which are the Lebesgue space and the Sobolev space of functions on M, respectively, when the degree of differential forms to be zero. After discussing the properties of these spaces, we obtain the existence and uniqueness of weak solution for Dirichlet problems of nonhomogeneous $p(m)$-harmonic equations with variable growth in $W_0^{1,p(m)}(\Lambda^k M)$.

1. Introduction

Gol'dshteĭn et al. introduced spaces of differential forms on Riemannian manifold in [1–3]. The study of spaces for differential forms has been developed rapidly. For example, L_p-Cohomology and $L_{p,q}$-Cohomology and applications to some nonlinear PDE were studied in [4–6]; L^p Hodge decomposition theory on the compact and complete Riemannian manifold were discussed in [7, 8]; properties of Riesz transforms of differential forms on complete Riemannian manifold were discussed in [9, 10]; the existence of minima of certain mean-coercive functionals is established in [11]. Many interesting results concerning A-harmonic equations have been established recently (see [12, 13] and the references therein).

After Kováčik and Rákosník first discussed the $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ spaces in [14], a lot of research has been done concerning these kinds of variable exponent spaces (see [15–19] and the references therein). The existence and uniqueness of solutions for $p(x)$-Laplacian Dirichlet problems with different types on bounded domains in $\mathbb{R}^n$ have been greatly discussed under various conditions (see [20] for the existence and [21] for the uniqueness). In recent years, the theory on problems with variable exponential growth conditions has important applications in nonlinear elastic mechanics (see [22]), electrorheological fluids (see [23, 24]).

The paper is organized as follows. In Section 2, we give the necessary definitions and some elementary properties of differential forms on Riemannian manifold. Moreover, we introduce the functional $\rho_{p(m),\Lambda^k M}$ on $\Lambda^k M$ and the spaces of differential forms $L^{p(m)}(\Lambda^k M)$ and $W^{1,p(m)}(\Lambda^k M)$, then discuss some important properties. In Section 3, we show the existence and uniqueness of weak solution for Dirichlet problems of nonhomogeneous $p(m)$-harmonic equations with variable growth in $W_0^{1,p(m)}(\Lambda^k M)$.

2. Preliminaries

Let M be an arbitrary smooth n-dimensional manifold (Hausdorff and with countable basis). Let $T^*M = \cup_{m\in M} T_m^* M$ be the cotangent bundle on M and $\Lambda^k T^*M$ (or $\Lambda^k M$) be the bundles of the exterior k-forms. We will call each fiber u of the bundle $\Lambda^k T^* M$ a exterior form of degree k on the manifold M. Here, $\Lambda^0 M = \mathbb{R}$ and $\Lambda^k M = \{0\}$ in the case $k > n$ or $k < 0$. Given a exterior k-form $u(m)$ and a local chart $f_\alpha : U_\alpha(\subset M) \to \mathbb{R}^n$, around $m \in U_\alpha$, we define the representation of $u(m)$ in this local coordinates system as the exterior k-forms u_α on $f_\alpha(U_\alpha) \subset \mathbb{R}^n$ given by

$$\begin{aligned} u_\alpha(f_\alpha(m))(X_1, X_2, \ldots, X_k) &= \left(\left(f_\alpha^{-1}\right)^* u\right)(f_\alpha(m))(X_1, X_2, \ldots, X_k) \\ &= u(m)\left(df_\alpha^{-1}(X_1), df_\alpha^{-1}(X_2), \ldots, df_\alpha^{-1}(X_k)\right), \end{aligned} \tag{2.1}$$

for any $X_1, X_2, \ldots, X_k \in \mathbb{R}^n$, where df_α^{-1} is the induced map by f_α^{-1} that takes vectors on $\mathrm{T}_{f_\alpha(m)}\mathbb{R}^n$ into vectors on $T_m M$ and $(f_\alpha^{-1})^*$ is the induced map by f_α^{-1} that takes exterior forms on $T_m M$ into exterior forms on $T_{f_\alpha(m)}\mathbb{R}^n$ (see [25]).

In this paper we will always assume (M, g) is an n-dimensional smooth orientable complete Riemannian manifold and $d\mu = \sqrt{\det(g_{ij})}dx$ is the Riemannian volume element on (M, g), where the g_{ij} are the components of the Riemannian metric g in the chart and dx is the Lebesgue volume element of $\mathbb{R}^n$. A Riemannian metric g on M induces a scalar product on each fiber of the bundle $\Lambda^k M$. Hence for any exterior forms u and v of the same degree k, the scalar product $\langle u, v\rangle = \langle u(m), v(m)\rangle$ is defined at each point $m \in M$ and the norm of u is given by the formula $|u| = \sqrt{\langle u, u\rangle}$. Let $\gamma : [\mathrm{a}, b] \to M$ be a curve of class C^1, the length of γ is

$$L(\gamma) = \int_a^b \sqrt{g(\gamma(t))\left(\left(\frac{d\gamma}{dt}\right)(t), \left(\frac{d\gamma}{dt}\right)(t)\right)}d\mu. \tag{2.2}$$

For $m_1, m_2 \in M$, let $C^1_{m_1,m_2}$ be the space of piecewise C^1 curves $\gamma : [a, b] \to M$ such that $\gamma(a) = m_1$ and $\gamma(b) = m_2$. One can define a distance $d_g(m_1, m_2) = \inf_{C^1_{m_1,m_2}} L(\gamma)$ on M.

The Grassman algebra $\Lambda^* M = \oplus \Lambda^k M$ is a graded algebra with respect to the exterior products. We denote by $L^1_{\mathrm{loc}}(\Lambda^k M)$ the space of locally integrable exterior forms of degree k (i.e., differential k-forms) on M. The local integrability of an exterior k-form means the local integrability of the components of its coordinate representation in each chart of the Riemannian manifold M. We denote by $C_c^\infty(\Lambda^k M)$ the vector space of smooth differential forms of degree k with compact support on M.

Let (M, g) be is an n-dimensional smooth orientable Riemannian manifold. We define the integral of u, a exterior n-form u with compact support on M (see [26]). Let (U_α, f_α) be a local chart of (M, g), we have a partition of unity $\{\pi_\alpha\}$ subordinate to this cover. Recall that $\operatorname{supp}(\pi_\alpha) \subseteq U_\alpha$ and $\sum_\alpha \pi_\alpha = 1$. Thus, every $\pi_\alpha u$ is an exterior n-form whose support is a subset of U_α and we may write $u = \sum_\alpha \pi_\alpha u$. By definition

$$\int_M u = \sum_\alpha \int_{U_\alpha} \pi_\alpha u = \sum_\alpha \int_{f_\alpha(U_\alpha)} \left(f_\alpha^{-1}\right)^* (\pi_\alpha u) = \sum_\alpha \int_{f_\alpha(U_\alpha)} \left(\sqrt{\det(g_{ij})} \pi_\alpha u\right) \circ f_\alpha^{-1} dx. \tag{2.3}$$

We will identify each exterior form of degree k on the n-dimensional Riemannian manifold M with an exterior $(n-k)$-form on M (see [27]). Using this identification, we can assume that each exterior form u has a weak exterior differential du.

Definition 2.1 (see [6]). We say that an exterior form $v \in L^1_{\mathrm{loc}}(\Lambda^k M)$ is the weak exterior differential of a form $u \in L^1_{\mathrm{loc}}(\Lambda^{k-1} M)$ and we write $du = v$ if for each $\varphi \in C_c^\infty(\Lambda^k M)$, one has

$$\int_M v \wedge \varphi = (-1)^k \int_M u \wedge d\varphi. \tag{2.4}$$

The operator $\star : \Lambda^k M \to \Lambda^{n-k} M$, also called Hodge star operator (see [27]), has the following properties: for $u, v \in \Lambda^k M$ and $\varphi, \psi \in C^\infty(M)$

(a_1) $\star(\varphi u + \psi v) = \varphi \star u + \psi \star v$,

(a_2) $\star \star u = (-1)^{k(n-k)} u$,

(a_3) $\star\varphi = \varphi d\mu$,

(a_4) $\langle u, v\rangle = \star(u \wedge \star v) = \langle \star u, \star v\rangle$,

(a_5) $u \wedge \star v = \langle u, v\rangle d\mu$.

By the operator $\star$ and the exterior differentiation d we define the codifferential operator δ by the formula

$$\delta u = (-1)^{n(k+1)+1} \star d \star u \in L^1_{\mathrm{loc}}\left(\Lambda^{k-1} M\right), \tag{2.5}$$

for any differential form $u \in L^1_{\mathrm{loc}}(\Lambda^k M)$.

The Riemannian measure and the characteristic function of a set $A \subseteq M$ will be denoted by $\mu(A)$ and χ_A, respectively.

