

## Research Article

# Hybrid Gradient-Projection Algorithm for Solving Constrained Convex Minimization Problems with Generalized Mixed Equilibrium Problems

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It is well known that the gradient-projection algorithm (GPA) for solving constrained convex minimization problems has been proven to have only weak convergence unless the underlying Hilbert space is finite dimensional. In this paper, we introduce a new hybrid gradient-projection algorithm for solving constrained convex minimization problems with generalized mixed equilibrium problems in a real Hilbert space. It is proven that three sequences generated by this algorithm converge strongly to the unique solution of some variational inequality, which is also a common element of the set of solutions of a constrained convex minimization problem, the set of solutions of a generalized mixed equilibrium problem, and the set of fixed points of a strict pseudocontraction in a real Hilbert space.

## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ . Let  $C$  be a nonempty closed convex subset of  $H$  and let  $P_C$  be the metric projection of  $H$  onto  $C$ . Recall that a  $\rho$ -Lipschitz continuous mapping  $T : C \rightarrow H$  is a mapping on  $C$  such that

$$\|Tx - Ty\| \leq \rho \|x - y\|, \quad \forall x, y \in C, \quad (1.1)$$

where  $\rho \geq 0$  is a constant. In particular, if  $\rho \in [0, 1)$  then  $T$  is called a contraction on  $C$ ; if  $\rho = 1$  then  $T$  is called a nonexpansive mapping on  $C$ . A mapping  $A : C \rightarrow H$  is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \quad (1.2)$$

A mapping  $A : C \rightarrow H$  is called  $\alpha$ -inverse strongly monotone if there exists a constant  $\alpha > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C; \quad (1.3)$$

see, for example, [1]. A self-mapping  $S : C \rightarrow C$  is called  $k$ -strictly pseudocontractive if there exists a constant  $k \in [0, 1)$  such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C; \quad (1.4)$$

see, for example, [2]. In particular, if  $k = 0$ , then  $S$  reduces to a nonexpansive self-mapping on  $C$ .

Consider the following constrained convex minimization problem:

$$\text{minimize} \{f(x) : x \in C\}, \quad (1.5)$$

where  $f : C \rightarrow \mathbf{R}$  is a real-valued convex function. If  $f$  is (Frechet) differentiable, then the gradient-projection method (for short, GPM) generates a sequence  $\{x_n\}$  via the recursive formula

$$x_{n+1} = P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \quad (1.6)$$

or more generally,

$$x_{n+1} = P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \quad (1.7)$$

where in both (1.6) and (1.7), the initial guess  $x_0$  is taken from  $C$  arbitrarily, the parameters,  $\lambda$  or  $\lambda_n$ , are positive real numbers, and  $P_C$  is the metric projection from  $H$  onto  $C$ . The convergence of the algorithms (1.6) and (1.7) depends on the behavior of the gradient  $\nabla f$ . As a matter of fact, it is known that if  $\nabla f$  is strongly monotone and Lipschitzian; namely, there are constants  $\eta, L > 0$  satisfying the properties

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \eta \|x - y\|^2, \quad (1.8)$$

$$\|\nabla f(x) - \nabla f(y)\| \leq L \|x - y\| \quad (1.9)$$

for all  $x, y \in C$ , then, for  $0 < \lambda < 2\eta/L^2$ , the operator

$$T := P_C(I - \lambda \nabla f) \quad (1.10)$$

is a contraction; hence, the sequence  $\{x_n\}$  defined by algorithm (1.6) converges in norm to the unique solution of the minimization (1.5). More generally, if the sequence  $\{\lambda_n\}$  is chosen to satisfy the property

$$0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{2\eta}{L^2} \quad (1.11)$$

then the sequence  $\{x_n\}$  defined by algorithm (1.7) converges in norm to the unique minimizer of (1.5). However, if the gradient  $\nabla f$  fails to be strongly monotone, the operator  $T$  defined in (1.10) would fail to be contractive; consequently, the sequence  $\{x_n\}$  generated by algorithm (1.6) may fail to converge strongly (see Section 4 in Xu [3]). The following theorem states that if the Lipschitz condition (1.9) holds, then the algorithms (1.6) and (1.7) can still converge in the weak topology.

**Theorem 1.1** (see [3, Theorem 3.2]). *Assume the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  satisfies the Lipschitz condition (1.9). Let the sequence of parameters,  $\{\lambda_n\}$ , satisfy the condition*

$$0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{2}{L}. \quad (1.12)$$

*Then the sequence  $\{x_n\}$  generated by the gradient-projection algorithm (1.7) converges weakly to a minimizer of (1.5).*

From the above theorem, it is known that the gradient-projection algorithm has weak convergence, in general, unless the underlying Hilbert space is finite dimensional. This gives naturally rise to a question how to appropriately modify the gradient-projection algorithm so as to have strong convergence. Xu [3] gave two such modifications, one of which is simply a convex combination of a contraction with the point generated by the projected gradient algorithm.

**Theorem 1.2** (see [3, Theorem 5.2]). *Assume the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  satisfies the Lipschitz condition (1.9). Let  $Q : C \rightarrow C$  be a  $\rho$ -contraction with  $\rho \in [0, 1)$ . Let a sequence  $\{x_n\}$  be generated by the following hybrid gradient-projection algorithm:*

$$x_{n+1} = \alpha_n Qx_n + (1 - \alpha_n)P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0. \quad (1.13)$$

*Assume the sequence  $\{\lambda_n\}$  satisfies the condition (1.12) and, in addition, the following conditions are satisfied for  $\{\lambda_n\}$  and  $\{\alpha_n\} \subset [0, 1]$ :*

- (i)  $\alpha_n \rightarrow 0$ ;
- (ii)  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (iii)  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ ;
- (iv)  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ .

