Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 678171, 21 pages
doi:10.1155/2012/678171

Research Article

Weighted Hardy and Potential Operators in Morrey Spaces

Natasha Samko

Department of Mathematics, Research Center CEAF, Instituto Superior Técnico, 1049-003 Lisbon, Portugal

Correspondence should be addressed to Natasha Samko, nsamko@gmail.com

Received 15 November 2009; Accepted 9 June 2010

Academic Editor: V. Stepanov

We study the weighted $p \to q$-boundedness of Hardy-type operators in Morrey spaces $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ (or $\mathcal{L}^{p,\lambda}(\mathbb{R}^1_+)$ in the one-dimensional case) for a class of almost monotonic weights. The obtained results are applied to a similar weighted $p \to q$-boundedness of the Riesz potential operator. The conditions on weights, both for the Hardy and potential operators, are necessary and sufficient in the case of power weights. In the case of more general weights, we provide separately necessary and sufficient conditions in terms of Matuszewska-Orlicz indices of weights.

1. Introduction

The well-known Morrey spaces $\mathcal{L}^{p,\lambda}$ introduced in [1] in relation to the study of partial differential equations and presented in various books, see [2–4], were widely investigated during the last decades, including the study of classical operators of harmonic analysis—maximal, singular, and potential operators—in these spaces; we refer for instance to papers [5–23], where Morrey spaces on metric measure spaces may be also found. Surprisingly, weighted estimates of these classical operators, in fact, were not studied. Just recently, in [24] we proved weighted $p \to p$-estimates in Morrey spaces for Hardy operators on $\mathbb{R}^1_+$ and one-dimensional singular operators (on $\mathbb{R}^1$ or on Carleson curves in the complex plane).

In this paper we develop an approach which allows us both to obtain weighted $p \to q$ – estimations of Hardy-type operators and potential operators and extend them to the multidimensional case, for the Hardy operators (related to integration over balls). Note that, in contrast with the case of Lebesgue spaces, Hardy inequalities in Morrey norms admit the value $p = 1$ for p; see Theorem 4.3. The progress in comparison with [24] is based on the pointwise estimation of the Hardy operators we present in Sections 3.2 and 4.1. Such an estimation is helpless in the case of Lebesgue spaces ($\lambda = 0$), but proved to be effective in the case of Morrey spaces ($0 < \lambda < n$). Roughly speaking, it is based on the simple fact that

$$|x|^{-n/p} \notin L^p(\lambda = 0), \quad \text{but } |x|^{(\lambda-n)/p} \in L^{p,\lambda}(\lambda > 0). \tag{1.1}$$

The admitted weights $\varphi(|x - x_0|)$ are generated by functions $\varphi(r)$ from the Bary-Stechkin-type class; they may be characterized as weights continuous and positive for $r \in (0, \infty)$, with possible decay or growth at $r = 0$ and $r = \infty$, which become almost increasing or almost decreasing after the multiplication by some power. Such weights are oscillating between two powers at the origin and infinity (with different exponents for the origin and infinity).

We also note that the obtained estimates show that the Hardy operators (with admitted weights) act boundedly not only in local and global Morrey spaces (see definitions in Section 3.1), but also from a larger local Morrey space into a more narrow global Morrey space (see Theorems 4.3 and 4.4).

The paper is organized as follows. In Section 2 we give necessary preliminaries on some classes of weight functions. In Section 3 we prove some statements on embedding of Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega)$ into some weighted $L^p(\Omega, \varrho)$-spaces. In Section 4 we prove theorems on the weighted $p \rightarrow q$-boundedness of Hardy operators in Morrey spaces. Finally, in Section 5 we apply the results of Section 4 to a similar weighted boundedness of potential operators. The conditions on weights, both for the Hardy and potential operators are necessary and sufficient in the case of power weights. In the case of more general weights, we provide separately necessary and sufficient conditions in terms of Matuszewska-Orlicz indices of weights.

The main results are given in Theorems 4.3, 4.4, 4.5, and 5.3 and Corollary 5.4.

2. Preliminaries on Weight Functions

2.1. Zygmund-Bary-Stechkin (ZBS) Classes and Matuszewska-Orlicz (MO) Type Indices

2.1.1. On Classes of the Type W_0 and W_∞

In the sequel, a nonnegative function f on $[0, \ell]$, $0 < \ell \leq \infty$, is called almost increasing (almost decreasing) if there exists a constant $C(\geq 1)$ such that $f(x) \leq Cf(y)$ for all $x \leq y$ ($x \geq y$, resp.). Equivalently, a function f is almost increasing (almost decreasing) if it is equivalent to an increasing (decreasing, resp.) function g, that is, $c_1 f(x) \leq g(x) \leq c_2 f(x)$, $c_1 > 0$, $c_2 > 0$.

Definition 2.1. Let $0 < \ell < \infty$.

(1) By $W_0 = W_0([0, \ell])$ one denotes the class of functions φ continuous and positive on $(0, \ell]$ such that there exists the finite limit $\lim_{x \to 0} \varphi(x)$, and $\varphi(x)$ is almost increasing on $(0, \ell)$;

(2) by $\widetilde{W}_0 = \widetilde{W}_0([0, \ell])$ one denotes the class of functions φ on $[0, \ell]$ such that $x^a \varphi(x) \in W_0$ for some $a = a(\varphi) \in \mathbb{R}^1$.

Definition 2.2. Let $0 < \ell < \infty$.

(1) By $W_\infty = W_\infty([\ell, \infty])$ one denotes the class of functions φ continuous and positive on $[\ell, \infty)$ which have the finite limit $\lim_{x\to\infty}\varphi(x)$, and $\varphi(x)$ is almost increasing on $[\ell, \infty)$;

(2) by $\widetilde{W}_\infty = \widetilde{W}_\infty([\ell, \infty))$ one denotes the class of functions $\varphi \in W_\infty$ such $x^a\varphi(x) \in W_\infty$ for some $a = a(\varphi) \in \mathbb{R}^1$.

2.1.2. ZBS Classes and MO Indices of Weights at the Origin

In this subsection we assume that $\ell < \infty$.

Definition 2.3. One says that a function $\varphi \in W_0$ belongs to the Zygmund class $\mathbb{Z}^\beta$, $\beta \in \mathbb{R}^1$, if

$$\int_0^x \frac{\varphi(t)}{t^{1+\beta}}dt \le c\frac{\varphi(x)}{x^\beta}, \qquad x \in (0, \ell), \tag{2.1}$$

and to the Zygmund class $\mathbb{Z}_\gamma$, $\gamma \in \mathbb{R}^1$, if

$$\int_x^\ell \frac{\varphi(t)}{t^{1+\gamma}}dt \le c\frac{\varphi(x)}{x^\gamma}, \qquad x \in (0, \ell). \tag{2.2}$$

One also denotes

$$\Phi_\gamma^\beta := \mathbb{Z}^\beta \bigcap \mathbb{Z}_\gamma, \tag{2.3}$$

the latter class being also known as Bary-Stechkin-Zygmund class [25].

It is known that the property of a function is to be almost increasing or almost decreasing after the multiplication (division) by a power function is closely related to the notion of the so called Matuszewska-Orlicz indices. We refer to [26, 27] [28, page 20], [29–32] for the properties of the indices of such a type. For a function $\varphi \in \widetilde{W}_0$, the numbers

$$\begin{aligned} m(\varphi) &= \sup_{0<x<1} \frac{\ln\left(\limsup_{h\to 0}(\varphi(hx)/\varphi(h))\right)}{\ln x} = \lim_{x\to 0}\frac{\ln\left(\limsup_{h\to 0}(\varphi(hx)/\varphi(h))\right)}{\ln x}, \\ M(\varphi) &= \sup_{x>1} \frac{\ln\left(\limsup_{h\to 0}(\varphi(hx)/\varphi(h))\right)}{\ln x} = \lim_{x\to \infty}\frac{\ln\left(\limsup_{h\to 0}(\varphi(hx)/\varphi(h))\right)}{\ln x}. \end{aligned} \tag{2.4}$$

are known as *the Matuszewska-Orlicz type lower and upper indices* of the function $\varphi(r)$. Note that in this definition $\varphi(x)$ needs not to be an N-function: only its behaviour at the origin is of importance. Observe that $0 \le m(\varphi) \le M(\varphi) \le \infty$ for $\varphi \in W_0$ and $-\infty < m(\varphi) \le M(\varphi) \le \infty$ for $\varphi \in \widetilde{W}_0$, and the formulas are valid

$$m[x^a\varphi(x)] = a + m(\varphi), \quad M[x^a\varphi(x)] = a + M(\varphi), \quad a \in \mathbb{R}^1, \tag{2.5}$$

$$m([\varphi(x)]^a) = am(\varphi), \quad M([\varphi(x)]^a) = aM(\varphi), \quad a \ge 0, \tag{2.6}$$

$$m\left(\frac{1}{\varphi}\right) = -M(\varphi), \qquad M\left(\frac{1}{\varphi}\right) = -m(\varphi), \tag{2.7}$$

$$m(uv) \ge m(u) + m(v), \qquad M(uv) \le M(u) + M(v) \tag{2.8}$$

for $\varphi, u, v \in \widetilde{W}_0$.

