Research Article

# Some Matrix Transformations of Convex and Paranormed Sequence Spaces into the Spaces of Invariant Means 

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We determine the necessary and sufficient conditions to characterize the matrices which transform convex sequences and Maddox sequences into $V_{\sigma}(\theta)$ and $V_{\sigma}^{\infty}(\theta)$.

## 1. Introduction and Preliminaries

By $w$, we denote the space of all real-valued sequences $x=\left(x_{k}\right)_{k=1}^{\infty}$. Any vector subspace of $w$ is called a sequence space. We write that $\ell_{\infty}, c$, and $c_{0}$ denote the sets of all bounded, convergent, and null sequences, respectively, and note that $c \subset \ell_{\infty}$; also cs and $\ell_{p}$ are the set of all convergent and $p$-absolutely convergent series, respectively, where $\ell_{p}:=\{x \in w$ : $\left.\sum_{k=0}^{\infty}\left|x_{k}\right|^{p}<\infty\right\}$ for $1 \leq p<\infty$. In the theory of sequence spaces, a beautiful application of the well-known Hahn-Banach extension theorem gave rise to the concept of the Banach limit. That is, the lim functional defined on $c$ can be extended to the whole $\ell_{\infty}$, and this extended functional is known as the Banach limit [1]. In 1948, Lorentz [2] used this notion of a weak limit to define a new type of convergence, known as the almost convergence. Later on, Raimi [3] gave a slight generalization of almost convergence and named it the $\sigma$-convergence.

A sequence space $X$ with a linear topology is called a $K$-space if each of the maps $p_{i}: X \rightarrow \mathbb{C}$ defined by $p_{i}(x)=x_{i}$ is continuous for all $i \in \mathbb{N}$. A $K$-space $X$ is called an $F K$-space if $X$ is a complete linear metric space. An $F K$-space whose topology is normable is called a $B K$-space. An $F K$-space $X$ is said to have $A K$ property if $X \supset \phi$ and $\left(e^{(k)}\right)$ is a basis for $X$, where $\left(e^{(k)}\right)$ is a sequence whose only nonzero term is a 1 in $k$ th place for each $k \in \mathbb{N}$ and $\phi=\operatorname{span}\left\{e^{(k)}\right\}$, the set of all finitely nonzero sequences. If $\phi$ is dense in $X$, then
$X$ is called an $A D$-space, thus $A K$ implies $A D$. For example, the spaces $c_{0}, c s$, and $\ell_{p}$ are $A K$-spaces.

Let $X$ and $Y$ be two sequence spaces, and let $A=\left(a_{n k}\right)_{n ; k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $A x=\left(A_{n}(x)\right), A_{n}(x)=\sum_{k} a_{n k} x_{k}$ provided that the series on the right converges for each $n$. If $x=\left(x_{k}\right) \in X$ implies that $A x \in Y$, then we say that $A$ defines a matrix transformation from $X$ into $Y$, and by $(X, Y)$, we denote the class of such matrices.

Let $\sigma$ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional $\varphi$ on the space $\ell_{\infty}$ is said to be an invariant mean or a $\sigma$-mean if and only if (i) $\varphi(x) \geq 0$ if $x \geq 0$ (i.e., $x_{k} \geq 0$ for all $k$ ), (ii) $\varphi(e)=1$, where $e=(1,1,1, \ldots$ ), and (iii) $\varphi(x)=\varphi\left(\left(x_{\sigma(k)}\right)\right)$ for all $x \in \ell_{\infty}$.

Throughout this paper, we consider the mapping $\sigma$ which has no finite orbits, that is, $\sigma^{p}(k) \neq k$ for all integer $k \geq 0$ and $p \geq 1$, where $\sigma^{p}(k)$ denotes the $p$ th iterate of $\sigma$ at $k$. Note that a $\sigma$-mean extends the limit functional on the space $c$ in the sense that $\varphi(x)=\lim x$ for all $x \in c$, (cf. [4]). Consequently, $c \subset V_{\sigma}$, the set of bounded sequences all of whose $\sigma$-means are equal. We say that a sequence $x=\left(x_{k}\right)$ is $\sigma$-convergent if and only if $x \in V_{\sigma}$. Using this concept, Schaefer [5] defined and characterized $\sigma$-conservative, $\sigma$-regular, and $\sigma$-coercive matrices. If $\sigma$ is translation, then $V_{\sigma}$ is reduced to the set $f$ of almost convergent sequences [2].