*Then the sequence  $\{x_n\}$  converges in norm to a minimizer of (1.5) which is also the unique solution of the variational inequality of finding  $x^* \in \Omega$  such that*

$$\langle (I - Q)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \quad (1.14)$$

*In other words,  $x^*$  is the unique fixed point of the contraction  $P_{\Omega}Q$ ,  $x^* = P_{\Omega}Qx^*$ .*

On the other hand, Peng and Yao [4] recently introduced the following generalized mixed equilibrium problem of finding  $\bar{x} \in C$  such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) + \langle F\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \quad (1.15)$$

where  $F : C \rightarrow H$  is a nonlinear mapping and  $\varphi : C \rightarrow \mathbf{R}$  is a function and  $\Theta : C \times C \rightarrow \mathbf{R}$  is a bifunction. The set of solutions of problem (1.15) is denoted by GMEP. Subsequently, Yao et al. [5] and Ceng and Yao [6] also considered problem (1.15).

The problem (1.15) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problems in noncooperative games, and others; see, for example, [7–15]. Here some special cases of problem (1.15) are stated as follows.

If  $F = 0$ , then problem (1.15) reduces to the following mixed equilibrium problem of finding  $\bar{x} \in C$  such that

$$\Theta(\bar{x}, y) + \varphi(y) - \varphi(\bar{x}) \geq 0, \quad \forall y \in C, \quad (1.16)$$

which was considered by Ceng and Yao [7] and Bigi et al. [16]. Very recently, Peng [10] further discussed this problem. The set of solutions of this problem is denoted by MEP.

If  $\varphi = 0$ , then problem (1.15) reduces to the following generalized equilibrium problem of finding  $\bar{x} \in C$  such that

$$\Theta(\bar{x}, y) + \langle F\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \quad (1.17)$$

which was studied by S. Takahashi and W. Takahashi [8].

If  $\varphi = 0$  and  $F = 0$ , then problem (1.15) reduces to the following equilibrium problem of finding  $\bar{x} \in C$  such that

$$\Theta(\bar{x}, y) \geq 0, \quad \forall y \in C. \quad (1.18)$$

If  $\Theta = 0$ ,  $\varphi = 0$  and  $F = A$ , then problem (1.15) reduces to the following classical variational inequality of finding  $\bar{x} \in C$  such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C, \quad (1.19)$$

whose solution set is denoted by  $VI(C, A)$ .

The variational inequalities have been extensively studied in the literature; see [14, 17–27] and the references therein. In 2006, Nadezhkina and Takahashi [22, 25] and Zeng and Yao [18] proposed some variants of Korpelevič's extragradient method [17] for finding an element of  $\text{Fix}(S) \cap VI(C, A)$ , where  $S$  is a nonexpansive self-mapping on  $C$ .

Very recently, Peng [10] also introduced a variant of Korpelevič's extragradient method [17] for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a strict pseudocontraction, and the set of solutions of a variational inequality for a monotone, Lipschitz continuous mapping.

**Theorem 1.3** (see [10, Theorem 3.1]). *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  a lower semicontinuous and convex function with assumptions (A1) or (A2), where*

(H1)  $\Theta(x, x) = 0$ , for all  $x \in C$ ;

(H2)  $\Theta$  is monotone, that is,  $\Theta(x, y) + \Theta(y, x) \leq 0$ , for all  $x, y \in C$ ;

- (H3) for each  $y \in C$ ,  $x \mapsto \Theta(x, y)$  is weakly upper semicontinuous;  
 (H4) for each  $x \in C$ ,  $y \mapsto \Theta(x, y)$  is convex and lower semicontinuous;  
 (A1) for each  $x \in H$  and  $r > 0$ , there exists a bounded subset  $D_x \subset C$  and  $y_x \in C$  such that for any  $z \in C \setminus D_x$ ,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0; \quad (1.20)$$

(A2)  $C$  is a bounded set.

Let  $A : C \rightarrow H$  be a monotone and  $L$ -Lipschitz-continuous mapping and  $S : C \rightarrow C$  be a  $k$ -strict pseudocontraction for some  $0 \leq k < 1$  such that  $\text{Fix}(S) \cap \text{VI}(C, A) \cap \text{MEP} \neq \emptyset$ . For given  $x_0 \in H$  arbitrarily, let  $\{x_n\}$ ,  $\{t_n\}$ ,  $\{y_n\}$ ,  $\{u_n\}$ ,  $\{z_n\}$  be sequences generated by

$$\begin{aligned} \Theta(t_n, y) + \varphi(y) - \varphi(t_n) + \frac{1}{r_n} \langle y - t_n, t_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(t_n - \lambda_n A t_n), \\ u_n &= P_C(t_n - \lambda_n A y_n), \\ z_n &= \alpha_n u_n + (1 - \alpha_n) S u_n, \end{aligned} \quad (1.21)$$

$$C_n = \left\{ z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \varepsilon) \|t_n - S t_n\|^2 \right\},$$

$$Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\},$$

$$x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 0.$$

Assume that  $\{\lambda_n\} \subset [a, b]$  for some  $a, b \in (0, 1/L)$ ,  $\{\alpha_n\} \subset [c, d]$  for some  $c, d \in (k, 1)$  and let  $\{r_n\} \subset (0, \infty)$  satisfy  $\liminf_{n \rightarrow \infty} r_n > 0$ . Then,  $\{x_n\}$ ,  $\{t_n\}$ ,  $\{y_n\}$ ,  $\{u_n\}$ ,  $\{z_n\}$  converge strongly to  $w = P_{\text{Fix}(S) \cap \text{VI}(C, A) \cap \text{MEP}} x$ .