The following statement is known see [26, Theorems 3.1, 3.2, and 3.5]. (In the formulation of Theorem 2.4 in [26], it was supposed that $\beta \ge 0, \gamma > 0$, and $\varphi \in W_0$. It is evidently true also for $\varphi \in \widetilde{W}_0$ and all $\beta, \gamma \in \mathbb{R}^1$, in view of formulas (2.5).)

Theorem 2.4. *Let $\varphi \in \widetilde{W}_0$ and $\beta, \gamma \in \mathbb{R}^1$. Then*

$$\varphi \in \mathbb{Z}^\beta \Longleftrightarrow m(\varphi) > \beta, \qquad \varphi \in \mathbb{Z}_\gamma \Longleftrightarrow M(\varphi) < \gamma. \tag{2.9}$$

Besides this,

$$m(\varphi) = \sup\left\{\delta > 0 : \frac{\varphi(x)}{x^\delta} \text{ is almost increasing}\right\}, \tag{2.10}$$

$$M(\varphi) = \inf\left\{\lambda > 0 : \frac{\varphi(x)}{x^\lambda} \text{ is almost decreasing}\right\}, \tag{2.11}$$

and for $\varphi \in \Phi_\gamma^\beta$ the inequalities

$$c_1 x^{M(\varphi)+\varepsilon} \le \varphi(x) \le c_2 x^{m(\varphi)-\varepsilon} \tag{2.12}$$

hold with an arbitrarily small $\varepsilon > 0$ and $c_1 = c_1(\varepsilon), c_2 = c_2(\varepsilon)$.

2.1.3. ZBS Classes and MO Indices of Weights at Infinity

Following [14, Section 4.1] and [29, Section 2.2], we introduce the following definitions.

Definition 2.5. Let $-\infty < \alpha < \beta < \infty$. One puts $\Psi_\alpha^\beta := \widehat{\mathbb{Z}}^\beta \cap \widehat{\mathbb{Z}}_\alpha$, where $\widehat{\mathbb{Z}}^\beta$ is the class of functions $\varphi \in \widetilde{W}_\infty$ satisfying the condition

$$\int_x^\infty \left(\frac{x}{t}\right)^\beta \frac{\varphi(t)dt}{t} \le c\varphi(x), \qquad x \in (\ell, \infty), \tag{2.13}$$

and $\widehat{\mathbb{Z}}_\alpha$ is the class of functions $\varphi \in W([\ell,\infty))$ satisfying the condition

$$\int_\ell^x \left(\frac{x}{t}\right)^\alpha \frac{\varphi(t)\ dt}{t} \le c\varphi(x), \qquad x \in (\ell,\infty), \tag{2.14}$$

where $c = c(\varphi) > 0$ does not depend on $x \in [\ell,\infty)$.

The indices $m_\infty(\varphi)$ and $M_\infty(\varphi)$ responsible for the behavior of functions $\varphi \in \Psi_\alpha^\beta([\ell,\infty))$ at infinity are introduced in the way similar to (2.4):

$$\begin{aligned} m_\infty(\varphi) &= \sup_{x>1} \frac{\ln\left[\liminf_{h\to\infty}(\varphi(xh)/\varphi(h))\right]}{\ln x}, \\ M_\infty(\varphi) &= \inf_{x>1} \frac{\ln\left[\limsup_{h\to\infty}(\varphi(xh)/\varphi(h))\right]}{\ln x}. \end{aligned} \tag{2.15}$$

Properties of functions in the class $\Psi_\alpha^\beta([\ell,\infty))$ are easily derived from those of functions in $\Phi_\beta^\alpha\ ([0,\ell])$ because of the following equivalence:

$$\varphi \in \Psi_\alpha^\beta([\ell,\infty)) \Longleftrightarrow \varphi_* \in \Phi_{-\alpha}^{-\beta}\ ([0,\ell^*]), \tag{2.16}$$

where $\varphi_*(t) = \varphi(1/t)$ and $\ell_* = 1/\ell^\bullet$. Direct calculation shows that

$$m_\infty(\varphi) = -M(\varphi_*), \qquad M_\infty(\varphi) = -m(\varphi_*), \qquad \varphi_*(t) := \varphi\left(\frac{1}{t}\right). \tag{2.17}$$

Making use of (2.16) and (2.17), one can easily reformulate properties of functions of the class Φ_α^β near the origin, given in Theorem 2.4 for the case of the corresponding behavior at infinity of functions of the class Ψ_α^β and obtain that

$$c_1 t^{m_\infty(\varphi)-\varepsilon} \le \varphi(t) \le c_2 t^{M_\infty(\varphi)+\varepsilon}, \quad t \ge \ell,\ \varphi \in \widetilde{W}_\infty, \tag{2.18}$$

$$m_\infty(\varphi) = \sup\left\{\mu \in \mathbb{R}^1 :\ t^{-\mu}\varphi(t) \text{ is almost increasing on } [\ell,\infty)\right\}, \tag{2.19}$$

$$M_\infty(\varphi) = \inf\left\{\nu \in \mathbb{R}^1 :\ t^{-\nu}\varphi(t) \text{ is almost decreasing on } [\ell,\infty)\right\}. \tag{2.20}$$

We say that a function φ continuous in $(0,\infty)$ is in the class $\widetilde{W}_{0,\infty}(\mathbb{R}_+^1)$ if its restriction to $(0,1)$ belongs to $\widetilde{W}_0([0,1])$ and its restriction to $(1,\infty)$ belongs to $\widetilde{W}_\infty([1,\infty])$. For functions in $\widetilde{W}_{0,\infty}(\mathbb{R}_+^1)$, the notation

$$\mathbb{Z}^{\beta_0,\beta_\infty}\left(\mathbb{R}_+^1\right) = \mathbb{Z}^{\beta_0}([0,1]) \cap \mathbb{Z}^{\beta_\infty}([1,\infty)), \qquad \mathbb{Z}_{\gamma_0,\gamma_\infty}\left(\mathbb{R}_+^1\right) = \mathbb{Z}_{\gamma_0}([0,1]) \cap \mathbb{Z}_{\gamma_\infty}([1,\infty)) \tag{2.21}$$

has an obvious meaning. In the case, where the indices coincide: $\beta_0 = \beta_\infty := \beta$, we will simply write $\mathbb{Z}^\beta(\mathbb{R}^1_+)$ and similarly for $\mathbb{Z}_\gamma(\mathbb{R}^1_+)$. We also denote

$$\Phi^\beta_\alpha\left(\mathbb{R}^1_+\right) := \mathbb{Z}^\beta\left(\mathbb{R}^1_+\right) \cap \mathbb{Z}_\gamma\left(\mathbb{R}^1_+\right). \tag{2.22}$$

Making use of Theorem 2.4 for $\Phi^\beta_\alpha([0,1])$ and relations (2.17), we easily arrive at the following statement.

Lemma 2.6. *Let $\varphi \in \widetilde{W}_0(\mathbb{R}^1_+)$. Then*

$$\begin{aligned} \varphi \in \mathbb{Z}^{\beta_0,\beta_\infty}\left(\mathbb{R}^1_+\right) &\Longleftrightarrow m(\varphi) > \beta_0, \quad m_\infty(\varphi) > \beta_\infty\ , \\ \varphi \in \mathbb{Z}_{\gamma_0,\gamma_\infty}\left(\mathbb{R}^1_+\right) &\Longleftrightarrow M(\varphi) < \gamma_0, \quad M_\infty(\varphi) < \gamma_\infty. \end{aligned} \tag{2.23}$$

2.2. *On Classes* $V^\mu_\pm$

Note that we slightly changed the notation of the class introduced in the following definition, in comparison with its notation in [32].

Definition 2.7. Let $0 < \mu \le 1$. By $\mathbf{V}^\mu_\pm$, One denotes the classes of functions φ nonnegative on $[0,\ell]$, $0 < \ell \le \infty$, defined by the following conditions:

$$\mathbf{V}^\mu_+ : \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\mu} \le C\frac{\varphi(x_+)}{x^\mu_+}, \tag{2.24}$$

$$\mathbf{V}^\mu_- : \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\mu} \le C\frac{\varphi(x_-)}{x^\mu_+}, \tag{2.25}$$

where $x, y \in (0,\ell]$ and $x_+ = \max(x,y)$, $x_- = \min(x,y)$.