The notion of $\sigma$-convergence for double sequences has been introduced in [6] and further studied in [7-9].

Recently, the sequence spaces $V_{\sigma}(\theta)$ and $V_{\sigma}^{\infty}(\theta)$ have been introduced in [10] which are related to the concept of $\sigma$-mean and the lacunary sequence $\theta=\left(k_{r}\right)$.

In this section, we establish the necessary and sufficient conditions on the matrix $A=$ $\left(a_{n k}\right)_{n, k=1}^{\infty}$ which transforms $r$-convex sequences in to the spaces $V_{\sigma}(\theta)$ and $V_{\sigma}^{\infty}(\theta)$.

By a lacunary sequence, we mean an increasing integer sequence $\theta=\left(k_{r}\right)$ such that $k_{0}=0$ and $h_{r}:=k_{r}-k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Throughout this paper, the intervals determined by $\theta$ will be denoted by $I_{r}:=\left(k_{r-1}, k_{r}\right]$, and the ratio $k_{r} / k_{r-1}$ will be abbreviated by $q_{r}$.

A bounded sequence $x=\left(x_{k}\right)$ of real numbers is said to be $\sigma$-lacunary convergent to a number $L$ if and only if $\lim _{r \rightarrow \infty}\left(1 / h_{r}\right) \sum_{j \in I_{r}} x_{\sigma^{j}(n)}=L$, uniformly in $n$, and let $V_{\sigma}(\theta)$ denote the set of all such sequences, that is,

$$
\begin{equation*}
V_{\sigma}(\theta)=\left\{x \in \ell_{\infty}: \lim _{m \rightarrow \infty} \frac{1}{h_{r}} \sum_{j \in I_{r}} x_{\sigma^{j}(n)}=L \text { uniformly in } n\right\} \tag{1.1}
\end{equation*}
$$

In this case, $L$ is called the $(\sigma, \theta)$-limit of $x$. We remark that
(i) if $\sigma(n)=n+1$, then $V_{\sigma}(\theta)$ is reduced to the space $f(\theta)$ (cf. [11]),
(ii) $c \subset V_{\sigma}(\theta) \subset \ell_{\infty}$.

A bounded sequence $x=\left(x_{k}\right)$ of real numbers is said to be $\sigma$-lacunary bounded if and only if $\sup _{r, n}\left|\left(1 / h_{r}\right) \sum_{j \in I_{r}} x_{\sigma^{j}(n)}\right|<\infty$, and let $V_{\sigma}^{\infty}(\theta)$ denote the set of all such sequences, that is,

$$
\begin{equation*}
V_{\sigma}^{\infty}(\theta)=\left\{x \in \ell_{\infty}: \sup _{r, n}\left|\frac{1}{h_{r}} \sum_{j \in I_{r}} x_{\sigma^{j}(n)}\right|<\infty\right\} \tag{1.2}
\end{equation*}
$$

We remark that $c \subset V_{\sigma}(\theta) \subset V_{\sigma}^{\infty}(\theta) \subset \ell_{\infty}$ and the spaces $V_{\sigma}^{\infty}(\theta)$ and $V_{\sigma}^{\infty}(\theta)$ are BK spaces with the norm

$$
\begin{equation*}
\|x\|=\sup _{r, n}\left|\frac{1}{h_{r}} \sum_{j \in I_{r}} x_{\sigma^{j}(n)}\right| . \tag{1.3}
\end{equation*}
$$

## 2. Convex Sequence Spaces

Pati and Sinha [12] defined $r$-convex sequences as follows: a real sequence $x=\left(x_{k}\right)_{k=0}^{\infty}$ is said to be $r$-convex, $r \in \mathbb{N}$, if $\Delta^{r} x_{k} \geq 0$ for all $k \in \mathbb{N}$, where $\Delta^{r} x_{k}$ is defined by

$$
\begin{equation*}
\Delta^{0} x_{k}=x_{k}, \quad \Delta^{1} x_{k}=x_{k}-x_{k+1}, \quad \Delta^{r} x_{k}=\Delta\left(\Delta^{r-1} x_{k}\right), \quad r \in \mathbb{N} \tag{2.1}
\end{equation*}
$$