Furthermore, related iterative methods for solving fixed point problems, variational inequalities, equilibrium problems, and optimization problems can be found in [1, 2, 6, 11, 13–16, 19, 20, 24, 26–35].

In this paper, let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Let the gradient  $\nabla f$  be  $L$ -Lipschitzian with constant  $L > 0$  and  $F : C \rightarrow H$  be an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ . Let  $Q : C \rightarrow C$  be a  $\rho$ -contraction with  $\rho \in [0, 1/2)$ . For given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  be generated iteratively by

$$\begin{aligned} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle F x_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (1.22)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$  and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that  $\beta_n + \gamma_n + \delta_n = 1$  for all  $n \geq 0$ . It is proven that under very mild conditions, the sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  converge strongly to the unique solution of the variational inequality of finding  $x^* \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}$  such that

$$\langle (I - Q)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}. \quad (1.23)$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}}Q$ ,  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}}Qx^*$ . The result presented in this paper generalizes and improves some well-known results in the literature. Indeed, compared with some well-known results in the literature, our result improves and extends them in the following aspects.

- (i) Compared with Xu [3, Theorem 3.2], a weak convergence result, our result is a strong convergence result.
- (ii) Our problem of finding an element of  $\text{Fix}(S) \cap \Omega \cap \text{GMEP}$  is more general than the problem of finding an element of  $\text{Fix}(S) \cap \text{VI}(C, A)$  in [14, 18, 22, 23, 25].
- (iii) In our algorithm (1.22), Xu's modified gradient-projection algorithm in [3, Theorem 5.2] is rewritten as the second iteration step

$$y_n = \alpha_n Qx_n + (1 - \alpha_n)P_C(z_n - \lambda_n \nabla f(z_n)). \quad (1.24)$$

Here the main purpose of the reason why we use such an iteration step is to play a convenience and efficiency role in the computation of an element of  $\Omega$ . Therefore, Xu's algorithm (1.13) is extended to develop our algorithm (1.22).

- (iv) Our problem of finding an element of  $\text{Fix}(S) \cap \Omega \cap \text{GMEP}$  is more general than the problem of finding an element of  $\Omega$  in Xu [3]. In addition, it is worth pointing out that Xu's conditions  $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$  and  $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$  in the above Theorem 1.2 are replaced by the weaker conditions  $\lim_{n \rightarrow \infty} (\alpha_n - \alpha_{n+1}) = 0$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$  in our result (see Theorem 3.2 in Section 3).

## 2. Preliminaries

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\|$  and  $C$  a nonempty closed convex subset of  $H$ . We write  $\rightarrow$  to indicate that the sequence  $\{x_n\}$  converges strongly to  $x$  and  $\rightharpoonup$  to indicate that the sequence  $\{x_n\}$  converges weakly to  $x$ . Moreover, we use  $\omega_w(x_n)$  to denote the weak  $\omega$ -limit set of the sequence  $\{x_n\}$ , that is,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}. \quad (2.1)$$

For every point  $x \in H$ , there exists a unique nearest point in  $C$ , denoted by  $P_C x$ , such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \quad (2.2)$$

$P_C$  is called the metric projection of  $H$  onto  $C$ . We know that  $P_C$  is a firmly nonexpansive mapping of  $H$  onto  $C$ ; that is, there holds the following relation:

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \quad (2.3)$$

Consequently,  $P_C$  is nonexpansive and monotone. It is also known that  $P_C$  is characterized by the following properties:  $P_C x \in C$  and

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \quad (2.4)$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad (2.5)$$

for all  $x \in H$ ,  $y \in C$ ; see [36] for more details. Let  $A : C \rightarrow H$  be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \text{VI}(C, A) \iff x = P_C(x - \lambda Ax) \quad \forall \lambda > 0. \quad (2.6)$$

A set-valued mapping  $T : H \rightarrow 2^H$  is called monotone if for all  $x, y \in H$ ,  $f \in Tx$  and  $g \in Ty$  imply  $\langle f - g, x - y \rangle \geq 0$ . A monotone mapping  $T : H \rightarrow 2^H$  is called maximal if its graph  $G(T)$  is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping  $T$  is maximal if and only if for  $(x, f) \in H \times H$ ,  $\langle f - g, x - y \rangle \geq 0$  for every  $(y, g) \in G(T)$  implies  $f \in Tx$ .

Let  $A : C \rightarrow H$  be a monotone,  $k$ -Lipschitz-continuous mapping and let  $N_C v$  be the normal cone to  $C$  at  $v \in C$ , that is,  $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$ . Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \quad (2.7)$$

Then,  $T$  is maximal monotone and  $0 \in Tv$  if and only if  $v \in \text{VI}(C, A)$ ; see [37].

Recall that a mapping  $S : C \rightarrow C$  is called a strict pseudocontraction if there exists a constant  $0 \leq k < 1$  such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \quad (2.8)$$

In this case, we also say that  $S$  is a  $k$ -strict pseudocontraction. A mapping  $A : C \rightarrow H$  is called  $\alpha$ -inverse strongly monotone if there exists a constant  $\alpha > 0$  such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (2.9)$$

It is obvious that any  $\alpha$ -inverse strongly monotone mapping is Lipschitz continuous. Meanwhile, observe that (2.8) is equivalent to

$$\langle Sx - Sy, x - y \rangle \leq \|x - y\|^2 - \frac{1 - k}{2} \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \quad (2.10)$$

It is easy to see that if  $S$  is a  $k$ -strictly pseudocontractive mapping, then  $I - S$  is  $((1 - k)/2)$ -inverse strongly monotone and hence  $(2/(1 - k))$ -Lipschitz continuous. Thus,  $S$  is Lipschitz continuous with constant  $(1 + k)/(1 - k)$ . We denote by  $\text{Fix}(S)$  the set of fixed points of  $S$ . It is clear that the class of strict pseudocontractions strictly includes the one of nonexpansive mappings which are mappings  $S : C \rightarrow C$  such that  $\|Sx - Sy\| \leq \|x - y\|$  for all  $x, y \in C$ .