Lemma 2.8. *Functions $\varphi \in \mathbf{V}^\mu_+$ are almost increasing on $[0,\ell]$, and functions $\varphi \in \mathbf{V}^\mu_-$ are almost decreasing on $[0,\ell]$.*

Proof. Let $\varphi \in \mathbf{V}^\mu_+$ and $y \le x$. By (2.24) we have $|\varphi(x) - \varphi(y)| \le C\varphi(x)(1 - y/x)^\mu \le C\varphi(x)$. Then $\varphi(y) \le |\varphi(x) - \varphi(y)| + \varphi(x) \le (C+1)\varphi(x)$. The case $\varphi \in \mathbf{V}^\mu_-$ is similarly treated. □

Corollary 2.9. *Functions $\varphi \in \mathbf{V}^\mu_+$ have non-negative indices $0 \le m(\varphi) \le M(\varphi)$, and functions $\varphi \in \mathbf{V}^\mu_-$ have non-positive indices $m(\varphi) \le M(\varphi) \le 0$, the same being also valid with respect to the indices $m_\infty(\varphi), M_\infty(\varphi)$ in the case $\ell = \infty$.*

Note that

$$V^\mu_+ \subset V^\nu_+, \quad V^\mu_- \subset V^\nu_-, \quad 0 < \nu < \mu \le 1, \tag{2.26}$$

the classes $V_{\pm}^{\mu}$ being trivial for $\mu > 1$. We also have

$$x^{\gamma} \in \bigcup_{\mu\in[0,1]} \mathbf{V}_{+}^{\mu} \Longleftrightarrow \gamma \geq 0, \qquad x^{\gamma} \in \bigcup_{\mu\in[0,1]} \mathbf{V}_{-}^{\mu} \Longleftrightarrow \gamma \leq 0, \tag{2.27}$$

which follows from the fact that $x^{\gamma} \in \mathbf{V}_{+}^{1} \Leftrightarrow \gamma \geq 0$ and $x^{\gamma} \in \mathbf{V}_{-}^{1} \Leftrightarrow \gamma \leq 0$ (see [24, Subsection 2.3, Remark 2.8]) and property (2.26).

An example of a function which is in $\mathbf{V}_{+}^{\mu}$ with some $\mu > 0$, but does not belong to the total intersection $\bigcap_{\mu\in[0,1]} \mathbf{V}_{+}^{\mu}$ is given by

$$\varphi(x) = ax^{\gamma} + b|x - x_0|^{\beta} \in \bigcap_{\mu\in[0,\beta]} \mathbf{V}_{+}^{\mu}, \tag{2.28}$$

where $x_0 > 0$ and $\gamma \geq 0,\ 0 < \beta < 1,\ a > 0,\ b > 0$.

The following lemmas (see [24], Lemmas 2.10 and 2.11) show that conditions (2.24) and (2.25) are fulfilled with $\mu = 1$ not only for power functions but also for an essentially larger class of functions (which in particular may oscillate between two power functions with different exponents). Note that the information about this class in Lemmas 2.10 and 2.11 is given in terms of increasing or decreasing functions, without the word "almost".

Lemma 2.10. *Let $\varphi \in W$. Then*

(i) *$\varphi \in \mathbf{V}_{+}^{1}$ in the case φ is increasing and the function $\varphi(x)/x^{\nu}$ is decreasing for some $\nu \geq 0$;*

(ii) *$\varphi \in \mathbf{V}_{-}^{1}$ in the case $\varphi(x)$ is decreasing and there exists a number $\mu \geq 0$ such that $x^{\mu}\varphi(x)$ is increasing.*

Lemma 2.11. *Let $\varphi \in W \cap C^1((0,\ell])$. If there exist $\varepsilon > 0$ and $\nu \geq 0$ such that $0 \leq \varphi'(x)/\varphi(x) \leq \nu/x$ for $0 < x \leq \varepsilon$, then $\varphi \in \mathbf{V}_{+}^{1}$. If there exist $\varepsilon > 0$ and $\mu \geq 0$ such that $-\mu/x \leq \varphi'(x)/\varphi(x) \leq 0$ for $0 < x \leq \varepsilon$, then $\varphi \in \mathbf{V}_{-}^{1}$.*

3. On Weighted Integrability of Functions in Morrey Spaces

3.1. Definitions and Belongness of Some Functions to Morrey Spaces

Let Ω be an open set in $\mathbb{R}^n$.

Definition 3.1. The Morrey spaces $\mathcal{L}^{p,\lambda}(\Omega), \mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\Omega)$ $1 \leq p < \infty,\ 0 \leq \lambda < n$, are defined as the space of functions $f \in L_{\mathrm{loc}}^{p}(\Omega)$ such that

$$\begin{aligned} \|f\|_{p,\lambda} &= \sup_{x\in\Omega, r>0} \left(\frac{1}{r^{\lambda}} \int_{\widetilde{B}(x,r)} |f(y)|^p dy \right)^{1/p}, \\ \|f\|_{p,\lambda;\,\mathrm{loc}} &= \sup_{r>0} \left(\frac{1}{r^{\lambda}} \int_{\widetilde{B}(0,r)} |f(y)|^p dy \right)^{1/p}, \end{aligned} \tag{3.1}$$

respectively, where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$.

Obviously,

$$\mathcal{L}^{p,\lambda}(\Omega) \subset \mathcal{L}_0^{p,\lambda}\operatorname{loc}(\Omega). \tag{3.2}$$

The spaces $\mathcal{L}^{p,\lambda}(\Omega)$, $\mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\Omega)$ are known under the names of *global and local Morrey spaces*; see for instance, [9, 10].

The weighted Morrey space is defined as

$$L^{p,\gamma}(\Omega,\omega) = \{f : \omega f \in L^{p,\gamma}(\Omega)\}. \tag{3.3}$$

Remark 3.2. As is well known, the space $\mathcal{L}^{p,\lambda}(\Omega)$ as defined above is not necessarily embedded into $L^p(\Omega)$, in the case when Ω is unbounded. A typical counterexample in the case $\Omega = \mathbb{R}^n$ is

$$f(x) = |x|^{(\lambda-n)/p} \in \mathcal{L}^{p,\lambda}(\mathbb{R}^n). \tag{3.4}$$

Indeed, we have

$$\|f\|_{p,\lambda} = \sup_{x,r}\left(\frac{1}{r^\lambda}\int_{B(x,r)} |y|^{\lambda-n} dy\right)^{1/p} \tag{3.5}$$

which is bounded (when $|x| \geq 2r$, take into account that $|y| \geq r$, and when $|x| \leq 2r$, make use of the inclusion $B(x,r) \subset B(0,3r)$).

Lemma 3.3. *Let* $\ell = \operatorname{diam}\Omega < \infty$, $u \in \widetilde{W}_0(0,\ell)$, *and* $x_0 \in \overline{\Omega}$. *The condition*

$$m(u) > \frac{\lambda - n}{p} \tag{3.6}$$

is sufficient for the function $f(x) = u(|x - x_0|)$ *to belong to* $\mathcal{L}^{p,\lambda}(\Omega), 0 \leq \lambda < n$. *In the case* $u(t) = t^\gamma$, *the inclusion* $|x - x_0|^\gamma \in \mathcal{L}^{p,\lambda}(\Omega)$ *with* $0 < \lambda < n$ *holds if* $\gamma \geq (\lambda - n)/p$, *the latter condition being necessary, when* x_0 *is an inner point of* Ω *or* $n = 1$ *and* $\Omega = (a,b), -\infty \leq a < b \leq \infty$.