The space of all bounded $r$-convex sequences with $r \geq 2$ is denoted by $\mathrm{SC}^{r}$, that is,

$$
\begin{gather*}
\mathrm{SC}^{r}:=\left\{x \in l_{\infty}: \Delta^{r} x_{k} \geq 0 \forall k \in \mathbb{N}\right\}, \\
\mathrm{SC}^{1}:=\left\{x \in l_{\infty}: x_{k}-x_{k+1} \geq 0\right\} . \tag{2.2}
\end{gather*}
$$

It is clear that $\mathrm{SC}^{1} \subseteq c$.
It is well known that (Zygmund [13]) a bounded convex sequence $\left(x_{k}\right)$ is nonincreasing. It is easy to prove the identity $\Delta^{(r+s)} x_{k}=\Delta^{r}\left(\Delta^{s} x_{k}\right), r, s \geq 0$, which shows that $\mathrm{SC}^{r} \subset \mathrm{SC}^{r-1}$, when $r \geq 2$. Properties of bounded $r$-convex sequences have been investigated by Rath [14]. Note that $\mathrm{SC}^{r} \subset v \subset c \subset \ell_{\infty}$. Recently, using the generalized difference operator $\Delta^{r}$, Çolak and Et [15], and Et and Çolak [16] defined and studied the sequence spaces $c_{0}\left(\Delta^{r}\right), c\left(\Delta^{r}\right)$, and $\ell_{\infty}\left(\Delta^{r}\right)$. In this section, we establish the necessary and sufficient conditions on the matrix $A=\left(a_{n k}\right)_{n, k=1}^{\infty}$ which transforms $r$-convex sequences into the spaces $V_{\sigma}(\theta)$ and $V_{\sigma}^{\infty}(\theta)$.

Write

$$
\begin{align*}
t(n, k, m) & =\frac{1}{h_{m}} \sum_{j \in I_{m}} a_{\sigma^{j}(n), k} \\
g^{(r)}(n, k, m) & =\frac{\sum_{j=1}^{k}\binom{r-1}{k-j} t(n, j, m)}{k^{r-1}},  \tag{2.3}\\
\lambda_{m n}(x) & =\sum_{k=1}^{\infty} g(n, k, m) x_{k},
\end{align*}
$$

where for our convenience, we use $g(n, k, m)$ instead of $g^{(r)}(n, k, m)$ for $r \geq 2$ throughout the paper.

Theorem 2.1. $A \in\left(S C^{r}, V_{\sigma}(\theta)\right)$ if and only if
(i) $\sup _{i, p}\left|\sum_{k=p}^{\infty} a_{i k}\right|<\infty$,
(ii) there exists a constant $M$ such that for $s, n=1,2, \ldots$,

$$
\begin{equation*}
\sup _{m} \sum_{k=s}^{\infty}|g(n, k, m)| \leq M, \tag{2.4}
\end{equation*}
$$

(iii) $\lim _{m} g(n, k, m)=\alpha_{k}$, uniformly in $n$, for each $k \in \mathbb{N}$,
(iv) $\lim _{m} \sum_{k} g(n, k, m)=\alpha$, uniformly in $n$.

Proof. In [17], a characterization of $A \in\left(\mathrm{SC}^{r}, F_{\mathcal{B}}\right)$ was given, where $F_{\mathcal{B}}$, in the sense of [18], is the bounded domain of a sequence $\bar{B}=\left(B^{(i)}\right)$ of matrices $B^{(i)}=\left(b_{r k}^{(i)}\right)$. Now, by the setting

$$
b_{r k}^{(i)}=\left\{\begin{array}{lc}
\frac{1}{h_{r}} & \text { if } k=\sigma^{j}(i), j \in I_{r},  \tag{2.5}\\
0 & \text { otherwise } .
\end{array} \quad(r, k, i \in \mathbb{N}),\right.
$$

then $V_{\sigma}(\theta)=F_{\mathbb{B}}$, and the proof follows from Theorem 2.1 of [17].
Theorem 2.2. $A \in\left(S C^{r}, V_{\sigma}^{\infty}(\theta)\right)$ if and only if the condition (i) of Theorem 2.1 holds and

$$
\begin{equation*}
\sup _{n, m} \sum_{k}|t(n, k, m)|<\infty . \tag{2.6}
\end{equation*}
$$