In order to prove our main result in the next section, we need the following lemmas and propositions.

**Lemma 2.1** (see [7]). *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and let  $\varphi : C \rightarrow \mathbf{R}$  be a lower semicontinuous and convex function. For  $r > 0$  and  $x \in H$ , define a mapping  $T_r^{(\Theta, \varphi)} : H \rightarrow C$  as follows:*

$$T_r^{(\Theta, \varphi)}(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (2.11)$$

for all  $x \in H$ . Assume that either (A1) or (A2) holds. Then the following conclusions hold:

- (i)  $T_r^{(\Theta, \varphi)}(x) \neq \emptyset$  for each  $x \in H$  and  $T_r^{(\Theta, \varphi)}$  is single-valued;
- (ii)  $T_r^{(\Theta, \varphi)}$  is firmly nonexpansive, that is, for any  $x, y \in H$ ,

$$\|T_r^{(\Theta, \varphi)}x - T_r^{(\Theta, \varphi)}y\|^2 \leq \langle T_r^{(\Theta, \varphi)}x - T_r^{(\Theta, \varphi)}y, x - y \rangle; \quad (2.12)$$

- (iii)  $\text{Fix}(T_r^{(\Theta, \varphi)}) = \text{MEP}(\Theta, \varphi)$ ;
- (iv)  $\text{MEP}(\Theta, \varphi)$  is closed and convex.

**Remark 2.2.** If  $\varphi = 0$ , then  $T_r^{(\Theta, \varphi)}$  is rewritten as  $T_r^\Theta$ .

The following lemma is an immediate consequence of an inner product.

**Lemma 2.3.** *In a real Hilbert space  $H$ , there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H. \quad (2.13)$$

**Proposition 2.4** (see [6, Proposition 2.1]). *Let  $C$ ,  $H$ ,  $\Theta$ ,  $\varphi$ , and  $T_r^{(\Theta, \varphi)}$  be as in Lemma 2.1. Then the following relation holds:*

$$\|T_s^{(\Theta, \varphi)}x - T_t^{(\Theta, \varphi)}x\|^2 \leq \frac{s-t}{s} \langle T_s^{(\Theta, \varphi)}x - T_t^{(\Theta, \varphi)}x, T_s^{(\Theta, \varphi)}x - x \rangle \quad (2.14)$$

for all  $s, t > 0$  and  $x \in H$ .

Recall that  $S : C \rightarrow C$  is called a quasi-strict pseudocontraction if the fixed point set of  $S$ ,  $\text{Fix}(S)$ , is nonempty and if there exists a constant  $0 \leq k < 1$  such that

$$\|Sx - p\|^2 \leq \|x - p\|^2 + k\|x - Sx\|^2 \quad \forall x \in C, p \in \text{Fix}(S). \quad (2.15)$$

We also say that  $S$  is a  $k$ -quasi-strict pseudocontraction if condition (2.15) holds.

**Proposition 2.5** (see [2, Proposition 2.1]). *Assume  $C$  is a nonempty closed convex subset of a real Hilbert space  $H$  and let  $S : C \rightarrow C$  be a self-mapping on  $C$ .*



(i) If  $S$  is a  $k$ -strict pseudocontraction, then  $S$  satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1+k}{1-k} \|x - y\|, \quad \forall x, y \in C. \quad (2.16)$$

(ii) If  $S$  is a  $k$ -strict pseudocontraction, then the mapping  $I - S$  is demiclosed (at 0). That is, if  $\{x_n\}$  is a sequence in  $C$  such that  $x_n \rightharpoonup \tilde{x}$  and  $(I - S)x_n \rightarrow 0$ , then  $(I - S)\tilde{x} = 0$ , that is,  $\tilde{x} \in \text{Fix}(S)$ .

(iii) If  $S$  is a  $k$ -quasi-strict pseudocontraction, then the fixed point set  $\text{Fix}(S)$  of  $S$  is closed and convex so that the projection  $P_{\text{Fix}(S)}$  is well defined.

The following lemma was proved by Suzuki [30].

**Lemma 2.6** (see [30]). Let  $\{x_n\}$  and  $\{y_n\}$  be bounded sequences in a Banach space  $X$  and let  $\{\beta_n\}$  be a sequence in  $[0, 1]$  with  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$ . Suppose  $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$  for all integers  $n \geq 0$  and  $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$ . Then,  $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$ .

**Lemma 2.7** (see [34]). Let  $\{a_n\}$  be a sequence of nonnegative numbers satisfying the condition

$$a_{n+1} \leq (1 - \delta_n)a_n + \delta_n \sigma_n, \quad \forall n \geq 0, \quad (2.17)$$

where  $\{\delta_n\}$ ,  $\{\sigma_n\}$  are sequences of real numbers such that

(i)  $\{\delta_n\} \subset [0, 1]$  and  $\sum_{n=0}^{\infty} \delta_n = \infty$ , or equivalently,

$$\prod_{n=0}^{\infty} (1 - \delta_n) := \lim_{n \rightarrow \infty} \prod_{j=0}^n (1 - \delta_j) = 0; \quad (2.18)$$

(ii)  $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ , or,

(iii)  $\sum_{n=0}^{\infty} \delta_n \sigma_n$  is convergent.