Proof. We have

$$\|f\|_{p,\lambda} = \sup_{x\in\Omega, r>0}\left(\frac{1}{r^\lambda}\int_{\widetilde{B}(x,r)} u^p(|y - x_0|)dy\right)^{1/p} \leq \sup_{x\in\Omega_{x_0}, r>0}\left(\frac{1}{r^\lambda}\int_{B(x,r)} u^p(|y|)dy\right)^{1/p}, \tag{3.7}$$

where $\Omega_{x_0} = \{x : x + x_0 \in \Omega\}$. Then

$$\|f\|_{p,\lambda} \leq C \sup_{x\in\Omega_{x_0}, r>0}\left(\frac{1}{r^\lambda}\int_{B(x,r)} |y|^{pm(u)-p\varepsilon}\, dy\right)^{1/p}, \tag{3.8}$$

by (2.12). If $m(u) > 0$, we choose $0 < \varepsilon < m(u)$ and then the right-hand side of the last inequality is bounded. So let $m(u) \leq 0$. We distinguish the cases (1) $|x| \geq 2r$ and (2) $|x| \leq 2r$. In the case (1), $|y| \geq |x| - |x-y| \geq r$. Therefore,

$$\|f\|_{p,\lambda} \leq C \sup_{x\in\Omega, r>0} \left(\frac{r^{pm(u)-p\varepsilon}}{r^{\lambda}} \int_{B(x,r)} dy \right)^{1/p} = C\sup_{r>0} r^{m(u)+(n-\lambda)/p-\varepsilon} \tag{3.9}$$

which is bounded under the choice $\varepsilon < m(u) + (n-\lambda)/p$. In the case (2), we observe that $B(x,r) \subset B(0,3r)$ and then the same estimate $\|f\|_{p,\lambda} \leq C\sup_{r>0} r^{m(u)+(n-\lambda)/p-\varepsilon}$ follows.

In the case $u(t) = t^{\gamma}$, the proof of the "if" part follows the same lines as above with $\varepsilon = 0$. To prove the "only if" part, it suffices to observe that

$$\|f\|_{p,\lambda} \geq \sup_{0<r<\delta} \left(\frac{1}{r^{\lambda}} \int_{B(x_0,r)} |y-x_0|^{p\gamma} dy \right)^{1/p} \geq C \sup_{0<r<\delta} r^{\gamma+(n-\lambda)/p}. \tag{3.10}$$

□

Corollary 3.4. *If $u \in \widetilde{W}_0(0,\ell)$ and there exists an $a < (n-\lambda)/p$ such that $t^a u(t)$ is almost increasing, then $u(|x-x_0|) \in \mathcal{L}^{p,\lambda}$, $x_0 \in \overline{\Omega}$.*

To derive this corollary from Lemma 3.3, it suffices to refer to formula (2.10).

3.2. *Some Weighted Estimates of Functions in Morrey Spaces*

Lemma 3.5. *Let $1 \leq p < \infty, 0 < s \leq p,\ 0 \leq \lambda < n$, and $v \in \widetilde{W}_0([0,\ell]),\ 0 < \ell \leq \infty$. Then*

$$\left(\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)}\, dz \right)^{1/s} \leq c\mathcal{A}(|y|) \|f\|_{p,\lambda;\,\mathrm{loc}}, \quad 0 < |y| \leq \ell, \tag{3.11}$$

where $C > 0$ does not depend on y and f and

$$\mathcal{A}(r) = \left(\int_0^r \frac{t^{n-1-((n-\lambda)/p)s} dt}{v(t)} \right)^{1/s} \tag{3.12}$$

under the assumption that the last integral converges.

Proof. We have

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)} dz = \sum_{k=0}^{\infty} \int_{B_k(y)} \frac{|f(z)|^s}{v(|z|)} dz, \tag{3.13}$$

where $B_k(y) = \{z : 2^{-k-1}|y| < |z| < 2^{-k}|y|\}$. Making use of the fact that there exists a β such that $t^{\beta}v(t)$ is almost increasing, we observe that

$$\frac{1}{v(|z|)} \leq \frac{C}{v(2^{-k-1}|y|)}. \tag{3.14}$$

Applying this in (3.13) and making use of the Hölder inequality with the exponent $p/s \geq 1$, we obtain

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)} dz \leq C \sum_{k=0}^{\infty} \frac{(2^{-k-1}|y|)^{n(1-s/p)}}{v(2^{-k-1}|y|)} \left(\int_{B_k(y)} |f(z)|^p dz \right)^{s/p}. \tag{3.15}$$

Hence,

$$\int_{|z|<|y|} \frac{|f(z)|^s}{v(|z|)} dz \leq C \sum_{k=0}^{\infty} \frac{(2^{-k-1}|y|)^{n-(n-\lambda)s/p}}{v(2^{-k-1}|y|)} \|f\|^s_{p,\lambda;\,\mathrm{loc}}. \tag{3.16}$$

It remains to prove that

$$\sum_{k=0}^{\infty} \frac{(2^{-k-1}|y|)^{n-(n-\lambda)s/p}}{v(2^{-k-1}|y|)} \leq C[\mathcal{A}(|y|)]^s. \tag{3.17}$$

We have

$$\int_0^{|y|} \frac{t^{n-1-((n-\lambda)/p)s} dt}{v(t)} = \sum_{k=0}^{\infty} \int_{2^{-k-1}|y|}^{2^{-k}|y|} \frac{t^{n-1-((n-\lambda)/p)s} dt}{v(t)}. \tag{3.18}$$

Making use of the fact that $t^{\beta}v(t)$ is almost increasing with some β, we easily obtain that

$$\int_0^{|y|} \frac{t^{n-1-((n-\lambda)/p)s} dt}{v(t)} \geq C \sum_{k=0}^{\infty} \frac{(2^{-k}|y|)^{n-((n-\lambda)/p)s}}{v(2^{-k}|y|)} \geq C \sum_{k=0}^{\infty} \frac{(2^{-k-1}|y|)^{n-((n-\lambda)/p)s}}{v(2^{-k-1}|y|)}, \tag{3.19}$$

which proves (3.17). □

Corollary 3.6. *Let $1 \leq p < \infty$, $0 < s \leq p$, $0 \leq \lambda < n$, and $a < n/s - (n-\lambda)/p$. Then*

$$\left(\int_{|t|<|y|} \left(\frac{|f(t)|}{|t|^a} \right)^s dt \right)^{1/s} \leq c|y|^{n/s-(n-\lambda)/p-a} \|f\|_{p,\lambda;\,\mathrm{loc}}, \quad 0 < |y| \leq \ell \leq \infty. \tag{3.20}$$

Lemma 3.7. *Let $1 \leq p < \infty$, $0 \leq s \leq p$, $0 \leq \lambda < n$, and $v \in \widetilde{W}_0(\mathbb{R}^1_+)$. Then*

$$\left(\int_{|z|>|y|} v(|z|)|f(z)|^s dz \right)^{1/s} \leq c\mathcal{B}(|y|)\|f\|_{p,\lambda;\mathrm{loc}}, \quad y \neq 0, \tag{3.21}$$

where $C > 0$ does not depend on y and f and

$$\mathcal{B}(y) = \left(\int_{|y|}^{\infty} t^{n-1-((n-\lambda)/p)s} v(t)\,dt \right)^{1/s}. \tag{3.22}$$

Proof. The proof is similar to that of Lemma 3.5. We have

$$\int_{|z|>|y|} v(z)\left|f(z)\right|^{s} dz = \sum_{k=0}^{\infty} \int_{B^k(y)} v(z)\left|f(z)\right|^{s} dz, \tag{3.23}$$

where $B^k(y) = \{z : 2^k|y| < |z| < 2^{k+1}|y|\}$. Since there exists a $\beta \in \mathbb{R}^1$ such that $t^{\beta}v(t)$ is almost increasing, we obtain

$$\int_{B^k(y)} v(|z|)\left|f(z)\right|^{s} dz \le C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right) \int_{B^k(y)} \left|f(z)\right|^{s} dz, \tag{3.24}$$

where C may depend on β, but does not depend on y and f. Applying the Hölder inequality with the exponent p/s, we get

$$\begin{aligned}\int_{|z|>|y|} v(|z|)\left|f(z)\right|^{s} dz &\le C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n(1-s/p)} \left(\int_{B^k(y)} \left|f(z)\right|^{p} dz \right)^{s/p} \\ &\le C\sum_{k=0}^{\infty} v(2^{k+1}|y|)(2^k|y|)^{n-((n-\lambda)/p)s} \|f\|_{p,\lambda;\mathrm{loc}}.\end{aligned} \tag{3.25}$$

It remains to prove that

$$\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n-((n-\lambda)/p)s} \le C\int_{|y|}^{\infty} t^{n-1-((n-\lambda)/p)s} v(t)\,dt. \tag{3.26}$$

We have

$$\begin{aligned}\int_{|y|}^{\infty} t^{n-1-((n-\lambda)/p)s} v(t)\,dt &= \sum_{k=0}^{\infty} \int_{2^k|y|}^{2^{k+1}|y|} t^{n-1-((n-\lambda)/p)s} v(t)\,dt \\ &\ge C\sum_{k=0}^{\infty} v\left(2^k|y|\right) \int_{2^k|y|}^{2^{k+1}|y|} t^{n-1-((n-\lambda)/p)s}\,dt \\ &= C\sum_{k=0}^{\infty} v\left(2^k|y|\right)\left(2^k|y|\right)^{n-((n-\lambda)/p)s} \\ &\ge C\sum_{k=0}^{\infty} v\left(2^{k+1}|y|\right)\left(2^k|y|\right)^{n-((n-\lambda)/p)s},\end{aligned} \tag{3.27}$$

which completes the proof. □

Remark 3.8. The analysis of the proof shows that estimate (3.21) remains in force, if the assumption $v \in \widetilde{W}_0(\mathbb{R}^1_+)$ is replaced by the condition that $1/v \in \widetilde{W}_0(\mathbb{R}^1_+)$ and v satisfies the doubling condition $v(2t) \le cv(t)$.