Let $\mathcal{P}(M)$ be the set of all measurable functions $p : M \to [1, \infty]$. For $p \in \mathcal{P}(M)$ we put $M_1 = M_1^p = \{m \in M : p(m) = 1\}$, $M_\infty = M_\infty^p = \{m \in M : p(m) = \infty\}$, $M_0 = M \setminus (M_1 \cup M_\infty)$, $p_* = \operatorname{essinf}_{M_0} p(m)$ and $p^* = \operatorname{esssup}_{M_0} p(m)$ if $\mu(M_0) > 0$, $p_* = p^* = 1$ if $\mu(M_0) = 0$, $c_p = \|\chi_{M_0}\|_{L^\infty(M)} + \|\chi_{M_1}\|_{L^\infty(M)} + \|\chi_{M_\infty}\|_{L^\infty(M)}$ and $r_p = c_p + 1/p_* + 1/p^*$. We always assume that $p \in \mathcal{P}(M)$, $\mathcal{P}_1(M) = \mathcal{P}(M) \cap L^\infty(M)$ and $\mathcal{P}_2(M) = \{p \in \mathcal{P}_1(M) : 1 < \operatorname{essinf}_M p(m)\}$. We use the convention $1/\infty = 0$.

For a differential k-form u on M we define the functional $\rho_{p(m),\Lambda^k M}$ by

$$\rho_{p(m),\Lambda^k M}(u) = \int_{M \setminus M_\infty} |u|^{p(m)} d\mu + \operatorname{esssup}_{M_\infty} |u|. \tag{2.6}$$

The Lebesgue space $L^{p(m)}(\Lambda^k M)$ is the space of differential forms u in $L^1_{\mathrm{loc}}(\Lambda^k M)$ such that

$$\rho_{p(m),\Lambda^k M}(\lambda u) < \infty \quad \text{for some } \lambda = \lambda(u) > 0, \tag{2.7}$$

with the following norm

$$\|u\|_{L^{p(m)}(\Lambda^k M)} = \inf\left\{\lambda > 0 : \rho_{p(m),\Lambda^k M}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.8}$$

The exterior Sobolev space $W^{1,p(m)}(\Lambda^k M)$ consists of such forms $u \in L^{p(m)}(\Lambda^k M)$ for which $du \in L^{p(m)}(\Lambda^{k+1} M)$. The norm is defined by

$$\|u\|_{W^{1,p(m)}(\Lambda^k M)} = \|u\|_{L^{p(m)}(\Lambda^k M)} + \|du\|_{L^{p(m)}(\Lambda^{k+1} M)}. \tag{2.9}$$

The space $W_0^{1,p(m)}(\Lambda^k M)$ is defined as the closure of $C_c^\infty(\Lambda^k M)$ in $W^{1,p(m)}(\Lambda^k M)$.

Note that $L^{p(m)}(\Lambda^0 M)$, $W^{1,p(m)}(\Lambda^0 M)$ and $W_0^{1,p(m)}(\Lambda^0 M)$ are spaces of functions on M. In this paper we denote them by $L^{p(m)}(M)$, $W^{1,p(m)}(M)$ and $W_0^{1,p(m)}(M)$.

Given $p \in \mathcal{P}(M)$ we define the conjugate function $p'(m) \in \mathcal{P}(M)$ by

$$p'(m) = \begin{cases} \infty & \text{if } m \in M_1, \\ 1 & \text{if } m \in M_\infty, \\ \dfrac{p(m)}{p(m) - 1} & \text{if } m \in M_0. \end{cases} \tag{2.10}$$

Similar to the proof of properties of $\rho_{p(m),\Omega}$ and $L^{p(m)}(\Omega)$ for $\Omega \subset \mathbb{R}^n$ (see [15, 16, 18]), it is easy to see that $\rho_{p(m),\Lambda^k M}$ and $L^{p(m)}(\Lambda^k M)$ has the following properties:

(b_1) $\rho_{p(m),\Lambda^k M}$ is convex.

(b_2) $\rho_{p(m),\Lambda^k M}(u\chi_A) \le \rho_{p(m),\Lambda^k M}(u)$ for every subset $A \subset M$ and differential forms u.

(b_3) If $|u(m)| \ge |v(m)|$ for a.e. $m \in M$ and if $\rho_{p(m),\Lambda^k M}(u) < \infty$, then $\rho_{p(m),\Lambda^k M}(u) \ge \rho_{p(m),\Lambda^k M}(v)$, the last inequality is strict if $|u| \ne |v|$.

(b_4) If $0 < \rho_{p(m),\Lambda^k M}(u) < \infty$, then the function $\lambda \to \rho_{p(m),\Lambda^k M}(u/\lambda)$ is continuous and decreasing on the interval $[1, \infty)$.

(b_5) If $0 < \|u\|_{L^{p(m)}(\Lambda^k M)} < \infty$, then $\rho_{p(m),\Lambda^k M}(u/\|u\|_{L^{p(m)}(\Lambda^k M)}) \le 1$.

(b_6) If $p^* < \infty$, then $\rho_{p(m),\Lambda^k M}(u/\|u\|_{L^{p(m)}(\Lambda^k M)}) = 1$ for every differential forms u with $0 < \|u\|_{L^{p(m)}(\Lambda^k M)} < \infty$.

(b_7) If $\|u\|_{L^{p(m)}(\Lambda^k M)} \le 1$, then $\rho_{p(m),\Lambda^k M}(u) \le \|u\|_{L^{p(m)}(\Lambda^k M)}$.

(b_8) If $p \in \mathcal{P}_1(M)$ and $\|u\|_{L^{p(m)}(\Lambda^k M)} > 1$, then

$$\|u\|^{p_*}_{L^{p(m)}(\Lambda^k M)} \le \rho_{p(m),\Lambda^k M}(u) \le \|u\|^{p^*}_{L^{p(m)}(\Lambda^k M)}. \tag{2.11}$$

(b_9) If $p \in \mathcal{P}_1(M)$ and $\|u\|_{L^{p(m)}(\Lambda^k M)} < 1$, then

$$\|u\|^{p_*}_{L^{p(m)}(\Lambda^k M)} \geq \rho_{p(m),\Lambda^k M}(u) \geq \|u\|^{p^*}_{L^{p(m)}(\Lambda^k M)}. \tag{2.12}$$

Lemma 2.2. *If $p(m) \in \mathcal{P}(M)$, then the inequality*

$$\int_M |\langle u, v\rangle| d\mu \leq r_p \|u\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^k M)} \tag{2.13}$$

holds for every $u \in L^{p(m)}(\Lambda^k M)$, $v \in L^{p'(m)}(\Lambda^k M)$.

Proof. Obviously, we can suppose that $\|u\|_{L^{p(m)}(\Lambda^k M)} \neq 0$, $\|v\|_{L^{p'(m)}(\Lambda^k M)} \neq 0$ and $\mu(M_0) > 0$. We have

$$1 < p(m) < \infty, \quad |u(m)| < \infty, \quad |v(m)| < \infty \quad \text{a.e. } m \in M_0. \tag{2.14}$$

By Young inequality, we have

$$\frac{|\langle u, v\rangle|}{\|u\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^k M)}} \leq \frac{1}{p(m)} \left(\frac{|u|}{\|u\|_{L^{p(m)}(\Lambda^k M)}} \right)^{p(m)} + \frac{1}{p'(m)} \left(\frac{|v|}{\|v\|_{L^{p'(m)}(\Lambda^k M)}} \right)^{p'(m)}. \tag{2.15}$$

Integrating over M_0 we obtain

$$\begin{aligned} &\int_{M_0} \frac{|\langle u, v\rangle|}{\|v\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^k M)}} d\mu \\ &\quad \leq \frac{1}{p_*} \int_{M_0} \left(\frac{|u|}{\|u\|_{L^{p(m)}(\Lambda^k M)}} \right)^{p(x)} d\mu + \left(1 - \frac{1}{p^*}\right) \int_{M_0} \left(\frac{|v|}{\|v\|_{L^{p'(m)}(\Lambda^k M)}} \right)^{p'(m)} d\mu \\ &\quad \leq 1 + \frac{1}{p_*} - \frac{1}{p^*}. \end{aligned} \tag{2.16}$$

Then by (b_2), we have

$$\begin{aligned} \int_M |\langle u, v\rangle| d\mu \leq{}& \left(1 + \frac{1}{p_*} - \frac{1}{p^*}\right) \|u\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^k M)} \left\|\chi_{M_0}\right\|_{L^\infty(M)} \\ &+ \left\|u\chi_{M_1}\right\|_{L^1(\Lambda^k M)} \left\|v\chi_{M_1}\right\|_{L^\infty(\Lambda^k M)} + \left\|u\chi_{M_\infty}\right\|_{L^\infty(\Lambda^k M)} \left\|v\chi_{M_\infty}\right\|_{L^1(\Lambda^k M)} \\ \leq{}& r_p \|u\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^k M)}, \end{aligned} \tag{2.17}$$

□

For differential k-forms u on M, we define

$$|||u|||_{L^{p(m)}(\Lambda^k M)} = \sup_{\rho_{p'(m),\Lambda^{n-k}M}(v)\le 1} \int_M u \wedge v. \tag{2.18}$$

We denote by $\Lambda(k,n)$ the set of ordered multi-indices $(i_1, i_2, \dots, i_k)$ of integers $1 \le i_1 < i_2 < \cdots < i_k \le n$. Let $I = (i_1, i_2, \dots, i_k)$ be a multi-index from $\Lambda(k,n)$. The complement I^* of the multi-index I is the multi-index $I^* = (i_{k+1}, i_{k+2}, \dots, i_n)$ in $\Lambda(n-k,n)$ where the components i_l are in $\{1,\dots,n\} \setminus \{i_1, i_2, \dots, i_k\}$ for all $l = k+1, \dots, n$.