## Proof. Sufficiency

Suppose that the conditions (i) and (2.6) hold and $x=\left(x_{k}\right) \in \mathrm{SC}^{r} \subset \ell_{\infty}$. Therefore, $A x$ is bounded, and we have

$$
\begin{equation*}
\left|\lambda_{m n}(x)\right| \leq \sum_{k}\left|g(n, k, m) x_{k}\right| \leq\left(\sum_{k}|g(n, k, m)|\right)\left(\sup _{k}\left|x_{k}\right|\right) . \tag{2.7}
\end{equation*}
$$

Taking the supremum over $n, m$ on both sides and using (2.6), we get $A x \in V_{\sigma}^{\infty}(\theta)$ for $x \in \mathrm{SC}^{r}$.

## Necessity

Let $A \in\left(\mathrm{SC}^{r}, V_{\sigma}^{\infty}(\theta)\right)$. Condition (i) follows as in the proof of Theorem 2.1. Write $q_{n}(x)=$ $\sup _{m}\left|\lambda_{m n}(A x)\right|$. It is easy to see that $q_{n}$ is a continuous seminorm on $\mathrm{SC}^{r}$, since for $x \in \mathrm{SC}^{r} \subset$ $\ell_{\infty}$,

$$
\begin{equation*}
\left|q_{n}(x)\right| \leq M\|x\|, \quad M>0 . \tag{2.8}
\end{equation*}
$$

Suppose that (2.6) is not true, then there exists $x \in \mathrm{SC}^{r}$ with $\sup _{n} q_{n}(x)=\infty$. By the principle of condensation of singularities (cf. [19]), the set

$$
\begin{equation*}
\left\{x \in \mathrm{SC}^{r}: \sup _{n} q_{n}(x)=\infty\right\} \tag{2.9}
\end{equation*}
$$

is of second category in $\mathrm{SC}^{r}$ and hence nonempty, that is, there is $x \in \mathrm{SC}^{r}$ with $\sup _{n} q_{n}(x)=$ $\infty$. But this contradicts the fact that $q_{n}$ is pointwise bounded on $\mathrm{SC}^{r}$. Now, by the BanachSteinhaus theorem, there is a constant $M$ such that

$$
\begin{equation*}
q_{n}(x) \leq M\|x\| . \tag{2.10}
\end{equation*}
$$

Now, we define a sequence $x=\left(x_{k}\right)$ by

$$
x_{k}= \begin{cases}\operatorname{sgn} \frac{g(n, k, m)}{k} & \text { for each } m, n\left(1 \leq k \leq k_{0}\right),  \tag{2.11}\\ 0 & \text { for } k>k_{0} .\end{cases}
$$

Then, $x \in \mathrm{SC}^{r}$. Applying this sequence to (2.10), we get (2.6).
This completes the proof of the theorem.

## 3. Maddox Sequence Spaces

A linear topological space $X$ over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \rightarrow \mathbb{R}$ such that $g(\theta)=0, g(x)=g(-x)$, and scalar multiplication is continuous, that is, $\left|\alpha_{n}-\alpha\right| \rightarrow 0$ and $g\left(x_{n}-x\right) \rightarrow 0$ imply $g\left(\alpha_{n} x_{n}-\alpha x\right) \rightarrow 0$ for all $x, x_{n}$ in $X$ and $\alpha, \alpha_{n}$ in $\mathbb{R}$, where $\theta$ is the zero vector in the linear space $X$. Assume here and after that $x=\left(x_{k}\right)$ is a sequence such that $x_{k} \neq 0$ for all $k \in \mathbb{N}$. Let $p=\left(p_{k}\right)_{k=0}^{\infty}$ be a bounded sequence of positive real numbers with $\sup _{k} p_{k}=H$ and $M=\max \{1, H\}$. The sequence spaces