Corollary 3.9. *Let $1 \le p < \infty$, $0 < s \le p$, and $b < (n-\lambda)/p - n/s$. Then*

$$\left(\int_{|z|>|y|} \left(|z|^b |f(z)|\right)^s dz\right)^{1/s} \le c y^{b+n/s-(n-\lambda)/p} \|f\|_{p,\lambda;\mathrm{loc}}, \quad y \ne 0. \tag{3.28}$$

4. On Weighted Hardy Operators in Morrey Spaces

4.1. Pointwise Estimations

We consider the generalized Hardy operators

$$H^{\alpha}_{\varphi} f(x) = |x|^{\alpha-n}\varphi(|x|)\int_{|y|<|x|} \frac{f(t)dt}{\varphi(|t|)}, \qquad \mathcal{H}^{\alpha}_{\varphi} f(x) = |x|^{\alpha}\varphi(|x|)\int_{\mathbb{R}^n} \frac{f(t)dt}{|t|^n\varphi(|t|)}. \tag{4.1}$$

In the sequel $\mathbb{R}^n$ with $n = 1$ may be read either as $\mathbb{R}^1$ or $\mathbb{R}^1_+$ with the operators interpreted as

$$H^{\alpha}_{\varphi} f(x) := x^{\alpha-1}\varphi(x)\int_0^x \frac{f(y)}{\varphi(y)}\, dy, \quad \mathcal{H}^{\alpha}_{\varphi} f(x) := x^{\alpha}\varphi(x)\int_x^\infty \frac{f(y)}{\varphi(y)y} dy, \quad x > 0. \tag{4.2}$$

In the case $\varphi(t)$ is a power function, we also use the notation

$$H^{\alpha}_{(\gamma)} f(x) := |x|^{\gamma+\alpha-n}\int_{|y|<|x|} \frac{f(y)}{|y|^{\gamma}} dy, \qquad \mathcal{H}^{v}_{(\gamma)} f(x) := |x|^{\gamma+\alpha}\int_{|y|>|x|} \frac{f(y)}{|y|^{\gamma+n}} dy \tag{4.3}$$

and their one-dimensional versions

$$H^{\alpha}_{(\gamma)} f(x) := x^{\gamma+\alpha-1}\int_0^x \frac{f(y)}{y^{\gamma}} dy, \quad \mathcal{H}^{\alpha}_{(\gamma)} f(x) := x^{\gamma+\alpha}\int_x^\infty \frac{f(y)}{y^{\gamma+1}} dy, \quad x > 0 \tag{4.4}$$

adjusted for the half-axis $\mathbb{R}^1_+$.

Lemma 4.1. *Let $1 \le p < \infty$ and $0 < \lambda < n$.*

(I) *Let $\varphi \in \widetilde{W}_0$. Then the Hardy operator H^{α}_{φ} is defined on the space $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ or on the space $\mathcal{L}^{p,\lambda}_{\mathrm{loc}}(\mathbb{R}^n)$, if and only if*

$$\int_{0+} \frac{t^{n-1-(n-\lambda)/p}}{\varphi(t)} dt < \infty, \tag{4.5}$$

and in this case

$$\left|H_{\varphi}^{\alpha}(x)\right| \leq C|x|^{\alpha-n}\varphi(|x|)\int_{0}^{|x|}\frac{t^{n-1-(n-\lambda)/p}}{\varphi(t)}dt\|f\|_{p,\lambda;\,\mathrm{loc}}. \tag{4.6}$$

(II) *Let* $1/\varphi \in \widetilde{W}_0$ *or* $\varphi \in \widetilde{W}_0$ *and* $\varphi(2t) \leq C\varphi(t)$. *Then the Hardy operator* $\mathcal{H}_{\varphi}^{\alpha}$ *is defined on the space* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ *or on the space* $\mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\mathbb{R}^n)$, *if and only if*

$$\int_{\varepsilon}^{\infty}\frac{t^{n-1-(n-\lambda)/p}}{\varphi(t)}dt < \infty \tag{4.7}$$

for every $\varepsilon > 0$ *and in this case*

$$\left|\mathcal{H}_{\varphi}^{\alpha}(x)\right| \leq C|x|^{\alpha}\varphi(|x|)\int_{|x|}^{\infty}\frac{t^{n-1-(n-\lambda)/p}}{\varphi(t)}dt\|f\|_{p,\lambda;\,\mathrm{loc}}. \tag{4.8}$$

Proof. (I) *The "If" Part*. The sufficiency of condition (4.5) and estimate (4.6) follow from (3.12) under the choice $s = 1$ and $v(t) = \varphi(t)$.

The "Only If" Part. We choose a function $f(x)$ equal to $|x|^{(\lambda-n)/p}$ in a neighborhood of the origin and zero beyond this neighborhood. Then $f \in \mathcal{L}^{p,\lambda}$ by Lemma 3.3. For this function f, the existence of the integral $H_{\varphi}^{\alpha}f$ is equivalent to condition (4.5).

(II) *The "If" part*. The sufficiency of condition (4.7) and estimate (4.8) follow from (3.21) under the choice $s = 1$ and $v(t) = 1/t^n\varphi(t)$.

The "Only If" Part. We choose a function $f(x)$ equal to $x^{(\lambda-n)/p}$ in a neighborhood of infinity and zero beyond this neighborhood. Then $f \in \mathcal{L}^{p,\lambda}$ by Remark 3.2. For this function f, the existence of the integral $\mathcal{H}_{\varphi}^{\alpha}f$ is nothing else but condition (4.7). □

Corollary 4.2. *(I) The Hardy operator* $H_{(\gamma)}^{\alpha}$ *is defined on the space* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ *or on the space* $\mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\mathbb{R}^n)$, *if and only if* $\gamma < n/p' + \lambda/p$, *and in this case*

$$\left|H_{(\gamma)}^{\alpha}f(x)\right| \leq C|x|^{\alpha-(n-\lambda)/p}\|f\|_{p,\lambda;\,\mathrm{loc}}. \tag{4.9}$$

(II) The Hardy operator $\mathcal{H}_{(\gamma)}^{\alpha}$ *is defined on the space* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ *or on the space* $\mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\mathbb{R}^n)$, *if and only if* $\gamma > \lambda - n/p$, *and in this case*

$$\left|\mathcal{H}_{(\gamma)}^{\alpha}f(x)\right| \leq C|x|^{\alpha-(n-\lambda)/p}\|f\|_{p,\lambda;\,\mathrm{loc}}. \tag{4.10}$$

4.2. Weighted $p \to q$-Estimates for Hardy Operators in Morrey Spaces

The statements of Theorem 4.3 are well known in the case of Lebesgue space $\lambda = 0$ when $1 < p < n/\alpha$; see, for instance, [33, p. 6, 54]. As can be seen from the results below, inequalities for the Hardy operators in Morrey spaces admit the case $p = 1$ when $\lambda > 0$.

4.2.1. The Case of Power Weights

Theorem 4.3. *Let $0 < \lambda < n$, $0 < \alpha < n - \lambda$, and $1 \le p < (n-\lambda)/\alpha$. The operator $H^{\alpha}_{(\gamma)}$ ($\mathcal{H}^{\alpha}_{(\gamma)}$, resp.) is bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ or $\mathcal{L}^{p,\lambda}_{\text{loc}}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$, where $1/q = 1/p - \alpha/(n-\lambda)$, if and only if $\gamma < n/p' + \lambda/p$ ($\gamma > (\lambda - n)/p$, resp.).*

Proof. The "only if" part follows from Corollary 4.2 and the "if part" from (4.9) and (4.10), since $|x|^{\alpha-(n-\lambda)/p} = |x|^{(n-\lambda)/p} \in \mathcal{L}^{q,\lambda}(\mathbb{R}^n)$; see Remark 3.2. □

4.2.2. The Case of General Weights

We first deal with Hardy operators on a ball $B(0,\ell)$, $0 < \ell < \infty$ of a finite radius ℓ.