Let $x^1, \dots, x^n$ be the orientable coordinates on M. Each differential k-form u can be written as the linear combination

$$u = \sum_{1\le i_1<\cdots<i_k\le n} u_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \sum_{I\in\Lambda(k,n)} u_I dx^I. \tag{2.19}$$

Here u_I are the components of u with respect to natural basis

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad I = (i_1, i_2, \dots, i_k) \in \Lambda(k,n). \tag{2.20}$$

For a differential $(n-k)$-form $v = \sum_{L\in\Lambda(k,n)} v_{L^*} dx^{L^*}$, we have

$$u \wedge v = (-1)^{k(n-k)} u \wedge \star\star v = (-1)^{k(n-k)} \langle u, \star v\rangle d\mu = \langle \star u, v\rangle d\mu. \tag{2.21}$$

Note that $\star dx^I = \sqrt{\det(g_{ij})} \sum_{J\in\Lambda(k,n)} \prod_{\gamma=1}^k g^{i_\gamma j_\gamma} \sigma(J) dx^{J^*}$, and hence

$$\langle \star u, v\rangle = \sqrt{\det(g_{ij})} \sum_{I,J,L\in\Lambda(k,n)} \prod_{\gamma=1}^k g^{i_\gamma j_\gamma} \prod_{\beta=k+1}^n g^{j_\beta l_\beta} \sigma(J) u_I v_{L^*} \quad \text{on } M, \tag{2.22}$$

where g^{ij} are the components of the inverse matrix of (g_{ij}) and $\sigma(J)$ is the signature of the permutation $(j_1 \cdots j_n)$ in the set $\{1 \cdots n\}$.

We consider an arbitrary local chart $f : V(\subset M) \to \mathbb{R}^n$ on M. Let U be any open set in M, whose closure is compact and is contained in V. Note that the components g_{ij} of g in (U, f) satisfy $1/2\delta_{ij} \le g_{ij} \le 2\delta_{ij}$ as bilinear forms. Then

$$\langle \star u, v\rangle = \sqrt{\prod_{l=1}^n g^{ll}} \sum_{I\in\Lambda(k,n)} \sigma(I) u_I v_{I^*} \quad \text{on } M. \tag{2.23}$$

Thus, if $\operatorname{sgn} v_{I^*} = \sigma(I) \operatorname{sgn} u_I$, $\omega = \sum_{I\in\Lambda(k,n)} \omega_{I^*} dx^{I^*}$ with $\rho_{p'(m),\Lambda^{n-k}M}(\omega) \le 1$ and $\omega_{I^*} = \pm v_{I^*}$, we have

$$\langle \star u, \omega\rangle \le \langle \star u, v\rangle, \quad 2^{-n/2} \sum_I |u_I||v_{I^*}| \le \langle \star u, v\rangle \le 2^{n/2} \sum_I |u_I||v_{I^*}| \quad \text{on } M. \tag{2.24}$$

Integrating on K and M, by (2.18) we have

$$0 \leq \left\| |u\chi_K| \right\|_{L^{p(m)}(\Lambda^k M)} \leq \||u\||_{L^{p(m)}(\Lambda^k M)}, \tag{2.25}$$

for any compact subset K on M. Furthermore, It is easy to see that it is a norm on the class of differential k-forms u with $\||u\||_{L^{p(m)}(\Lambda^k M)} < \infty$.

Lemma 2.3. *Let* $\||u\||_{L^{p(m)}(\Lambda^k M)} < \infty$ *and* $\rho_{p'(m),\Lambda^{n-k}M}(v) < \infty$. *Then*

$$\left| \int_M u \wedge v \right| \leq \begin{cases} \||u\||_{L^{p(m)}(\Lambda^k M)} & \text{if } \rho_{p'(m),\Lambda^{n-k}M}(v) \leq 1, \\ \rho_{p'(m),\Lambda^{n-k}M}(v) \||u\||_{L^{p(m)}(\Lambda^k M)} & \text{if } \rho_{p'(m),\Lambda^{n-k}M}(v) > 1. \end{cases} \tag{2.26}$$

Proof. The first case follows from (2.18). Assume that $\rho_{p'(m),\Lambda^{n-k}M}(v) > 1$, we have

$$\rho_{p'(m),\Lambda^{n-k}M}\left(\frac{v}{\rho_{p'(m),\Lambda^{n-k}M}(v)} \right) \leq \frac{\rho_{p'(m),\Lambda^{n-k}M}(v)}{\rho_{p'(m),\Lambda^{n-k}M}(v)} = 1, \tag{2.27}$$

and so

$$\left| \int_M u \wedge v \right| = \rho_{p'(m),\Lambda^{n-k}M}(v) \left| \int_M u \wedge \frac{v}{\rho_{p'(m),\Lambda^{n-k}M}(v)} \right| \leq \rho_{p'(m),\Lambda^{n-k}M}(v) \||u\||_{L^{p(m)}(\Lambda^k M)}. \tag{2.28}$$

□

Lemma 2.4. *If* $\mu(M) = \mu(M_0)$, $\rho_{p(m),\Lambda^k M}(u) < \infty$ *and* $\||u\||_{L^{p(m)}(\Lambda^k M)} \leq 1$, *then* $\rho_{p(m),\Lambda^k M}(u) \leq 1$.

Proof. If this is not true, we may assume that $\rho_{p(m),\Lambda^k M}(u) > 1$, by (b_4) there exist $\lambda > 1$ such that $\rho_{p(m),\Lambda^k M}(u/\lambda) = 1$. Set

$$v = \frac{|u|^{p(m)-2}}{\lambda^{p(m)-1}} (\star u), \quad m \in M, \tag{2.29}$$

we have $\rho_{p'(m),\Lambda^{n-k}M}(v) = \rho_{p(m),\Lambda^k M}(u/\lambda) = 1$ and so

$$\||u\||_{L^{p(m)}(\Lambda^k M)} \geq \int_M u \wedge v = \lambda \rho_{p(m),\Lambda^k M}\left(\frac{u}{\lambda}\right) = \lambda > 1, \tag{2.30}$$

which is a contradiction. □

Lemma 2.5. *If* $\||u\||_{L^{p(m)}(\Lambda^k M)} \leq 1$, *then* $\rho_{p(m),\Lambda^k M}(u) \leq c_p \||u\||_{L^{p(m)}(\Lambda^k M)}$.

Proof. First, suppose that $\rho_{p(m),\Lambda^k M}(u) < \infty$. We have

$$\rho_{p(m),\Lambda^k M}(u) = \sum_{j=0,1,\infty} \rho_{p(m),\Lambda^k M}(u_j), \tag{2.31}$$

where $u_j = u\chi_{M_j}$, $j = 0, 1, \infty$. Set

$$v_1 = \begin{cases} |u|^{-1}(\star u_1) & \text{if } |u| \neq 0, \\ 0 & \text{if } |u| = 0, \end{cases} \qquad v_0 = \begin{cases} |u|^{p(m)-2}(\star u_0) & \text{if } |u| \neq 0, \\ 0 & \text{if } |u| = 0. \end{cases} \tag{2.32}$$

Then $\rho_{p'(m),\Lambda^{n-k}M}(v_1) = \operatorname{esssup}_{M_1}|v_1| = 1$ and due to Lemma 2.4,

$$\rho_{p'(m),\Lambda^{n-k}M}(v_0) = \int_{M_0} |u_0|^{p(m)} d\mu \leq 1. \tag{2.33}$$

Hence, Lemma 2.3 yields

$$\rho_{p(m),\Lambda^k M}(u_j) = \int_{M \setminus M_\infty} u \wedge v_j \leq |||u|||_{L^{p(m)}(\Lambda^k M)}, \quad j = 0, 1. \tag{2.34}$$

If $\mu(M_\infty) > 0$, then for every $\varepsilon \in (0, 1)$ there exists a set $D \subset M_\infty$ such that $0 < \mu(D) < \infty$ and $|u(m)| \geq \operatorname{esssup}_{M_\infty}|u|\varepsilon$, $m \in D$. Take

$$v_\infty = \begin{cases} \mu(D)^{-1}\chi_D|u|^{-1}(\star u) & \text{if } |u| \neq 0, \\ 0 & \text{if } |u| = 0, \end{cases} \tag{2.35}$$

we have $\rho_{p'(m),\Lambda^{n-k}M}(v_\infty) = \int_D \mu(D)^{-1}|u|^{-1}| \star u| d\mu \leq 1$ and so

$$|||u|||_{L^{p(m)}(\Lambda^k M)} \geq \int_M u \wedge v_\infty = \mu(D)^{-1} \int_D |u| d\mu \geq \varepsilon \operatorname*{esssup}_{M_\infty}|u| = \varepsilon \rho_{p(m),\Lambda^k M}(u_\infty). \tag{2.36}$$

Letting $\varepsilon \to 1$ we obtain

$$\rho_{p(m),\Lambda^k M}(u_\infty) \leq |||u|||_{L^{p(m)}(\Lambda^k M)}. \tag{2.37}$$

Hence, (2.31)–(2.37) yield the desired results.