$$
\begin{gather*}
c_{0}(p):=\left\{x \in \omega: \lim _{k}\left|x_{k}\right|^{p_{k}}=0\right\}, \\
c(p):=\left\{x \in \omega: x-l e \in c_{0}(p) \text { for some } l \in \mathbb{C}\right\}, \\
l_{\infty}(p):=\left\{x \in \omega: \sup _{k}\left|x_{k}\right|^{p_{k}}<\infty\right\},  \tag{3.1}\\
l(p):=\left\{x \in \omega: \sum_{k=0}^{\infty}\left|x_{k}\right|^{p_{k}}<\infty\right\},
\end{gather*}
$$

were defined and studied by Et and Çolak [16] and Pati and Sinha [12]. If $p_{k}=p(k=0,1, \ldots)$ for some constant $p>0$, then these sets reduce to $c_{0}, c, l_{\infty}$, and $l_{p}$, respectively. Note that $c_{0}(p)$ is a linear metric space paranormed by

$$
\begin{equation*}
g(x)=\sup _{k}\left|x_{k}\right|^{p k / M} \tag{3.2}
\end{equation*}
$$

where $M=\max \left(1, \sup p_{k}\right) . l_{\infty}(p)$ and $c(p)$ fail to be linear metric spaces because the continuity of scalar multiplication does not hold for them, but these two turn out to be linear metric spaces if and only if $\inf _{k} p_{k}>0 . l(p)$ is linear metric space paranormed by $h_{1}(x)=\left(\sum_{k}\left|x_{k}\right|^{p_{k}}\right)^{1 / M}$. All these spaces are complete in their respective topologies; however, these are not normed spaces in general (cf. [20]).

In this section, we characterize the matrix classes $\left(l(p), V_{\sigma}(\theta)\right)$ and $\left(l_{\infty}(p), V_{\sigma}(\theta)\right)$.
Let $A x$ be defined, then, for all $r, n$, we write

$$
\begin{equation*}
\tau_{r}(A x)=\sum_{k=1}^{\infty} t(n, k, r) x_{k} \tag{3.3}
\end{equation*}
$$

where

$$
\begin{equation*}
t(n, k, r)=\frac{1}{h_{r}} \sum_{j \in I_{r}} a\left(\sigma^{j}(n), k\right) \tag{3.4}
\end{equation*}
$$

and $a(n, k)$ denotes the element $a_{n k}$ of the matrix $A$.
Theorem 3.1. $A \in\left(\ell(p), V_{\sigma}(\theta)\right)$ if and only if there exists $B>1$ such that for every $n$,
(i)

$$
\begin{gather*}
\sup _{r} \sum_{k}|t(n, k, r)|^{q_{k}} B^{-q_{k}}<\infty, \quad\left(1<p_{k}<\infty\right),\left(p_{k}^{-1}+q_{k}^{-1}=1\right)  \tag{3.5}\\
\sup _{r, k}|t(n, k, r)|^{p_{k}}<\infty, \quad\left(0<p_{k} \leq 1\right)
\end{gather*}
$$

(ii) $a_{(k)}=\left\{a_{n k}\right\}_{n=1}^{\infty} \in V_{\sigma}(\theta)$ for each $k$, that is, $\lim _{r} t(n, k, r)=u_{k}$ uniformly in $n$.

In this case, the $(\sigma, \theta)$-limit of $A x$ is $\sum_{k} u_{k} x_{k}$.

## Proof. Necessity

We consider the case $1<p_{k}<\infty$. Let $A \in\left(\ell(p), V_{\sigma}(\theta)\right)$. Since $e_{k} \in \ell(p)$, the condition (ii) holds. Put $f_{r n}(x)=\tau_{r n}(A x)$, since $\tau_{r n}(A x)$ exists for each $r$ and $x \in l(p)$, therefore $\left\{f_{r n}(x)\right\}_{r}$ is a sequence of continuous real functionals on $l(p)$ and further $\sup _{r}\left|f_{r n}(x)\right|<\infty$ on $l(p)$. Now condition (i) follows by arguing with uniform boundedness principle. The case $0<p_{k} \leq 1$ can be proved similarly.