Theorem 4.4. *Let $0 < \lambda < n$, $0 < \alpha < n - \lambda$, and $1 \le p < (1-\lambda)/\alpha$ and $\varphi \in \widetilde{W}_0$. Then the weighted Hardy operators H^{α}_{φ} and $\mathcal{H}^{\alpha}_{\varphi}$ are bounded from $\mathcal{L}^{p,\lambda}(B(0,\ell))$ or $\mathcal{L}^{p,\lambda}_{\text{loc}}(B(0,\ell))$ to $\mathcal{L}^{q,\lambda}(B(0,\ell))$, where $1/q = 1/p - \alpha/(n-\lambda)$, if*

$$\varphi \in \mathbb{Z}_{\lambda/p+n/p'}, \qquad \varphi \in \mathbb{Z}^{(\lambda-n)/p}, \tag{4.11}$$

respectively, or, equivalently,

$$M(\varphi) < \frac{\lambda}{p} + \frac{n}{p'} \quad \text{for the operator } H^{\alpha}_{\varphi}, \tag{4.12}$$

$$m(\varphi) > \frac{\lambda - n}{p} \quad \text{for the operator } \mathcal{H}^{\alpha}_{\varphi}. \tag{4.13}$$

The conditions

$$m(\varphi) \le \frac{\lambda}{p} + \frac{n}{p'}, \qquad M(\varphi) \ge \frac{\lambda}{p} - \frac{n}{p} \tag{4.14}$$

are necessary for the boundedness of the operators H^{α}_{φ} and $\mathcal{H}^{\alpha}_{\varphi}$, respectively.

Proof. By (2.10) and (2.11), the function $\varphi(t)/t^{m(\varphi)-\varepsilon}$ is almost increasing, while $\varphi(t)/t^{M(\varphi)+\varepsilon}$ is almost decreasing for every $\varepsilon > 0$. Consequently,

$$C_1 \frac{r^{m(\varphi)-\varepsilon}}{t^{m(\varphi)-\varepsilon}} \le \frac{\varphi(r)}{\varphi(t)} \le C_2 \frac{r^{M(\varphi)+\varepsilon}}{t^{M(\varphi)+\varepsilon}} \tag{4.15}$$

for $0 < t \le r$ and then

$$C_1|x|^{m(\varphi)-\varepsilon+\alpha-n} \int_{B(0,|x|)} \frac{f(y)dt}{|y|^{m(\varphi)-\varepsilon}} \le H_\varphi^\alpha f(x) \le C_2|x|^{M(\varphi)+\varepsilon+\alpha-n} \int_{B(0,|x|)} \frac{f(y)dy}{|y|^{M(\varphi)+\varepsilon}} \tag{4.16}$$

supposing that $f(y) \ge 0$. From the right-hand side inequality in (4.16) and Theorem 4.3, we obtain that the operator H_φ^α is bounded if $M(\varphi)+\varepsilon < \lambda/p + 1/p'$, which is satisfied under the choice of $\varepsilon > 0$ sufficiently small, the latter being possible by (4.12). It remains to recall that condition (4.12) is equivalent to the assumption $\varphi \in \mathbb{Z}_{\lambda/p+1/p'}$ by Theorem 2.4. The necessity of the condition $m(\varphi) \le \lambda/p + n/p'$ follows from the left-hand side inequality in (4.16).
The case of the operator $\mathcal{H}_\varphi$ is similarly treated. □

In the case of the whole space ($\ell = \infty$), we admit that the weight $\varphi(|x|)$ may have an "oscillation between power functions" different at the origin and infinity. Correspondingly, the behavior at the origin and infinity is characterized by different indices $m(\varphi), M(\varphi)$ and $m_\infty(\varphi), M_\infty(\varphi)$, as described in Section 2.1.3.

Theorem 4.5. *Let $0 < \lambda < n$, $0 < \alpha < n - \lambda$, and $1 \le p < (1-\lambda)/\alpha$ and $\varphi \in \widetilde{W}_{0,\infty}(\mathbb{R}_+^1)$. Then the weighted Hardy operators H_φ^α and $\mathcal{H}_\varphi^\alpha$ are bounded from $\mathcal{L}^{p,\lambda}(\mathbb{R}^n)$ or $\mathcal{L}_{\mathrm{loc}}^{p,\lambda}(\mathbb{R}^n)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^n)$, $1/q = 1/p - \alpha/(n-\lambda)$, if*

$$\varphi \in \mathbb{Z}_{\lambda/p+n/p'}\left(\mathbb{R}_+^1\right) \qquad \varphi \in \mathbb{Z}^{(\lambda-n)/p}\left(\mathbb{R}_+^1\right), \tag{4.17}$$

respectively, or, equivalently,

$$\begin{aligned} \max(M(\varphi), M_\infty(\varphi)) &< \frac{\lambda}{p} + \frac{n}{p'} \quad \text{for the operator } H_\varphi^\alpha, \\ \min(m(\varphi), m_\infty(\varphi)) &> \frac{\lambda - n}{p} \quad \text{for the operator } \mathcal{H}_\varphi^\alpha. \end{aligned} \tag{4.18}$$

The conditions

$$\max(m(\varphi), m_\infty(\varphi)) \le \frac{\lambda}{p} + \frac{n}{p'}, \qquad \min(M(\varphi), M_\infty(\varphi)) \ge \frac{\lambda}{p} - \frac{n}{p} \tag{4.19}$$

are necessary for the boundedness of the operators H_φ^α and $\mathcal{H}_\varphi^\alpha$, respectively.

Proof. The restriction of $H_\varphi^\alpha f(x)$ to $B(0,1)$ is covered by Theorem 4.4, so that it suffices to estimate $\|H_\varphi^\alpha f\|_{\mathcal{L}^{p,\lambda}(\mathbb{R}^n \setminus B(0,1))}$. For $|x| > 1$ we have

$$H_\varphi^\alpha f(x) = C(f)|x|^{\alpha-n}\varphi(|x|) + |x|^{\alpha-n} \int_{1<|y|<|x|} \frac{\varphi(|x|)}{\varphi(|y|)} f(y)dy, \tag{4.20}$$

where $C(f)=\int_{B(0,1)} f(y)/\varphi(|y|)dy$. By Lemma 3.5 we have $|C(f)|\leq c\|f\|_{p,\lambda;\mathrm{loc}}\int_0^1 t^{n-1-(1-\lambda)/p}\,dt/\varphi(t)$, where the integral converges since $\varphi(t)\geq Ct^{M(\varphi)+\varepsilon}$ with an arbitrarily small $\varepsilon>0$ and $(n-\lambda)/p+M(\varphi)<1$. Then

$$\left|C(f)x^{\alpha-n}\varphi(x)\right|\leq cx^{\alpha-n+M_\infty(\varphi)+\varepsilon}\|f\|_{p,\lambda;\mathrm{loc}} \tag{4.21}$$

by (2.18). Here $x^{\alpha-n+M_\infty(\varphi)+\varepsilon}\in\mathcal{L}^{q,\lambda}(1,\infty)$, since $\alpha-1+M_\infty(\varphi)+\varepsilon<(\lambda-n)/q$ for sufficiently small ε; see Remark 3.2.

To deal with the second term in (4.20), it suffices to observe that for $1\leq|y|\leq|x|<\infty$ we have inequality (4.15) with $m(\varphi),M(\varphi)$ replaced by $m_\infty(\varphi),M_\infty(\varphi)$ and then the proof follows the same lines as in Theorem (4.4) after formula (4.16).

The operator $\mathcal{H}_\varphi^\alpha$ is considered in a similar way. □

5. Application to Potential Operators

We consider the potential operator

$$I^\alpha f(x):=\int_{\mathbb{R}^n}\frac{f(y)dy}{|x-y|^{n-\alpha}},\qquad 0<\alpha<n, \tag{5.1}$$

and in Theorem 5.3 show that its weighted boundedness in Morrey spaces—in the case of weights $\varphi\in V_+^\mu\cup V_-^\mu$ with $\mu=\min\{1,n-\alpha\}$—is a consequence of the nonweighted boundedness due to Adams [5] and the weighted boundedness of Hardy operators provided by Theorem 4.5.

The necessity of the boundedness of the Hardy operators for that of potential operators is a consequence of the following simple fact, where $X=X(\mathbb{R}^n)$ and $Y=Y(\mathbb{R}^n)$ are arbitrary Banach function spaces in the sense of Luxemburg (cf., e.g., [34]).