To avoid the assumption $\rho_{p(m),\Lambda^k M}(u) < \infty$ we define differential k-forms

$$u_t = \begin{cases} u\chi_{G_t} & \text{if } |u| \leq t, \\ \dfrac{tu\chi_{G_t}}{|u|} & \text{if } |u| > t, \end{cases} \tag{2.38}$$

where $\{G_t\}$ is a sequence of compact sets such that $G_t \subset G_{t+1} \subset M$, $\mu(G_t) < \infty$ for $t \in \mathbb{N}$ and $M = \cup_{t=1}^{\infty} G_t$. Then for every u_t we have $\rho_{p(m),\Lambda^k M}(u_t) < \infty$, $|||u_t|||_{L^{p(m)}(\Lambda^k M)} \leq |||u|||_{L^{p(m)}(\Lambda^k M)} \leq 1$. By the first part of the proof, $\rho_{p(m),\Lambda^k M}(u_t) \leq c_p|||u|||_{L^{p(m)}(\Lambda^k M)}$. It follows let $t \to \infty$. □

Lemma 2.6. *For every $u \in L^{p(m)}(\Lambda^k M)$, the following inequalities hold*

$$c_p^{-1}\|u\|_{L^{p(m)}(\Lambda^k M)} \le |||u|||_{L^{p(m)}(\Lambda^k M)} \le r_p\|u\|_{L^{p(m)}(\Lambda^k M)}. \tag{2.39}$$

Furthermore, we have

$$L^{p(m)}\left(\Lambda^k M\right) = \left\{u \in L^1_{\mathrm{loc}}\left(\Lambda^k M\right) : |||u|||_{L^{p(m)}(\Lambda^k M)} < \infty\right\}. \tag{2.40}$$

Proof. Let $u \in L^{p(m)}(\Lambda^k M)$. If $\rho_{p'(m),\Lambda^{n-k}M}(v) \le 1$, then $\|v\|_{L^{p'(m)}(\Lambda^{n-k}M)} \le 1$ and Hölder inequality yields

$$\int_M u \wedge v \le r_p\|u\|_{L^{p(m)}(\Lambda^k M)}\|v\|_{L^{p'(m)}(\Lambda^{n-k}M)} \le r_p\|u\|_{L^{p(m)}(\Lambda^k M)}. \tag{2.41}$$

This gives the second inequality in (2.39) and, consequently, $|||u|||_{L^{p(m)}(\Lambda^k M)} < \infty$.

Conversely, we can suppose that $0 < |||u|||_{L^{p(m)}(\Lambda^k M)} < \infty$. By Lemma 2.5 and following inequalitiy

$$0 < \left|\left|\left|\frac{u}{\left(c_p|||u|||_{L^{p(m)}(\Lambda^k M)}\right)}\right|\right|\right|_{L^{p(m)}(\Lambda^k M)} = c_p^{-1} \le 1, \tag{2.42}$$

we get $\rho_{p(m),\Lambda^k M}(u/(c_p|||u|||_{L^{p(m)}(\Lambda^k M)})) \le c_p c_p^{-1} = 1$. The first inequality in (2.39) follows and then $u \in L^{p(m)}(\Lambda^k M)$. □

We shall say that differential k-forms $u_t \in L^{p(m)}(\Lambda^k M)$ *converge modularly* to a differential k-form $u \in L^{p(m)}(\Lambda^k M)$ if $\lim_{t\to\infty}\rho_{p(m),\Lambda^k M}(u_t - u) = 0$.

Next, we consider the relationship between convergence in norm, convergence in modular, and convergence in measure. For the corresponding results for domains in $\mathbb{R}^n$, readers can be referred to [15, 16].

Lemma 2.7. *If $p \in \mathcal{P}_1(M)$, then $\rho_{p(m),\Lambda^k M}(u_t) \to 0$ if and only if $\|u_t\|_{L^{p(m)}(\Lambda^k M)} \to 0$.*

Proof. According to Lemmas 2.5 and 2.6, the norm convergence is stronger than the modular convergence. Suppose that $\rho_{p(m),\Lambda^k M}(u_t) \to 0$, and take $\varepsilon \in (0, 1]$. For sufficiently large t we have $\rho_{p(m),\Lambda^k M}(u_t) < \varepsilon \le 1$ and so

$$\rho_{p(m),\Lambda^k M}\left(\frac{u_t}{\left(\rho_{p(m),\Lambda^k M}(u_t)\right)^{1/p^*}}\right) \le \frac{\rho_{p(m),\Lambda^k M}(u_t)}{\rho_{p(m),\Lambda^k M}(u_t)} = 1, \tag{2.43}$$

that is, $\|u_t\|_{L^{p(m)}(\Lambda^k M)} \le (\rho_{p(m),\Lambda^k M}(u_t))^{1/p^*} \le \varepsilon^{1/p^*}$. Hence, $\|u_t\|_{L^{p(m)}(\Lambda^k M)} \to 0$. □

Lemma 2.8. *If $p \in \mathcal{P}_1(M)$ and $\mu(M) < \infty$, then $\|u_t - u\|_{L^{p(m)}(\Lambda^k M)} \to 0$ if and only if u_t converges to u on M in measure and* $\lim_{t\to\infty}\rho_{p(m),\Lambda^k M}(u_t) = \rho_{p(m),\Lambda^k M}(u)$.

Proof. If $\|u_t - u\|_{L^{p(m)}(\Lambda^k M)} \to 0$, by Lemma 2.7

$$\lim_{t\to\infty}\int_M |u_t - u|^{p(m)} d\mu = 0, \tag{2.44}$$

then it is easy to see that u_t converges to u on M in measure. Hence by $\mu(M) < \infty$, $|u_t|^{p(m)}$ converges to $|u|^{p(m)}$ on M in measure and the integrals of the functions $|u_t - u|^{p(m)}$ possess absolutely equicontinuity on M. Since

$$|u_t|^{p(m)} \le 2^{p^*-1}\left(|u_t - u|^{p(m)} + |u|^{p(m)}\right), \tag{2.45}$$

the integrals of the $|u_t|^{p(m)}$ are also absolutely equicontinuous on M. By Vitali convergence theorem (see [28]), we deduce that $\lim_{t\to\infty}\rho_{p(m),\Lambda^k M}(u_t) = \rho_{p(m),\Lambda^k M}(u)$.

Conversely, if u_t converges to u on M in measure, we can deduce that $|u_t - u|^{p(m)}$ converges to 0 on M in measure. Similar to the above proof, by the inequality

$$|u_t - u|^{p(m)} \le 2^{p^*-1}\left(|u_t|^{p(m)} + |u|^{p(m)}\right), \tag{2.46}$$

and $\lim_{t\to\infty}\rho_{p(m),\Lambda^k M}(u_t) = \rho_{p(m),\Lambda^k M}(u)$, we get $\lim_{t\to\infty}\rho_{p(m),\Lambda^k M}(u_t - u) = 0$. □

Lemma 2.9. *If $p \in \mathcal{P}_1(M)$, then $L^\infty(\Lambda^k M) \cap L^{p(m)}(\Lambda^k M)$ is dense in $L^{p(m)}(\Lambda^k M)$.*

Proof. Let m_0 be some point of M, d_g be the distance associated to g and $G_t = \{m \in M : d_g(m_0, m) < t,\ t \in \mathbb{N}\}$. Given $u \in L^{p(m)}(\Lambda^k M)$, we define sequence of differential k-forms by

$$u_t = \begin{cases} u\chi_{G_t} & \text{if } |u| \le t, \\ \dfrac{tu\chi_{G_t}}{|u|} & \text{if } |u| > t. \end{cases} \tag{2.47}$$

Then $u_t \in L^\infty(\Lambda^k M)$ and by Lebesgue dominated convergence theorem, we have $\rho_{p(m),\Lambda^k M}(u - u_t) \to 0$. Hence, by Lemma 2.7 $\|u - u_t\|_{L^{p(m)}(\Lambda^k M)} \to 0$. □

Lemma 2.10. *If $p \in \mathcal{P}_1(M)$, then $C_c^\infty(\Lambda^k M)$ is dense in $L^{p(m)}(\Lambda^k M)$.*