## Sufficiency

Suppose that the conditions (i) and (ii) hold and $x \in \ell(p)$. Now for every $m \geq 1$, we have

$$
\begin{equation*}
\sum_{k=1}^{m}|t(n, k, r)|^{q_{k}} B^{-q_{k}} \leq \sup _{r} \sum_{k}|t(n, k, r)|^{q_{k}} B^{-q_{k}} \tag{3.6}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\sum_{k}\left|u_{k}\right|^{q_{k}} B^{-q_{k}}=\lim _{m} \lim _{r} \sum_{k=1}^{m}|t(n, k, r)|^{q_{k}} B^{-q_{k}} \leq \sup _{r} \sum_{k}|t(n, k, r)|^{q_{k}} B^{-q_{k}}<\infty . \tag{3.7}
\end{equation*}
$$

Thus, the series $\sum_{k} t(n, k, r) x_{k}$ and $\sum_{k} u_{k} x_{k}$ converge for each $r$ and $x \in \ell(p)$. For a given $\varepsilon>0$ and $x \in \ell(p)$, choose $k_{0}$ such that

$$
\begin{equation*}
\left(\sum_{k=k_{0}+1}^{\infty}\left|x_{k}\right|^{p_{k}}\right)^{1 / H}<\varepsilon, \tag{3.8}
\end{equation*}
$$

where $H=\sup _{k} p_{k}$. Since (ii) holds, therefore there exists $r_{0}$ such that

$$
\begin{equation*}
\left|\sum_{k=1}^{k_{0}}\left(t(n, k, r)-u_{k}\right)\right|<\varepsilon, \tag{3.9}
\end{equation*}
$$

for every $r>r_{0}$. Hence, by the condition (ii), it follows that

$$
\begin{equation*}
\left|\sum_{k=k_{0}+1}^{\infty}\left(t(n, k, r)-u_{k}\right)\right| \tag{3.10}
\end{equation*}
$$

is arbitrary small, and we have

$$
\begin{equation*}
\lim _{r} \sum_{k} t(n, k, r) x_{k}=\sum_{k} u_{k} x_{k}, \tag{3.11}
\end{equation*}
$$

uniformly in $n$.
This completes the proof of the theorem.
Theorem 3.2. $A \in\left(\ell_{\infty}(p), V_{\sigma}(\theta)\right)$ if and only if there exists $N>1$ such that
(i) $M_{n}=\sup _{r} \sum_{k}|t(n, k, r)| N^{1 / p_{k}}<\infty$ for every $n$,
(ii) $a_{(k)}=\left\{a_{n k}\right\}_{n=1}^{\infty} \in V_{\sigma}(\theta)$ for each $k$, that is, $\lim _{r} t(n, k, r)=u_{k}$ uniformly in $n$,
(iii) $\lim _{r} \sum_{k}\left|t(n, k, r)-u_{k}\right|=0$ uniformly in $n$.

In this case, the $(\sigma, \theta)$-limit of $A x$ is $\sum_{k} u_{k} x_{k}$.

## Proof. Necessity

Let $A \in\left(\ell_{\infty}(p), V_{\sigma}(\theta)\right)$, then $A \in\left(\ell_{\infty}, V_{\sigma}(\theta)\right)$, and the conditions (ii) and (iii) follow from Theorem 3 of Schaefer [5]. Now on the contrary, suppose that (i) does not hold, then there exists $N>1$ such that $M_{n}=\infty$. Therefore, by Theorem 3 of Schaefer [5], the matrix

$$
\begin{equation*}
B=\left(b_{n k}\right)=\left(a_{n k} N^{1 / p_{k}}\right) \notin\left(\ell_{\infty}, V_{\sigma}(\theta)\right), \tag{3.12}
\end{equation*}
$$

that is, there exists $x \in \ell_{\infty}$ such that $B x \notin V_{\sigma}(\theta)$. Now, let $y=\left(N^{1 / p_{k}} x_{k}\right)$, then $y \in \ell_{\infty}(p)$ and $B x=A y \notin V_{\sigma}(\theta)$, which contradicts that $A \in\left(\ell_{\infty}(p), V_{\sigma}(\theta)\right)$. Therefore, (i) must hold.