Lemma 5.1. *Let $w=w(x)$ be any weight function. For the boundedness of the weighted potential operator $wI^\alpha(1/w)$ from X to Y, it is necessary that the Hardy operators H_w^α and $\mathcal{H}_{w_\alpha}^\alpha$ are bounded from X to Y, where $w_\alpha(x)=|x|^{-\alpha}w(x)$.*

The proof of the sufficiency of the obtained conditions is based on the pointwise estimate of the following lemma.

Lemma 5.2. *Let $w\in\mathbf{V}_-^\mu\cup\mathbf{V}_+^\mu$ with $\mu=\min\{1,n-\alpha\}$ be a weight and f a non-negative function. Then the following pointwise estimate holds:*

$$wI^\alpha\frac{1}{w}f(x)\leq I^\alpha f(x)+c\begin{cases}H_w^\alpha f(x)+\mathcal{H}_{-\alpha}^\alpha f(x), & \text{if } w\in\mathbf{V}_+^\mu,\\[2ex] H^\alpha f(x)+\mathcal{H}_{w_\alpha}^\alpha f(x), & \text{if } w\in\mathbf{V}_-^\mu.\end{cases} \tag{5.2}$$

Proof. We have

$$\left(wI^{\alpha}\frac{1}{w}-I^{\alpha}\right)f(x)=\int_{\mathbb{R}^n}\mathcal{K}(x,y)f(y)dy,\qquad \mathcal{K}(x,y)=\frac{w(|x|)-w(|y|)}{w(|y|)|x-y|^{n-\alpha}}. \tag{5.3}$$

We first consider the case $n-\alpha\le 1$. For $w\in\mathbf{V}_-^{\mu}\cup\mathbf{V}_+^{\mu}$ with $\mu=n-\alpha$ in this case, by the definition of the classes $\mathbf{V}_{\pm}^{n-\alpha}$, we have

$$\begin{aligned}
&\mathcal{K}(x,y)\le c\begin{cases}|x|^{\alpha-n}\dfrac{w(|x|)}{w(|y|)}, & |y|<|x|,\\ |y|^{\alpha-n}, & |y|>|x|,\end{cases}\quad when\ w\in\mathbf{V}_+^{n-\alpha},\\
&\mathcal{K}(x,y)\le c\begin{cases}|x|^{\alpha-n}, & |y|<|x|,\\ |y|^{\alpha-n}\dfrac{w(|x|)}{w(|y|)}, & |y|>|x|,\end{cases}\quad when\ w\in\mathbf{V}_-^{n-\alpha},
\end{aligned} \tag{5.4}$$

which yield

$$\left|\left(wI^{\alpha}\frac{1}{w}-I^{\alpha}\right)f(x)\right|\le c\begin{cases}H_w^{\alpha}f(x)+\mathcal{H}_{-\alpha}^{\alpha}f(x), & \text{if } w\in\mathbf{V}_+^{n-\alpha},\\[2ex] H^{\alpha}f(x)+\mathcal{H}_{w_\alpha}^{\alpha}f(x), & \text{if } w\in\mathbf{V}_-^{n-\alpha},\end{cases} \tag{5.5}$$

with $\mathcal{H}_{-\alpha}^{\alpha}=\mathcal{H}_w^{\alpha}|_{w\equiv|x|^{-\alpha}}$ and prove (5.2).

Let now $n-\alpha>1$. We denote $n-\alpha=m+\{n-\alpha\}$, where $m=[n-\alpha]$ and $\{n-\alpha\}$ stands for the fractional part of $n-\alpha$. Now

$$w\in\mathbf{V}_-^1\cup\mathbf{V}_+^1\subset\mathbf{V}_-^{\{n-\alpha\}}\cup\mathbf{V}_+^{\{n-\alpha\}}. \tag{5.6}$$

The procedure is similar to the previous case; we can first manage with the fractional part $\{n-\alpha\}$, treating w as a function in $\mathbf{V}_-^{\{n-\alpha\}}\cup\mathbf{V}_+^{\{n-\alpha\}}$ like in the previous case, and then repeat a similar procedure m times treating w as a function in $\mathbf{V}_-^1\cup\mathbf{V}_+^1$.

For definiteness we consider the case where $w\in\mathbf{V}_+^1$; the case of $w\in\mathbf{V}_-^1$ is similarly treated. By the definition of the class $\mathbf{V}_+^{\{n-\alpha\}}$, we have

$$|K(x,y)|\le\frac{C}{|x-y|^m}\begin{cases}\dfrac{w(|x|)}{w(|y|)}|x|^{-\{n-\alpha\}}, & |y|<|x|,\\[2ex] |y|^{-\{n-\alpha\}}, & |y|>|x|\end{cases} \tag{5.7}$$

(this step should be omitted when $n-\alpha$ is an integer), that is,

$$|K(x,y)|\le K_+(x,y)+K_-(x,y), \tag{5.8}$$

where

$$K_+(x,y) = C\frac{w(|x|)}{w(|y|)} \cdot \frac{\theta(|x|-|y|)}{|x|^{\{n-\alpha\}}|x-y|^m},$$
$$K_-(x,y) = C\frac{\theta(|y|-|x|)}{|y|^{\{n-\alpha\}}|x-y|^m} \leq \frac{C2^{\{n-\alpha\}}}{|x-y|^{n-\alpha}} \tag{5.9}$$

and $\theta(t) = \chi_{\mathbb{R}^1_+}(t)$. We only have to take care about the kernel $K_+(x,y)$. We have

$$K_+(x,y) = C\frac{w(|x|)-w(|y|)}{w(|y|)} \cdot \frac{\theta(|x|-|y|)}{|x|^{\{n-\alpha\}}|x-y|^m} + C\frac{\theta(|x|-|y|)}{|x|^{\{n-\alpha\}}|x-y|^m}. \tag{5.10}$$

We make use of the fact that $w \in V^1_+$ and obtain

$$K_+(x,y) \leq C\frac{w(|x|)}{w(|y|)} \cdot \frac{\theta(|x|-|y|)}{|x|^{\{n-\alpha\}+1}|x-y|^{m-1}} + \frac{C}{|x-y|^{n-\alpha}}, \tag{5.11}$$

where again only the first term must be studied. We repeat the same procedure $m-1$ times more and finally arrive at the kernel

$$\frac{w(|x|)}{w(|y|)} \cdot \frac{\theta(|x|-|y|)}{|x|^{\{n-\alpha\}+m}} = \frac{|x|^{\alpha-n}w(|x|)}{w(|y|)} \cdot \theta(|x|-|y|), \tag{5.12}$$

which is the kernel of the Hardy operator H^α_w. □

We are ready for the following statement, where notation (2.22) is used.

Theorem 5.3. *Let $0<\alpha<n$, $0 \leq \lambda < n$, and $1<p<(n-\lambda)/\alpha$.*

(i) *Let $\varphi \in \widetilde{W}_{0,\infty}(\mathbb{R}^1_+) \cap (\mathbf{V}^\mu_- \cup \mathbf{V}^\mu_+)$ with $\mu = \min(1, n-\alpha)$. Then the condition*

$$\varphi \in \Phi^{\alpha+(\lambda-n)/p}_{n/p'+\lambda/p}\left(\mathbb{R}^1_+\right), \tag{5.13}$$

or equivalently

$$\alpha - \frac{n-\lambda}{p} < \min(m(\varphi), m_\infty(\varphi)), \qquad \max(M(\varphi), M_\infty(\varphi)) < \frac{n}{p'} + \frac{\lambda}{p}, \tag{5.14}$$

is sufficient for the boundedness of the potential operator (5.1) from the weighted space $\mathcal{L}^{p,\lambda}(\mathbb{R}^1_+,\varphi)$ to the space $\mathcal{L}^{q,\lambda}(\mathbb{R}^1_+,\varphi)$, where $1/q = 1/p - \alpha/(n-\lambda)$.

(ii) *Let* $\varphi \in \widetilde{W}_{0,\infty}(\mathbb{R}^1_+)$. *Then the condition*

$$\alpha - \frac{n-\lambda}{p} \le \min(M(\varphi), M_\infty(\varphi)), \qquad \max(m(\varphi), m_\infty(\varphi)) \le n/p' + \lambda/p \tag{5.15}$$

is necessary for the boundedness of the potential operator (5.1) *from* $\mathcal{L}^{p,\lambda}(\mathbb{R}^1_+,\varphi)$ *to* $\mathcal{L}^{q,\lambda}(\mathbb{R}^1_+,\varphi)$.

Proof. The necessity part (ii) follows from Lemma 5.1 and Theorem 4.5.