Proof. Since $p \in \mathcal{P}_1(M)$, we have $C_c^\infty(\Lambda^k M) \subset L^{p(m)}(\Lambda^k M)$. By Lemma 2.9, there is a differential k-form $u_{t_0} \in L^\infty(\Lambda^k M) \cap L^{p(m)}(\Lambda^k M)$ such that

$$\|u - u_{t_0}\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon. \tag{2.48}$$

By Luzin theorem there exists a continuous k-form $\varphi \in C(\Lambda^k M)$ and an open set $D \subset M$ such that

$$\mu(D) < \min\left\{1, \left(\frac{\varepsilon}{2\|u_{t_0}\|_{L^\infty(\Lambda^k M)}}\right)^{p^*}\right\}, \tag{2.49}$$

$\varphi = u_{t_0}$ on $M \setminus D$ and $\sup_M|\varphi| = \sup_{M\setminus D}|u_{t_0}| \le \|u_{t_0}\|_{L^\infty(\Lambda^k M)}$. Thus,

$$\rho_{p(m),\Lambda^k M}\left(\frac{u_{t_0}-\varphi}{\varepsilon}\right) \le \max\left\{1, \left(\frac{2\|u_{t_0}\|_{L^\infty(\Lambda^k M)}}{\varepsilon}\right)^{p^*}\right\}\mu(D) \le 1, \tag{2.50}$$

that is,

$$\left\|u_{t_0}-\varphi\right\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon. \tag{2.51}$$

Since $\varphi \in L^{p(m)}(\Lambda^k M)$, we have $\rho_{p(m),\Lambda^k M}(\varphi) < \infty$ and there exists a bounded open set $G \subset M$ such that $\rho_{p(m),\Lambda^k M}(\varphi\chi_{M\setminus G}/\varepsilon) \le 1$, that is,

$$\left\|\varphi - \varphi\chi_G\right\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon. \tag{2.52}$$

Let h be a polynomial differential k-form with $\sup_G|\varphi - h| < \varepsilon\min\{1, \mu(G)^{-1}\}$. The polynomial differential k-form means the components of its coordinate representation in each chart of the manifold M are polynomial functions. Then $\rho_{p(m),\Lambda^k M}((\varphi\chi_G - h\chi_G)/\varepsilon) \le \min\{1, \mu(G)^{-1}\}\mu(G) \le 1$, that is,

$$\left\|\varphi\chi_G - h\chi_G\right\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon. \tag{2.53}$$

Finally, there exists a compact set $K \subset G$ such that $\|h\chi_G - h\chi_K\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon$. Let $\pi \in C_c^\infty(G)$ with $0 \le \pi \le 1$ in G and $\pi = 1$ on K we obtain the estimate

$$\left\|h\chi_G - \pi h\right\|_{L^{p(m)}(\Lambda^k M)} \le \left\|h\chi_G - h\chi_K\right\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon. \tag{2.54}$$

From (2.48)–(2.54), we get

$$\|u - \pi h\|_{L^{p(m)}(\Lambda^k M)} \le 5\varepsilon. \tag{2.55}$$

Obviously, $\pi h \in C_c^\infty(\Lambda^k M)$. □

Theorem 2.11. *If $p \in \mathcal{P}_1(M)$, then the space $L^{p(m)}(\Lambda^k M)$ is separable.*

Proof. Let $u \in L^{p(m)}(\Lambda^k M)$, $\varepsilon > 0$. By the proof of Lemma 2.10, we can fine a continuous k-form $\varphi \in C(\Lambda^k M)$ and a set $G_{t_0} = \{m \in M : d_g(m_0, m) < t_0\}$ such that

$$\|u - \varphi\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon, \qquad \left\|\varphi\chi_{M\setminus G_{t_0}}\right\|_{L^{p(m)}(\Lambda^k M)} \le \varepsilon, \tag{2.56}$$

Let h be a polynomial differential k-form with $\sup_{G_{t_0}} |\varphi - h| < \varepsilon \min\{1, \mu(G_{t_0})^{-1}\}$, v be a polynomial differential k-form with rational coefficients and $\sup_{G_{t_0}} |h-v| < \varepsilon \min\{1, \mu(G_{t_0})^{-1}\}$. Then we have

$$\left\|\varphi\chi_{G_{t_0}} - h\chi_{G_{t_0}}\right\|_{L^{p(m)}(\Lambda^k M)} \leq \varepsilon, \qquad \left\|v\chi_{G_{t_0}} - h\chi_{G_{t_0}}\right\|_{L^{p(m)}(\Lambda^k M)} \leq \varepsilon. \tag{2.57}$$

Thus,

$$\left\|v\chi_{G_{t_0}} - u\right\|_{L^{p(m)}(\Lambda^k M)} \leq 4\varepsilon. \tag{2.58}$$

Therefore, we conclude that the set of all differential k-forms $v\chi_{G_t}$ is dense in $L^{p(m)}(\Lambda^k M)$. □

Theorem 2.12. *If $p \in \mathcal{P}(M)$, then the space $L^{p(m)}(\Lambda^k M)$ is complete.*

Proof. Let $\{u_t : u_t = \sum_I (u_t)_I dx^I\}$ be a Cauchy sequence of differential k-forms in $L^{p(m)}(\Lambda^k M)$ and $\varepsilon > 0$. Let $\{G_l\}$ be a sequence of compact sets such that $G_l \subset G_{l+1} \subset M$ for $l \in \mathbb{N}$ and $M = \cup_{l=1}^{\infty} G_l$. There exists $t_0 \in \mathbb{N}$ such that

$$\sup_{\rho_{p'(m),\Lambda^{n-k}M}(v)\leq 1} \int_{G_l} (u_t - u_\tau) \wedge v \leq \varepsilon, \tag{2.59}$$

for every $t, \tau \geq t_0$ and $l \in \mathbb{N}$. By (2.24) we have

$$\int_{G_l} \sum_I |(u_t - u_\tau)_I||v_{I^*}| d\mu \leq 2^{n/2}\varepsilon, \tag{2.60}$$

for every $v = \sum_I v_{I^*} dx^{I^*}$, $\rho_{p'(m),\Lambda^{n-k}M}(v) \leq 1$ and $\operatorname{sgn} v_{I^*} = \sigma(I)\operatorname{sgn}(u_t - u_\tau)_I$. We define $v_l = \varphi_l \chi_{G_l}$ where $|\varphi_l| = (1 + \mu(G_l))^{-1}$ for $l \in \mathbb{N}$. Then

$$\rho_{p'(m),\Lambda^{n-k}M}(v_l) \leq \int_{G_l} \left(1 + \mu(G_l)\right)^{-p'(m)} d\mu + \left(1 + \mu(G_l)\right)^{-1} \leq 1, \tag{2.61}$$

thus, by (2.60) we get

$$\int_{G_l} |u_t - u_\tau| d\mu \leq 2^{k/2} \int_{G_l} \sum_I |(u_t - u_\tau)_I| d\mu \leq \varepsilon 2^n \left(1 + \mu(G_l)\right), \quad \text{for } t, \tau \geq t_0,\ l \in \mathbb{N}. \tag{2.62}$$

This means that the sequence $\{u_t\}$ is Cauchy in each $L^1(\Lambda^k G_l)$. By induction we may find subsequences $\{u_t^{(l)}\}_t$ and differential k-forms $u^{(l)} \in L^1(\Lambda^k G_l)$ such that $u_t^{(l)} \to u^{(l)}$ a.e. onG_l for $l \in \mathbb{N}$, and $u^{(l+1)}\chi_{G_l} = u^{(l)}$. Thus, $\lim_{\tau\to\infty} u_\tau^{(\tau)} = \lim_{\tau\to\infty} u^{(\tau)}\chi_{G_\tau} = u$ a.e. on M. Replacing u_τ by $u_\tau^{(\tau)}$ in (2.60) and using the Fatou lemma we obtain

$$\int_{G_l} \sum_I |(u_t - u)_I||v_{I^*}| d\mu \leq \sup_{\tau > t_0} \int_{G_l} \sum_I \left|\left(u_t - u_\tau^{(\tau)}\right)_I\right| |v_{I^*}| d\mu \leq 2^{n/2}\varepsilon. \tag{2.63}$$

Let $l \to \infty$, together with (2.24) we have

$$\int_M (u_t - u) \wedge v \le 2^n \varepsilon. \tag{2.64}$$

Therefore, by (2.18) and (2.24), we obtain $|||u_t - u|||_{L^{p(m)}(\Lambda^k M)} \le 2^n \varepsilon$. □

Theorem 2.13. *If $p \in \mathcal{P}_2(M)$, then the space $L^{p(m)}(\Lambda^k M)$ is reflexive.*

Proof. Let $[L^{p(m)}(\Lambda^k M)]'$ denote the dual space to $L^{p(m)}(\Lambda^k M)$. We will show that $[L^{p(m)}(\Lambda^k M)]' = L^{p'(m)}(\Lambda^{n-k} M)$ in steps.