Then  $\lim_{n \rightarrow \infty} a_n = 0$ .

### 3. Strong Convergence Theorem

In order to prove our main result, we shall need the following lemma given in [21].

**Lemma 3.1.** Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$ . Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping. Let  $\gamma$  and  $\delta$  be two nonnegative real numbers. Assume  $(\gamma + \delta)k \leq \gamma$ . Then

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C. \quad (3.1)$$

We are now in a position to state and prove our main result.

**Theorem 3.2.** Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  a lower semicontinuous

and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $F : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ . Let  $Q : C \rightarrow C$  be a  $\rho$ -contraction with  $\rho \in [0, 1/2)$ . For given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated iteratively by

$$\begin{aligned} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \alpha_n Qx_n + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.2)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$ , and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 2\alpha$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \delta_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to the unique solution of the variational inequality of finding  $x^* \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}$  such that

$$\langle (I - Q)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}. \quad (3.3)$$

In other words,  $x^*$  is the unique fixed point of the contraction  $P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} Q$ ,  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} Q x^*$ .

*Proof.* First it is obvious that there hold the following assertions:

- (a)  $x^* \in C$  solves the minimization (1.5);
- (b)  $x^*$  solves the fixed point equation

$$x^* = P_C(I - \lambda \nabla f)x^*, \quad (3.4)$$

where  $\lambda > 0$  is any fixed positive number;

- (c)  $x^*$  solves the variational inequality of finding  $x^* \in C$  such that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \quad (3.5)$$

where its solution set is denoted by  $\text{VI}(C, \nabla f)$ .

We divide the proof into several steps.

*Step 1.* We claim that  $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ .

Indeed, first, we can write (3.2) as  $x_{n+1} = \beta_n x_n + (1 - \beta_n)u_n$ , for all  $n \geq 0$ , where  $u_n = (x_{n+1} - \beta_n x_n) / (1 - \beta_n)$ . It follows that

$$\begin{aligned}
 u_{n+1} - u_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
 &= \frac{\gamma_{n+1}P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) + \delta_{n+1}Sy_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n Sy_n}{1 - \beta_n} \\
 &= \frac{\gamma_{n+1}[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n \nabla f(z_n))] + \delta_{n+1}(Sy_{n+1} - Sy_n)}{1 - \beta_{n+1}} \\
 &\quad + \left( \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right) P_C(z_n - \lambda_n \nabla f(z_n)) + \left( \frac{\delta_{n+1}}{1 - \beta_{n+1}} - \frac{\delta_n}{1 - \beta_n} \right) Sy_n.
 \end{aligned} \tag{3.6}$$

From Lemma 3.1 and (3.2), we get

$$\begin{aligned}
 &\|\gamma_{n+1}[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n \nabla f(z_n))] + \delta_{n+1}(Sy_{n+1} - Sy_n)\| \\
 &\leq \|\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(Sy_{n+1} - Sy_n)\| \\
 &\quad + \gamma_{n+1}\|[P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - y_{n+1}] + [y_n - P_C(z_n - \lambda_n \nabla f(z_n))]\| \\
 &\leq (\gamma_{n+1} + \delta_{n+1})\|y_{n+1} - y_n\| + \gamma_{n+1}\alpha_{n+1}\|Qx_{n+1} - P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}))\| \\
 &\quad + \gamma_{n+1}\alpha_n\|Qx_n - P_C(z_n - \lambda_n \nabla f(z_n))\|.
 \end{aligned} \tag{3.7}$$

Let  $\{T_{r_n}^{(\Theta, \varphi)}\}$  be a sequence of mappings defined as in Lemma 2.1. Note that the  $L$ -Lipschitz continuity of  $\nabla f$  implies that the gradient  $\nabla f$  is  $(1/L)$ -ism [31]. Since  $\nabla f$  and  $F$  are  $(1/L)$ -inverse strongly monotone mapping and  $\alpha$ -inverse strongly monotone mapping, respectively, then we have

$$\begin{aligned}
 &\|(I - \lambda \nabla f)x - (I - \lambda \nabla f)y\|^2 \\
 &= \|x - y\|^2 - 2\lambda \langle \nabla f(x) - \nabla f(y), x - y \rangle + \lambda^2 \|\nabla f(x) - \nabla f(y)\|^2 \\
 &\leq \|x - y\|^2 + \lambda \left( \lambda - \frac{2}{L} \right) \|\nabla f(x) - \nabla f(y)\|^2, \\
 &\|(I - \mu F)x - (I - \mu F)y\|^2 \leq \|x - y\|^2 + \mu(\mu - 2\alpha) \|Fx - Fy\|^2.
 \end{aligned} \tag{3.8}$$

It is clear that if  $0 \leq \lambda \leq 2/L$  and  $0 \leq \mu \leq 2\alpha$ , then  $(I - \lambda \nabla f)$  and  $(I - \mu F)$  are nonexpansive. It follows from that

$$\begin{aligned}
 &\|P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n \nabla f(z_n))\| \\
 &\leq \|z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}) - (z_n - \lambda_n \nabla f(z_n))\| \\
 &\leq \|z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}) - (z_n - \lambda_{n+1}\nabla f(z_n))\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\|
 \end{aligned}$$

$$\begin{aligned}
&\leq \|z_{n+1} - z_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&= \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_{n+1} - r_{n+1}Fx_{n+1}) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_{n+1} - r_{n+1}Fx_{n+1}) - T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|(x_{n+1} - r_{n+1}Fx_{n+1}) - (x_n - r_nFx_n)\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|(x_{n+1} - r_{n+1}Fx_{n+1}) - (x_n - r_{n+1}Fx_n)\| + |r_{n+1} - r_n| \|Fx_n\| \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\leq \|x_{n+1} - x_n\| + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| \\
&\quad + |r_{n+1} - r_n| \|Fx_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\|.
\end{aligned} \tag{3.9}$$