Part (i). We have to prove the boundedness of the operator $\varphi I^\alpha 1/\varphi$ from $\mathcal{L}^{p,\lambda}(\mathbb{R}^1_+)$ to $\mathcal{L}^{q,\lambda}(\mathbb{R}^1_+)$. Since the non-weighted $\mathcal{L}^{p,\lambda}(\mathbb{R}^1_+) \rightarrow \mathcal{L}^{q,\lambda}(\mathbb{R}^1_+)$-boundedness of the potential operator I^α is known [5], it suffices to show the boundedness of the operator $\varphi I^\alpha 1/\varphi - I^\alpha$. For that it remains to make use of Theorem 4.5. This completes the proof. □

Corollary 5.4. *Let* $0 \le \lambda < n, 0 < \alpha < n-\lambda, 1 < p < (n-\lambda)/\alpha$, *and* $\varrho(x) = |x-x_0|^\gamma,\ x_0 \in \mathbb{R}^1$. *Then the potential operator* (5.1) *is bounded from* $\mathcal{L}^{p,\lambda}(\mathbb{R}^n,\varrho)$ *into* $\mathcal{L}^{q,\lambda}(\mathbb{R}^n,\varrho)$, $1/q = 1/p - \alpha/(n-\lambda)$, *if and only if*

$$\alpha - \frac{n}{p} < \gamma - \frac{\lambda}{p} < \frac{n}{p'}. \tag{5.16}$$

Remark 5.5. As can be seen from the proof of Theorem 5.3, its statement remains valid under the condition

$$\varphi \in \begin{cases} \mathbb{Z}^{\alpha+(\lambda-n)/p}, & \text{if } \varphi \in \mathbf{V}^\mu_+, \\ \mathbb{Z}_{n/p'+\lambda/p}(\mathbb{R}^1_+), & \text{if } \varphi \in \mathbf{V}^\mu_-, \end{cases} \quad \mu = \min(1, n-\alpha), \tag{5.17}$$

more general than (5.13). Correspondingly, condition (5.14) may be written in a more general form:

$$\begin{aligned} \max(M(\varphi), M_\infty(\varphi)) &< \frac{n}{p'} + \frac{\lambda}{p}, \ \text{ if } \varphi \in \mathbf{V}^\mu_+, \\ \alpha - \frac{n-\lambda}{p} &< \min(m(\varphi), m_\infty(\varphi)), \ \text{ if } \varphi \in \mathbf{V}^\mu_-. \end{aligned} \tag{5.18}$$

(Recall that $\min(M(\varphi), M_\infty(\varphi)) \ge 0$ in the case $\varphi \in \mathbf{V}^\mu_+$ and $\max(m(\varphi), m_\infty(\varphi)) \le 0$ in the case $\varphi \in \mathbf{V}^\mu_-$; see Corollary 2.9.)

Acknowledgment

This work was supported by Research Grant SFRH/BPD/34258/2006, FCT, Portugal.

References

[1] C. B. Morrey,, "On the solutions of quasi-linear elliptic partial differential equations," *Transactions of the American Mathematical Society*, vol. 43, no. 1, pp. 126–166, 1938.

[2] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, vol. 105, Princeton University Press, Princeton, NJ, USA, 1983.

[3] A. Kufner, O. John, and S. Fučík, *Function Spaces*, Noordhoff International, 1977.

[4] M. E. Taylor, *Tools for Pde: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials*, vol. 81 of *Mathematical Surveys and Monographs*, American Mathematical Society, Providence, RI, USA, 2000.

[5] D. R. Adams, "A note on Riesz potentials," *Duke Mathematical Journal*, vol. 42, no. 4, pp. 765–778, 1975.

[6] D. R. Adams and J. Xiao, "Nonlinear potential analysis on Morrey spaces and their capacities," *Indiana University Mathematics Journal*, vol. 53, no. 6, pp. 1631–1666, 2004.

[7] J. Alvarez, "The distribution function in the Morrey space," *Proceedings of the American Mathematical Society*, vol. 83, no. 4, pp. 693–699, 1981.

[8] H. Arai and T. Mizuhara, "Morrey spaces on spaces of homogeneous type and estimates for $\square_b$ and the Cauchy-Szegö projection," *Mathematische Nachrichten*, vol. 185, no. 1, pp. 5–20, 1997.

[9] V. I. Burenkov, H. V. Guliyev, and V. S. Guliyev, "Necessary and sufficient conditions for the boundedness of fractional maximal operators in local Morrey-type spaces," *Journal of Computational and Applied Mathematics*, vol. 208, no. 1, pp. 280–301, 2007.

[10] V. I. Burenkov and H. V. Guliyev, "Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces," *Studia Mathematica*, vol. 163, no. 2, pp. 157–176, 2004.

[11] F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function," *Rendiconti di Matematica*, vol. 7, no. 3-4, pp. 273–279, 1987.

[12] G. Di Fazio and M. A. Ragusa, "Commutators and Morrey spaces," *Bollettino dell'Unione Matematica Italiana*, vol. 7, no. 5, pp. 323–332, 1991.

[13] Y. Ding and S. Lu, "Boundedness of homogeneous fractional integrals on L^p for $n/a < p$," *Nagoya Mathematical Journal*, vol. 167, pp. 17–33, 2002.

[14] E. Nakai, "Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces," *Mathematische Nachrichten*, vol. 166, pp. 95–103, 1994.

[15] E. Nakai, "On generalized fractional integrals," *Taiwanese Journal of Mathematics*, vol. 5, no. 3, pp. 587–602, 2001.

[16] E. Nakai and H. Sumitomo, "On generalized Riesz potentials and spaces of some smooth functions," *Scientiae Mathematicae Japonicae*, vol. 54, no. 3, pp. 463–472, 2001.

[17] D. K. Palagachev and L. G. Softova, "Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's," *Potential Analysis*, vol. 20, no. 3, pp. 237–263, 2004.

[18] J. Peetre, "On convolution operators leaving $\mathcal{L}^{p,\lambda}$ spaces invariant," *Annali di Matematica Pura ed Applicata*, vol. 72, no. 1, pp. 295–304, 1966.

[19] J. Peetre, "On the theory of $\mathcal{L}_{p,\lambda}$ spaces," *Journal of Functional Analysis*, vol. 4, pp. 71–87, 1969.

[20] M. A. Ragusa, "Commutators of fractional integral operators on vanishing-Morrey spaces," *Journal of Global Optimization*, vol. 40, no. 1–3, pp. 361–368, 2008.

[21] S. Shirai, "Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces," *Hokkaido Mathematical Journal*, vol. 35, no. 3, pp. 683–696, 2006.

[22] S. Spanne, "Some function spaces defined using the mean oscillation over cubes," *Annali della Scuola Normale Superiore di Pisa*, vol. 19, pp. 593–608, 1965.

[23] G. Stampacchia, "The spaces $L^{p,\lambda}$, $N^{(p,\lambda)}$ and interpolation," *Annali della Scuola Normale Superiore di Pisa*, vol. 3, no. 19, pp. 443–462, 1965.

[24] N. G. Samko, "Weighted Hardy and singular operators in Morrey spaces," *Journal of Mathematical Analysis and Applications*, vol. 350, no. 1, pp. 56–72, 2009.

[25] N. K. Bari and S. B. Stečkin, "Best approximations and differential properties of two conjugate functions," *Proceedings of the Moscow Mathematical Society*, vol. 5, pp. 483–522, 1956 (Russian).

[26] N. K. Karapetyants and N. Samko, "Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^{\omega}(p)$ via the indices m_ω and M_ω," *Fractional Calculus & Applied Analysis*, vol. 7, no. 4, pp. 437–458, 2004.

[27] S. G. Krein, Y. I. Petunin, and E. M. Semënov, *Interpolyatsiya Lineinykh Operatorov*, Nauka, Moscow, Russia, 1978.

[28] L. Maligranda, "Indices and interpolation," *Dissertationes Mathematicae*, vol. 234, article 49, 1985.

[29] L. Maligranda, *Orlicz Spaces and Interpolation*, vol. 5, Departamento de Matemática, Universidade Estadual de Campinas, Campinas SP, Brazil, 1989.

[30] W. Matuszewska and W. Orlicz, “On some classes of functions with regard to their orders of growth,” *Studia Mathematica*, vol. 26, pp. 11–24, 1965.

[31] N. G. Samko, “Singular integral operators in weighted spaces with generalized Hölder condition,” *Proceedings of A. Razmadze Mathematical Institute*, vol. 120, pp. 107–134, 1999.

[32] N. Samko, “On non-equilibrated almost monotonic functions of the Zygmund-Bary-Stechkin class,” *Real Analysis Exchange*, vol. 30, no. 2, pp. 727–746, 2004.

[33] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, River Edge, NJ, USA, 2003.

[34] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129, Academic Press, Boston, Mass, USA, 1988.