## Sufficiency

Suppose that the conditions hold and $x \in \ell_{\infty}(p)$, then for every $n$,

$$
\begin{equation*}
\left|\sum_{k} t(n, k, r) x_{k}\right| \leq\left(\sup _{k}\left|x_{k}\right|^{p_{k}}\right)\left(\sup _{r} \sum_{k}|t(n, k, r)| N^{1 / p_{k}}\right)<\infty \tag{3.13}
\end{equation*}
$$

Therefore $A x$ is defined. Now arguing as in Theorem 3.1, we get $A x \in V_{\sigma}(\theta)$, and the series $\sum_{k} t(n, k, r) x_{k}$ and $\sum_{k} u_{k} x_{k}$ converge for $x \in \ell_{\infty}(p)$. Hence, by the condition (iii), we get

$$
\begin{equation*}
\lim _{r} \sum_{k} t(n, k, r) x_{k}=\sum_{k} u_{k} x_{k} \tag{3.14}
\end{equation*}
$$

uniformly in $n$.
This completes the proof of the theorem.
Theorem 3.3. Let $1<p_{k}<\sup _{k} p_{k}=H<\infty$ for every $k$, then $A \in\left(\ell(p), V_{\sigma}^{\infty}(\theta)\right)$ if and only if there exists an integer $N>1$ such that

$$
\begin{equation*}
\sup _{r, n} \sum_{k}|t(n, k, r)|^{q_{k}} N^{-q_{k}}<\infty . \tag{3.15}
\end{equation*}
$$

## Proof. Sufficiency

Let (3.15) hold and that $x \in \ell(p)$ using the following inequality (see [21]):

$$
\begin{equation*}
|a b| \leq C\left(|a|^{q} C^{-q}+|b|^{p}\right), \tag{3.16}
\end{equation*}
$$

for $C>0$ and $a, b$, are two complex numbers $\left(q^{-1}+p^{-1}=1\right)$, we have

$$
\begin{equation*}
\left|\tau_{r}(A x)\right|=\sum_{k}\left|t(n, k, r) x_{k}\right| \leq \sum_{k} N\left[|t(n, k, r)|^{q_{k}} N^{-q_{k}}+\left|x_{k}\right|^{p_{k}}\right] \tag{3.17}
\end{equation*}
$$

where $q_{k}^{-1}+p_{k}^{-1}=1$. Taking the supremum over $r, n$ on both sides and using (3.15), we get $\left.A x \in V_{\sigma}^{\infty}(\theta)\right)$ for $x \in \ell(p)$, that is, $A \in\left(\ell(p), V_{\sigma}^{\infty}(\theta)\right)$.

## Necessity

Let $A \in\left(\ell(p), V_{\sigma}^{\infty}(\theta)\right)$. Write $q_{n}(x)=\sup _{r}\left|\tau_{r}(A x)\right|$. It is easy to see that for $n \geq 0, q_{n}$ is a continuous seminorm on $\ell(p)$, and $\left(q_{n}\right)$ is pointwise bounded on $\ell(p)$. Suppose that (3.15) is
not true, then there exists $x \in \ell(p)$ with $\sup _{n} q_{n}(x)=\infty$. By the principle of condensation of singularities [19], the set

$$
\begin{equation*}
\left\{x \in \ell(p): \sup _{n} q_{n}(x)=\infty\right\} \tag{3.18}
\end{equation*}
$$

is of second category in $\ell(p)$ and hence nonempty, that is, there is $x \in \ell(p)$ with $\sup _{n} q_{n}(x)=$ $\infty$. But this contradicts the fact that $\left(q_{n}\right)$ is pointwise bounded on $\ell(p)$. Now, by the BanachSteinhaus theorem, there is constant $M$ such that

$$
\begin{equation*}
q_{n}(x) \leq M g(x) \tag{3.19}
\end{equation*}
$$

Now, we define a sequence $x=\left(x_{k}\right)$ by

$$
x_{k}= \begin{cases}\delta^{M / p_{k}}(\operatorname{sgn} t(n, k, r))|t(n, k, r)|^{q_{k}-1} S^{-1} N^{-q_{k} / p_{k}}, & 1 \leq k \leq k_{0}  \tag{3.20}\\ 0, & \text { for } k>k_{0}\end{cases}
$$

where $0<\delta<1$ and

$$
\begin{equation*}
S=\sum_{k=1}^{k_{0}}|t(n, k, r)|^{q_{k}} N^{-q_{k}} \tag{3.21}
\end{equation*}
$$

Then it is easy to see that $x \in \ell(p)$ and $g(x) \leq \delta$. Applying this sequence to (3.19), we get the condition (3.15).

This completes the proof of the theorem.

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