Then,

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&\leq \|P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1})) - P_C(z_n - \lambda_n\nabla f(z_n))\| \\
&\quad + \alpha_{n+1} \|Qx_{n+1} - P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}))\| + \alpha_n \|Qx_n - P_C(z_n - \lambda_n\nabla f(z_n))\| \\
&\leq \|x_{n+1} - x_n\| + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| \\
&\quad + |r_{n+1} - r_n| \|Fx_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\| \\
&\quad + \alpha_n \|Qx_n - P_C(z_n - \lambda_n\nabla f(z_n))\| + \alpha_{n+1} \|Qx_{n+1} - P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}))\|.
\end{aligned} \tag{3.10}$$

So, from (3.6), (3.7), and (3.10), we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \left(1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\right) \alpha_n \|Qx_n - P_C(z_n - \lambda_n\nabla f(z_n))\| \\
&\quad + \left(1 + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}\right) \alpha_{n+1} \|Qx_{n+1} - P_C(z_{n+1} - \lambda_{n+1}\nabla f(z_{n+1}))\| \\
&\quad + \left| \frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n} \right| (\|P_C(z_n - \lambda_n\nabla f(z_n))\| + \|Sy_n\|) \\
&\quad + \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_nFx_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_nFx_n) \right\| \\
&\quad + |r_{n+1} - r_n| \|Fx_n\| + |\lambda_{n+1} - \lambda_n| \|\nabla f(z_n)\|.
\end{aligned} \tag{3.11}$$

Utilizing Proposition 2.4 and condition (ii), we have

$$\lim_{n \rightarrow \infty} \left\| T_{r_{n+1}}^{(\Theta, \varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta, \varphi)}(x_n - r_n F x_n) \right\| = 0. \quad (3.12)$$

This implies that

$$\limsup_{n \rightarrow \infty} (\|u_{n+1} - u_n\| - \|x_{n+1} - x_n\|) \leq 0. \quad (3.13)$$

Hence by Lemma 2.6, we get  $\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0$ . Consequently,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|u_n - x_n\| = 0. \quad (3.14)$$

*Step 2.* We claim that  $\lim_{n \rightarrow \infty} \|\nabla f(z_n) - \nabla f(x^*)\| = 0$  and  $\lim_{n \rightarrow \infty} \|F x_n - F x^*\| = 0$ .

Indeed, let  $x^* \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}$ . Then we have  $x^* = S x^*$ ,  $x^* = P_C(x^* - \lambda_n \nabla f(x^*))$  and

$$x^* = T_{r_n}^{(\Theta, \varphi)}(x^* - r_n F x^*). \quad (3.15)$$

Hence from (3.8), we have

$$\begin{aligned} \|P_C(z_n - \lambda_n \nabla f(z_n)) - P_C(x^* - \lambda_n \nabla f(x^*))\|^2 &\leq \|(z_n - \lambda_n \nabla f(z_n)) - (x^* - \lambda_n \nabla f(x^*))\|^2 \\ &\leq \|z_n - x^*\|^2 + \lambda_n \left( \lambda_n - \frac{2}{L} \right) \|\nabla f(z_n) - \nabla f(x^*)\|^2, \end{aligned} \quad (3.16)$$

$$\begin{aligned} \|z_n - x^*\|^2 &= \left\| T_{r_n}^{(\Theta, \varphi)}(x_n - r_n F x_n) - T_{r_n}^{(\Theta, \varphi)}(x^* - r_n F x^*) \right\|^2 \\ &\leq \|(x_n - r_n F x_n) - (x^* - r_n F x^*)\|^2 \\ &\leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha) \|F x_n - F x^*\|^2. \end{aligned} \quad (3.17)$$

It follows from (3.2), (3.16), and (3.17) that

$$\begin{aligned} \|y_n - x^*\|^2 &\leq (1 - \alpha_n) \|P_C(z_n - \lambda_n \nabla f(z_n)) - P_C(x^* - \lambda_n \nabla f(x^*))\|^2 + \alpha_n \|Q x_n - x^*\|^2 \\ &\leq \alpha_n \|Q x_n - x^*\|^2 + \|z_n - x^*\|^2 + \lambda_n \left( \lambda_n - \frac{2}{L} \right) \|\nabla f(z_n) - \nabla f(x^*)\|^2 \\ &\leq \alpha_n \|Q x_n - x^*\|^2 + \|x_n - x^*\|^2 + r_n(r_n - 2\alpha) \|F x_n - F x^*\|^2 \\ &\quad + \lambda_n \left( \lambda_n - \frac{2}{L} \right) \|\nabla f(z_n) - \nabla f(x^*)\|^2. \end{aligned} \quad (3.18)$$