(i) For fixed $v \in L^{p'(m)}(\Lambda^{n-k} M)$, we define a linear functional F_v on $L^{p(m)}(\Lambda^k M)$

$$F_v(u) = \int_M u \wedge v = \int_M \langle \star u, v \rangle d\mu. \tag{2.65}$$

By Lemma 2.2, we have $|F_v(u)| \le r_p \|u\|_{L^{p(m)}(\Lambda^k M)} \|v\|_{L^{p'(m)}(\Lambda^{n-k} M)}$, that is,

$$\|F_v\| \le r_p \|v\|_{L^{p'(m)}(\Lambda^{n-k} M)}. \tag{2.66}$$

Thus, F_v is a bounded linear functional on $L^{p(m)}(\Lambda^k M)$ and so F_v belongs to $[L^{p(m)}(\Lambda^k M)]'$.

(ii) We consider an arbitrary local chart $f : V(\subset M) \to \mathbb{R}^n$ on M. Let U be any open set in M, whose closure is compact and contained in V. We define

$$h_I(\varphi dx_I) = \varphi \quad \text{for } I \in \Lambda(k,n), \;\; \varphi \in L^{p(f^{-1}(x))}(f(U)). \tag{2.67}$$

Since each continuous linear functional $\tilde{f} \in [L^{p(f^{-1}(x))}(f(U))]'$ can be represented uniquely in the form $\tilde{f}(\varphi) = \int_{f(U)} \varphi \psi_{\tilde{f}} dx$ for some $\psi_{\tilde{f}} \in L^{p'(f^{-1}(x))}(f(U))$, then for each continuous linear functional $\overline{f} \in [L^{p(f^{-1}(x))}(\Lambda^k f(U))]'$, we have

$$\begin{aligned} \overline{f}(\omega) &= \sum_{I \in \Lambda(k,n)} \overline{f}(\omega_I dx_I) = \sum_{I \in \Lambda(k,n)} \overline{f} \circ h_I^{-1}(\omega_I) = \sum_{I \in \Lambda(k,n)} \int_{f(U)} \omega_I \psi_{\overline{f} \circ h_I^{-1}} dx \\ &= \int_{f(U)} \omega \wedge \left(\sum_{I \in \Lambda(k,n)} \sigma(I) \psi_{\overline{f} \circ h_I^{-1}} dx_{I^*} \right), \end{aligned} \tag{2.68}$$

that is, $\overline{f}$ can be represented in the form

$$\overline{f}(\omega) = \int_{f(U)} \omega \wedge \varpi_{\overline{f}}, \tag{2.69}$$

where $\varpi_{\overline{f}} = \sum_{I\in\Lambda(k,n)} \sigma(I)\varphi_{\overline{f}\circ h_I^{-1}} dx_{I^*} \in L^{p'(f^{-1}(x))}(f(U))$. If $\varpi_1 = \sum_I \varpi_{1I} dx_{I^*}, \varpi_2 = \sum_I \varpi_{2I} dx_{I^*}$ such that

$$\overline{f}(\omega) = \int_{f(U)} \omega \wedge \varpi_1 = \int_{f(U)} \omega \wedge \varpi_2, \tag{2.70}$$

for every $\omega \in L^{p(f^{-1}(x))}(\Lambda^k f(U))$. Taking $\omega = \varphi dx_I$ for $I \in \Lambda(k,n)$, we have $\overline{f} \circ h_I^{-1}(\varphi) = \overline{f}(\omega) = \int_{f(U)} \varphi\varpi_{1I} dx = \int_{f(U)} \varphi\varpi_{2I} dx$, then $\varpi_{1I} = \varpi_{2I}$, that is, $\varpi_1 = \varpi_2$. Hence $\varpi_{\overline{f}}$ is uniquely determined.

For fixed $F \in [L^{p(m)}(\Lambda^k M)]'$ and any $u \in L^{p(m)}(\Lambda^{n-k} M)$ with compact support we have

$$F(\chi_U u) = F \circ f^*\left(\left(f^{-1}\right)^*(\chi_U u)\right) = \int_{f(U)} \left(f^{-1}\right)^*(\chi_U u) \wedge v_{F\circ f^*} = \int_U \chi_U u \wedge f^*(v_{F\circ f^*}), \tag{2.71}$$

where $v_U = f^*(v_{F\circ f^*}) \in L^{p'(m)}(\Lambda^{n-k} U)$ is uniquely determined. For any two sets U_1 and U_2, the differential forms v_{U_1} and v_{U_2} coincide on $U_1 \cap U_2$ because of the uniqueness of the differential form $v_{U_1\cap U_2}$. Thus, all the differential forms v_U, defined for different U, are compatible with one another, and hence defines a differential form v_F on M. The differential form v_F locally belongs to the space $L^{p'(m)}(\Lambda^{n-k} U)$ and satisfies

$$F(u) = \int_M u \wedge v_F, \tag{2.72}$$

for every $u \in L^{p(m)}(\Lambda^k M)$ with compact support, and is uniquely determined.

Let $\{G_t\}$ be a sequence of compact sets such that $G_t \subset G_{t+1} \subset M$ for $t \in \mathbb{N}$ and $M = \cup_{t=1}^{\infty} G_t$. Then

$$F(u) = F\left(\lim_{t\to\infty} \chi_{G_t} u\right) = \lim_{t\to\infty} F(\chi_{G_t} u) = \lim_{t\to\infty} \int_M \chi_{G_t} u \wedge v_F = \int_M u \wedge v_F. \tag{2.73}$$

If v_1, v_2 such that

$$F(u) = \int_M u \wedge v_1 = \int_M u \wedge v_2, \tag{2.74}$$

for every $u \in L^{p(m)}(\Lambda^k M)$. Then for any U, we have $F(\chi_U u) = \int_M \chi_U u \wedge v_1 = \int_M \chi_U u \wedge v_2$. Thus $\chi_U v_1 = \chi_U v_2$ for any U, that is, $v_1 = v_2$.

Therefore, we conclude that each continuous linear functional $F \in [L^{p(x)}(\Lambda^k M)]'$ can be uniquely represented in the form (2.72).

(iii) We shall show $\|v_F\|_{L^{p'(m)}(\Lambda^{n-k} M)} \le C\|F\|$ with the constant C dependent only on $p(m)$. We define a differential form u on M

$$u(m) = \begin{cases} \|v_F(m)\|_{L^{p'(m)}(\Lambda^{n-k} M)}^{1/(1-p(m))} |v_F(m)|^{p'(m)-2}(\star v_F(m)) & \text{if } |v_F(m)| \neq 0, \\ 0 & \text{if } |v_F(m)| = 0, \end{cases} \tag{2.75}$$

then by (b_4) and (b_6), we have

$$\|u\|_{L^{p(m)}(\Lambda^k M)} = \inf\left\{\lambda > 0 : \int_M \left(\frac{|v_F|}{\lambda^{p(m)-1}\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)}}\right)^{p'(m)} d\mu \le 1\right\} = 1. \tag{2.76}$$

Moreover

$$\begin{aligned} |F(u)| &= \left|\int_M u \wedge v_F\right| = \int_M \left(\frac{|v_F|}{\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)}}\right)^{p'(m)} \|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)} d\mu \\ &\ge \frac{\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)}}{2^{p_*/(p_*-1)}} \int_M \left(\frac{|v_F|}{(1/2)\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)}}\right)^{p'(m)} d\mu \\ &\ge \frac{\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)}}{2^{p_*/(p_*-1)}}. \end{aligned} \tag{2.77}$$

Hence, we assert that $\|v_F\|_{L^{p'(m)}(\Lambda^{n-k}M)} \le 2^{p_*/(p_*-1)}\|F\|$.