Utilizing the convexity of  $\|\cdot\|$ , we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
&= \left\| \beta_n(x_n - x^*) + (1 - \beta_n) \frac{1}{1 - \beta_n} [\gamma_n(P_C(z_n - \lambda_n \nabla f(z_n)) - x^*) + \delta_n(Sy_n - x^*)] \right\|^2 \\
&\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left\| \frac{\gamma_n}{1 - \beta_n} (P_C(z_n - \lambda_n \nabla f(z_n)) - x^*) + \frac{\delta_n}{1 - \beta_n} (Sy_n - x^*) \right\|^2 \\
&= \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left\| \frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1 - \beta_n} + \frac{\alpha_n \gamma_n}{1 - \beta_n} (P_C(z_n - \lambda_n \nabla f(z_n)) - Qx_n) \right\|^2 \\
&\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left\| \frac{\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)}{1 - \beta_n} \right\|^2 + M\alpha_n \\
&\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|y_n - x^*\|^2 + M\alpha_n,
\end{aligned} \tag{3.19}$$

where  $M > 0$  is some appropriate constant. So, from (3.18) and (3.19), it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 + r_n(r_n - 2\alpha)(1 - \beta_n) \|Fx_n - Fx^*\|^2 \\
&\quad + \lambda_n \left( \lambda_n - \frac{2}{L} \right) (1 - \beta_n) \|\nabla f(z_n) - \nabla f(x^*)\|^2 + \left( M + \|Qx_n - x^*\|^2 \right) \alpha_n.
\end{aligned} \tag{3.20}$$

Therefore,

$$\begin{aligned}
& \lambda_n \left( \frac{2}{L} - \lambda_n \right) (1 - \beta_n) \|\nabla f(z_n) - \nabla f(x^*)\|^2 + r_n(2\alpha - r_n)(1 - \beta_n) \|Fx_n - Fx^*\|^2 \\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \left( M + \|Qx_n - x^*\|^2 \right) \alpha_n \\
&\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_n - x_{n+1}\| + \left( M + \|Qx_n - x^*\|^2 \right) \alpha_n.
\end{aligned} \tag{3.21}$$

Since  $\liminf_{n \rightarrow \infty} \lambda_n(2/L - \lambda_n)(1 - \beta_n) > 0$ ,  $\liminf_{n \rightarrow \infty} r_n(2\alpha - r_n)(1 - \beta_n) > 0$ ,  $\|x_n - x_{n+1}\| \rightarrow 0$  and  $\alpha_n \rightarrow 0$ , we have

$$\lim_{n \rightarrow \infty} \|\nabla f(z_n) - \nabla f(x^*)\| = 0, \quad \lim_{n \rightarrow \infty} \|Fx_n - Fx^*\| = 0. \tag{3.22}$$

*Step 3.* We claim that  $\lim_{n \rightarrow \infty} \|Sy_n - y_n\| = 0$ .

Indeed, set  $v_n = P_C(z_n - \lambda_n \nabla f(z_n))$ . Noticing the firm nonexpansivity of  $T_{r_n}^{(\Theta, \varphi)}$ , we have

$$\begin{aligned}
& \|z_n - x^*\|^2 \\
&= \left\| T_{r_n}^{(\Theta, \varphi)}(x_n - r_n Fx_n) - T_{r_n}^{(\Theta, \varphi)}(x^* - r_n Fx^*) \right\|^2 \\
&\leq \langle (x_n - r_n Fx_n) - (x^* - r_n Fx^*), z_n - x^* \rangle \\
&= \frac{1}{2} \left( \|x_n - x^* - r_n(Fx_n - Fx^*)\|^2 + \|z_n - x^*\|^2 - \|(x_n - x^*) - r_n(Fx_n - Fx^*) - (z_n - x^*)\|^2 \right) \\
&\leq \frac{1}{2} \left( \|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - r_n(Fx_n - Fx^*)\|^2 \right) \\
&= \frac{1}{2} \left( \|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 + 2r_n \langle x_n - z_n, Fx_n - Fx^* \rangle - r_n^2 \|Fx_n - Fx^*\|^2 \right),
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
& \|v_n - x^*\|^2 \\
&= \left\| P_C(z_n - \lambda_n \nabla f(z_n)) - P_C(x^* - \lambda_n \nabla f(x^*)) \right\|^2 \\
&\leq \langle z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*)), v_n - x^* \rangle \\
&= \frac{1}{2} \left( \|z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*))\|^2 + \|v_n - x^*\|^2 \right. \\
&\quad \left. - \|z_n - \lambda_n \nabla f(z_n) - (x^* - \lambda_n \nabla f(x^*)) - (v_n - x^*)\|^2 \right) \\
&\leq \frac{1}{2} \left( \|z_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n\|^2 \right. \\
&\quad \left. + 2\lambda_n \langle \nabla f(z_n) - \nabla f(x^*), z_n - v_n \rangle - \lambda_n^2 \|\nabla f(z_n) - \nabla f(x^*)\|^2 \right) \\
&\leq \frac{1}{2} \left( \|x_n - x^*\|^2 + \|v_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \langle \nabla f(z_n) - \nabla f(x^*), z_n - v_n \rangle \right).
\end{aligned} \tag{3.24}$$

Thus, we have

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2r_n \langle x_n - z_n, Fx_n - Fx^* \rangle - r_n^2 \|Fx_n - Fx^*\|^2, \tag{3.25}$$

$$\|v_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \|\nabla f(z_n) - \nabla f(x^*)\| \|z_n - v_n\|. \tag{3.26}$$

It follows that

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \alpha_n \|Qx_n - x^*\|^2 + (1 - \alpha_n) \|v_n - x^*\|^2 \\
&\leq \alpha_n \|Qx_n - x^*\|^2 + \|v_n - x^*\|^2 \\
&\leq \alpha_n \|Qx_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - v_n\|^2 + 2\lambda_n \|\nabla f(z_n) - \nabla f(x^*)\| \|z_n - v_n\|.
\end{aligned} \tag{3.27}$$