Now we reach the conclusion $[L^{p(m)}(\Lambda^k M)]' = L^{p'(m)}(\Lambda^{n-k}M)$, and hence $L^{p(m)}(\Lambda^k M)$ is reflexive. □

Theorem 2.14. *If $p \in \mathcal{P}_2(M)$, then the exterior Sobolev space $W^{1,p(m)}(\Lambda^k M)$ is a separable, reflexive Banach space.*

Proof. We treat $W^{1,p(m)}(\Lambda^k M)$ in a natural way as a subspace of the Cartesian product space $L^{p(m)}(\Lambda^k M) \times L^{p(m)}(\Lambda^{k+1}M)$. Then we need only to show that $W^{1,p(m)}(\Lambda^k M)$ is a closed subspace of $L^{p(m)}(\Lambda^k M) \times L^{p(m)}(\Lambda^{k+1}M)$. Let $\{u_t\} \subset W^{1,p(m)}(\Lambda^k M)$ be a convergent sequence. Then $\{u_t\}$ is a convergent sequence in $L^{p(m)}(\Lambda^k M)$. In view of Theorem 2.12, there exists $u \in L^{p(m)}(\Lambda^k M)$ such that $u_t \to u$ in $L^{p(m)}(\Lambda^k M)$. Similarly there exists $\tilde{u} \in L^{p(m)}(\Lambda^{k+1}M)$ such that $du_t \to \tilde{u}$ in $L^{p(m)}(\Lambda^{k+1}M)$. Then it is easy to see that u_t converges to u and du_t converges to $\tilde{u}$ on M in measure. For any $\varphi \in C_c^\infty(\Lambda^{n-k-1}M) \subset L^{p'(m)}(\Lambda^{n-k-1}M)$, we have

$$\int_M u_t \wedge d\varphi = (-1)^{k+1}\int_M du_t \wedge \varphi. \tag{2.78}$$

Obviously, $|u_t \wedge d\varphi| \le |(u_t - u)\wedge d\varphi| + |u \wedge d\varphi|$ and $|du_t \wedge \varphi| \le |(du_t - \tilde{u})\wedge\varphi| + |\tilde{u}\wedge\varphi|$, then integrals of the functions $|u_t \wedge d\varphi|$ and $|du_t \wedge \varphi|$ possess absolutely equicontinuity on M. Hence, by Vitali convergence theorem (see [28]), we get

$$\int_M u \wedge d\varphi = (-1)^{k+1}\int_M \tilde{u} \wedge \varphi. \tag{2.79}$$

Thus, we obtain that $du = \tilde{u}$. Then it is immediate that $W^{1,p(m)}(\Lambda^k M)$ is a closed subspace of $L^{p(m)}(\Lambda^k M) \times L^{p(m)}(\Lambda^{k+1}M)$. □

Given two Banach spaces X and Y, the symbol $X \curvearrowright Y$ means that X is continuously embedded in Y.

Theorem 2.15. *Let $0 < \mu(M) < \infty$. If $p(m), q(m) \in \mathcal{P}(M)$ and $p(m) \leq q(m)$ a.e. $m \in M$, then*

$$L^{q(m)}\left(\Lambda^k M\right) \hookrightarrow L^{p(m)}\left(\Lambda^k M\right). \tag{2.80}$$

The norm of the embedding operator (2.80) *does not exceed $\mu(M) + 1$.*

Proof. Since $p(m) \leq q(m)$ a.e. $m \in M$, then $M_\infty^p \subset M_\infty^q$. We may assume that $u \in L^{q(m)}(\Lambda^k m)$ with $\|u\|_{L^{q(m)}(\Lambda^k M)} \leq 1$. Otherwise we can consider $u/\|u\|_{L^{q(m)}(\Lambda^k M)}$. By (b_7) we have $\rho_{q(m),\Lambda^k M}(u) \leq 1$, in particular, $|u(m)| \leq 1$ a.e. $m \in M_\infty^q$. Then we can write

$$\begin{aligned}\rho_{p(m),\Lambda^k M}(u) \leq \mu\left(\left\{m \in M \setminus M_\infty^q : |u| \leq 1\right\}\right) + \int_{M \setminus M_\infty^q} |u|^{q(m)} d\mu \\ + \mu\left(M_\infty^q \setminus M_\infty^p\right) + \operatorname*{esssup}_{M_\infty^p} |u| \leq \mu(M) + 1.\end{aligned} \tag{2.81}$$

Thus, we have $\rho_{p(m),\Lambda^k M}(u/(\mu(M)+1)) \leq (\mu(M)+1)^{-1} \rho_{p(m),\Lambda^k M}(u) \leq 1$. Therefore

$$\|u\|_{L^{p(m)}(\Lambda^k M)} \leq (\mu(M) + 1)\|u\|_{L^{q(m)}(\Lambda^k M)}. \tag{2.82}$$

□

Theorem 2.16. *Let M be a compact Riemannian manifold with a smooth boundary or without boundary and $p(m), q(m) \in C(\overline{M}) \cap \mathcal{P}_1(M)$. Assume that*

$$p(m) < n, \quad q(m) < \frac{np(m)}{n - p(m)}, \quad \text{for } m \in \overline{M}. \tag{2.83}$$

Then

$$W^{1,p(m)}(M) \hookrightarrow L^{q(m)}(M) \tag{2.84}$$

is a continuous and compact embedding.

Proof. We consider an arbitrary local chart $f : V(\subset M) \to \mathbb{R}^n$ on M. Let U be any open set in M, whose closure is compact and is contained in V. Choosing a finite subcovering $\{U_\alpha\}_{\alpha=1,2,\ldots,s}$ of M such that U_α is homeomorphic to the open unit ball $B_0(1)$ of $\mathbb{R}^n$ and for any α the components g_{ij}^α of g in (U_α, f_α) satisfy $1/C\delta_{ij} \leq g_{ij}^\alpha \leq C\delta_{ij}$ as bilinear forms, where constant $C > 1$ is given. Let $\{\pi_\alpha\}$ be a smooth partition of unity subordinate to the finite covering $\{U_\alpha\}$. It is obvious that if $u \in W^{1,p(m)}(M$, then $\pi_\alpha u \in W^{1,p(m)}(U_\alpha)$ and $(f_\alpha^{-1})^*(\pi_\alpha u) \in W^{1,p(f_\alpha^{-1}(x))}(B_0(1))$. By the definition of integral for differential n-forms on M and Sobolev embedding theorem in [16], we have the following continuous and compact embedding:

$$W^{1,p(m)}(U_\alpha) \hookrightarrow L^{q(m)}(U_\alpha), \quad \text{for each } \alpha = 1, 2, \ldots, s. \tag{2.85}$$

Since $u = \sum_{\alpha=1}^{s} \pi_\alpha u$, we can assert that $W^{1,p(m)}(M) \subset L^{q(m)}(M)$, and the embedding is continuous and compact. □

Let $u \in L^{p(m)}(\Lambda^k M)$, we say that u is *absolutely continuous* with respect to the norm $\|\cdot\|_{L^{p(m)}(\Lambda^k M)}$, if $G \subset M$ be a measurable subset, we have

$$\lim_{\mu(G)\to 0} \left\| u\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} = 0. \tag{2.86}$$

Theorem 2.17. *If $p \in \mathcal{P}_1(M)$, $u \in L^{p(m)}(\Lambda^k M)$ is absolutely continuous with respect to the norm $\|\cdot\|_{L^{p(m)}(\Lambda^k M)}$.*

Proof. By Lemma 2.9, there is a differential k-form $u_{t_0} \in L^\infty(\Lambda^k M) \cap L^{p(m)}(\Lambda^k M)$ such that

$$\|u - u_{t_0}\|_{L^{p(m)}(\Lambda^k M)} < \frac{\varepsilon}{2}. \tag{2.87}$$

Since u_{t_0} is bounded, we can find $\varepsilon_0 > 0$ such that when $\mu(G) < \varepsilon_0$, the following inequalities hold

$$\left\| u_{t_0}\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} < \frac{\varepsilon}{2}. \tag{2.88}$$

Hence, we get

$$\begin{aligned} \left\| u\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} &\le \left\| (u - u_{t_0})\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} + \left\| u_{t_0}\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} \\ &\le \|u - u_{t_0}\|_{L^{p(m)}(\Lambda^k M)} + \left\| u_{t_0}\chi_G \right\|_{L^{p(m)}(\Lambda^k M)} < \varepsilon. \end{aligned} \tag{2.89}$$

□

3. Applications

In this section, we shall show some applications of the exterior Sobolev space to Dirichlet problems with variable growth on Riemannian manifold. We shall assume that $\Omega \subset M$ is a bounded domain with smooth boundary and $p(m) \in \mathcal{P}_2(\Omega)$.

The nonhomogeneous $p(m)$-harmonic equation for differential forms with variable growth on Ω belong to the nonlinear elliptic equations which take the form

$$\delta\left(du|du|^{p(m)-2} \right) + u|u|^{p(m)-2} = f(m). \tag{3.1}$$

Definition 3.1. A differential form ω is a weak solution for the following Dirichlet problems

$$\begin{aligned} \delta\left(du|du|^{p(m)-2} \right) + u|u|^{p(m)-2} &= f(m), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.2}$$

where $f(m) \in L^{p'(m)}(\Lambda^{k-1}\Omega)$, if $\omega \in W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$ satisfies

$$\int_\Omega \left\langle d\omega|d\omega|^{p(m)-2}, dv \right\rangle + \left\langle \omega|\omega|^{p(m)-2}, v \right\rangle d\mu = \int_\Omega \langle f(m), v\rangle d\mu, \tag{3.3}$$

for every $v \in W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$.

We are now ready to show an application of exterior Sobolev spaces $W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$ to Dirichlet problems (3.2).

Let $X = W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$, X' be the dual space to X and $(\cdot,\cdot)$ denote a dual between X and X'. Consider the following functional:

$$I(u) = \int_\Omega \frac{1}{p(m)}\left(|du|^{p(m)} + |u|^{p(m)}\right) d\mu, \quad u \in X. \tag{3.4}$$

We denote $J = I' : X \to X'$, then

$$(J(u), v) = \int_\Omega \left\langle du|du|^{p(m)-2}, dv \right\rangle d\mu + \int_\Omega \left\langle u|u|^{p(m)-2}, v \right\rangle d\mu := (J_1(u), v) + (J_2(u), v), \tag{3.5}$$

where $u, v \in X$. Here,

$$(J_1(u), v) = \int_\Omega \left\langle du|du|^{p(m)-2}, dv \right\rangle d\mu, \qquad (J_2(u), v) = \int_\Omega \left\langle u|u|^{p(m)-2}, v \right\rangle d\mu. \tag{3.6}$$

Lemma 3.2. *$J = I' : X \to X'$ is a continuous, bounded, and strictly monotone operator.*

Proof. It is obvious that J is continuous and bounded. For any $y, z \in \mathbb{R}^N$, we have the following inequalities (see [29]) from which we can get the strictly monotonicity of J:

(h_1) $(|z|^{p-2}z - |y|^{p-2}y) \cdot (z - y) \geq (1/2)^p |z - y|^p$, $p \in [2, \infty)$,

(h_2) $[(|z|^{p-2}z - |y|^{p-2}y) \cdot (z - y)](|z|^p + |y^p|)^{(2-p)/p} \geq (p-1)^2 |z - y|^2$, $p \in (1, 2)$. □

Lemma 3.3. *$J = I' : X \to X'$ is a mapping of type (S_+), that is, if $u_t \rightharpoonup u$ weakly in X and $\limsup_{t\to\infty}(J(u_t) - J(u), u_t - u) \leq 0$, then $u_t \to u$ strongly in X.*

Proof. By Lemma 3.2, if $u_t \rightharpoonup u$ weakly in X and $\limsup_{t\to\infty}(J(u_t) - J(u), u_t - u) \leq 0$, we have $\lim_{t\to\infty}(J(u_t) - J(u), u_t - u) = 0$. In view of (h_1) and (h_2), $\lim_{t\to\infty}(J_i(u_t) - J_i(u), u_t - u) = 0$ $(i = 1, 2)$. Let $\Omega_1 = \{m \in \Omega : p(m) < 2\}$, $\Omega_2 = \{m \in \Omega : p(m) \geq 2\}$ and $v_t = \langle |u_t|^{p(m)-2}u_t - |u|^{p(m)-2}u, u_t - u\rangle$. Then there is a constant $C > 0$ such that

$$\int_{\Omega_2} |u_t - u|^{p(m)} d\mu \leq C \int_{\Omega_2} v_t d\mu \longrightarrow 0,$$

$$\begin{aligned} &\int_{\Omega_1} |u_t - u|^{p(m)} d\mu \\ &\quad \leq C \int_{\Omega_1} v_t^{p(m)/2} \left(|u_t|^{p(m)} + \left|u^{p(m)}\right|\right)^{(2-p(m))/2} d\mu \\ &\quad \leq C \left\| v_t^{p(m)/2} \chi_{\Omega_1} \right\|_{L^{2/p(m)}(\Omega)} \left\| \left(|u_t|^{p(m)} + \left|u^{p(m)}\right|\right)^{(2-p(m))/2} \chi_{\Omega_1} \right\|_{L^{2/(2-p(m))}(\Omega)} \longrightarrow 0. \end{aligned} \tag{3.7}$$

Therefore, by (3.7)

$$\lim_{t\to\infty} \int_{\Omega} |u_t - u|^{p(m)} d\mu = 0. \tag{3.8}$$

Similar to the proof above, we can obtain

$$\lim_{t\to\infty} \int_{\Omega} |du_t - du|^{p(m)} d\mu = 0. \tag{3.9}$$

From Lemma 2.8, we have $u_t \to u$ strongly in X, that is, J is a mapping of type (S_+). □

Lemma 3.4. *The mapping J is coercive, that is,*

$$\frac{(J(u), u)}{\|u\|_X} \longrightarrow \infty \quad \textit{as } \|u\|_X \longrightarrow \infty. \tag{3.10}$$

Proof. Taking $\varepsilon_0 = (1/2)\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}$, we have

$$\begin{aligned} \frac{\int_\Omega |u|^{p(m)} d\mu}{\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}} &= \int_\Omega \left(\frac{|u|}{\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)} - \varepsilon_0} \right)^{p(m)} \frac{\left(\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)} - \varepsilon_0\right)^{p(m)}}{\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}} d\mu \\ &\geq \frac{\left(\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)} - \varepsilon_0\right)^{p_*}}{\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}} \geq \frac{\|u\|^{p_*}_{L^{p(m)}(\Lambda^{k-1}\Omega)}}{2^{p^*}\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}} \longrightarrow \infty, \end{aligned} \tag{3.11}$$

as $\|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)} \to \infty$. Similarly, we also obtain

$$\frac{\int_\Omega |du|^{p(m)} d\mu}{\|du\|_{L^{p(m)}(\Lambda^k\Omega)}} \longrightarrow \infty \quad \text{as } \|du\|_{L^{p(\mathrm{m})}(\Lambda^k,\Omega)} \longrightarrow \infty. \tag{3.12}$$

Thus, for fixed constant $K > 0$, there exists $N = N(K)$ such that

$$\begin{aligned} &\frac{\int_\Omega |u|^{p(m)} d\mu}{\|u\|_{L^{p(m)}(\Lambda^{k-1}\Omega)}} > 2K, \quad \text{if } \|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)} > N, \\ &\frac{\int_\Omega |du|^{p(m)} d\mu}{\|du\|_{L^{p(m)}(\Lambda^{k}\Omega)}} > 2K, \quad \text{if } \|du\|_{L^{p(m)}(\Lambda^{k},\Omega)} > N. \end{aligned} \tag{3.13}$$

We take $N_0 = 2N$, if $\|u\|_X > N_0$ and $\|du\|_{L^{p(m)}(\Lambda^k,\Omega)} \geq \|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)}$, then

$$\frac{(J(u),u)}{\|u\|_X} = \frac{\int_\Omega |du|^{p(m)} d\mu + \int_\Omega |u|^{p(m)} d\mu}{\|du\|_{L^{p(m)}(\Lambda^k,\Omega)} + \|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)}} \geq \frac{\int_\Omega |du|^{p(m)} d\mu}{2\|du\|_{L^{p(m)}(\Lambda^k,\Omega)}} > K, \tag{3.14}$$

if $\|u\|_X > N_0$ and $\|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)} > \|du\|_{L^{p(m)}(\Lambda^k,\Omega)}$, then

$$\frac{(J(u),u)}{\|u\|_X} \geq \frac{\int_\Omega |u|^{p(m)} d\mu}{2\|u\|_{L^{p(m)}(\Lambda^{k-1},\Omega)}} > K. \tag{3.15}$$

Hence, $(J(u),u)/\|u\|_X \to \infty$ as $\|u\|_X \to \infty$, that is, the mapping J is coercive. □

Lemma 3.5. *$J : X \to X'$ is a homeomorphism.*

Proof. By Lemmas 3.2 and 3.4 and the theorem of Minty-Browder (see [30]), J is a bijection. Hence J has an inverse mapping $J^{-1} : X' \to X$. Therefore, the continuity of J^{-1} is sufficient to ensure J to be a homeomorphism.

If $v_t, v \in X'$ and $v_t \to v$ strongly in X', let $u_t = J^{-1}(v_t)$, $u = J^{-1}(v)$, then $J(u_t) = v_t$ and $J(u) = v$. As J is coercive, we have $\{u_t\}$ is bounded in X. Without loss of generality, we can assume that $u_t \rightharpoonup \overline{u}$ weakly in X. Since $v_t \to v$ strongly in X', then

$$\lim_{t\to\infty} (J(u_t) - J(\overline{u}), u_t - \overline{u}) = \lim_{t\to\infty} (J(u_t), u_t - \overline{u}) = \lim_{t\to\infty} (J(u_t) - J(u), u_t - \overline{u}) = 0. \tag{3.16}$$

Since J is a mapping of type (S_+), $u_t \to \overline{u}$ strongly in X. By Lemma 3.2, we conclude that $u_t \to u$ strongly in X, so J^{-1} is continuous. □

It is immediate to obtain the following conclusion from the above lemmas.

Theorem 3.6. *If $f(m) \in [W_0^{1,p(m)}(\Lambda^{k-1}\Omega)]'$, then Dirichlet problems* (3.2) *has a unique weak solution in $W_0^{1,p(m)}(\Lambda^{k-1}\Omega)$.*

If $k = 1$, that is, u is a function on Ω, let ∇ be the gradient operator on M. One has the following corollary.

Corollary 3.7. *If $f(m) \in [W_0^{1,p(m)}(\Omega)]'$, then Dirichlet problems*

$$-\operatorname{div}\left(\nabla u|\nabla u|^{p(m)-2}\right) + u|u|^{p(m)-2} = f(m), \quad in\ \Omega,$$
$$u = 0, \quad on\ \partial\Omega, \tag{3.17}$$

has a unique weak solution in $W_0^{1,p(m)}(\Omega)$.

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