From (3.18), (3.19), and (3.25), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \alpha_n \|Qx_n - x^*\|^2 + (1 - \beta_n) \|z_n - x^*\|^2 + M\alpha_n \\ &\leq \|x_n - x^*\|^2 - (1 - \beta_n) \|x_n - z_n\|^2 + 2(1 - \beta_n) r_n \|x_n - z_n\| \|Fx_n - Fx^*\| \\ &\quad + (M + \|Qx_n - x^*\|^2) \alpha_n. \end{aligned} \quad (3.28)$$

It follows that

$$\begin{aligned} (1 - \beta_n) \|x_n - z_n\|^2 &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| + (M + \|Qx_n - x^*\|^2) \alpha_n \\ &\quad + 2(1 - \beta_n) r_n \|x_n - z_n\| \|Fx_n - Fx^*\|. \end{aligned} \quad (3.29)$$

Note that  $\|x_{n+1} - x_n\| \rightarrow 0$ ,  $\alpha_n \rightarrow 0$  and  $\|Fx_n - Fx^*\| \rightarrow 0$ . Then we immediately deduce that

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (3.30)$$

From (3.19) and (3.27), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - (1 - \beta_n) \|z_n - v_n\|^2 \\ &\quad + 2\lambda_n (1 - \beta_n) \|\nabla f(z_n) - \nabla f(x^*)\| \|z_n - v_n\| + (M + \|Qx_n - x^*\|^2) \alpha_n. \end{aligned} \quad (3.31)$$

So, we obtain

$$\begin{aligned} (1 - \beta_n) \|z_n - v_n\|^2 &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \|x_{n+1} - x_n\| \\ &\quad + (M + \|Qx_n - x^*\|^2) \alpha_n + 2\lambda_n (1 - \beta_n) \|\nabla f(z_n) - \nabla f(x^*)\| \|z_n - v_n\|. \end{aligned} \quad (3.32)$$

Note that  $\|x_{n+1} - x_n\| \rightarrow 0$ ,  $\alpha_n \rightarrow 0$  and  $\|\nabla f(z_n) - \nabla f(x^*)\| \rightarrow 0$ . Then we immediately conclude that

$$\lim_{n \rightarrow \infty} \|z_n - v_n\| = 0. \quad (3.33)$$

This together with  $\|y_n - v_n\| \leq \alpha_n \|Qx_n - v_n\| \rightarrow 0$ , implies that

$$\lim_{n \rightarrow \infty} \|z_n - y_n\| = 0. \quad (3.34)$$

Thus, from (3.30) and (3.34), we deduce that

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \quad (3.35)$$



Since

$$\begin{aligned} \|\delta_n(Sy_n - x_n)\| &\leq \|x_{n+1} - x_n\| + \gamma_n \|P_C(z_n - \lambda_n \nabla f(z_n)) - x_n\| \\ &\leq \|x_{n+1} - x_n\| + \gamma_n \|y_n - x_n\| + \gamma_n \alpha_n \|Qx_n - P_C(z_n - \lambda_n \nabla f(z_n))\|. \end{aligned} \quad (3.36)$$

Therefore,

$$\lim_{n \rightarrow \infty} \|Sy_n - x_n\| = 0, \quad \lim_{n \rightarrow \infty} \|Sy_n - y_n\| = 0. \quad (3.37)$$

*Step 4.* We claim that  $\limsup_{n \rightarrow \infty} \langle Qx^* - x^*, x_n - x^* \rangle \leq 0$  where  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} Qx^*$ .

Indeed, since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  such that  $x_{n_i} \rightharpoonup u$  and

$$\limsup_{n \rightarrow \infty} \langle Qx^* - x^*, x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle Qx^* - x^*, x_{n_i} - x^* \rangle = \langle Qx^* - x^*, u - x^* \rangle. \quad (3.38)$$

We can obtain that  $u \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}$ . First, we show  $u \in \Omega (= \text{VI}(C, \nabla f))$ . Since  $x_n - z_n \rightarrow 0$  and  $v_n - z_n \rightarrow 0$ , we conclude that  $z_{n_i} \rightharpoonup u$  and  $v_{n_i} \rightharpoonup u$ . Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \quad (3.39)$$

where  $N_C v$  is the normal cone to  $C$  at  $v \in C$ . We have already mentioned that in this case, the mapping  $T$  is maximal monotone, and  $0 \in Tv$  if and only if  $v \in \text{VI}(C, \nabla f) (= \Omega)$ ; see [37]. Let  $G(T)$  be the graph of  $T$  and let  $(v, w) \in G(T)$ . Then, we have  $w \in Tv = \nabla f(v) + N_C v$  and hence  $w - \nabla f(v) \in N_C v$ . So, we have  $\langle v - t, w - \nabla f(v) \rangle \geq 0$  for all  $t \in C$ . On the other hand, from  $v_n = P_C(z_n - \lambda_n \nabla f(z_n))$  and  $v \in C$ , we have

$$\langle z_n - \lambda_n \nabla f(z_n) - v_n, v_n - v \rangle \geq 0 \quad (3.40)$$

and hence

$$\left\langle v - v_n, \frac{v_n - z_n}{\lambda_n} + \nabla f(z_n) \right\rangle \geq 0. \quad (3.41)$$

From  $\langle v - t, w - \nabla f(v) \rangle \geq 0$  for all  $t \in C$  and  $v_{n_i} \in C$ , we have

$$\begin{aligned} &\langle v - v_{n_i}, w \rangle \\ &\geq \langle v - v_{n_i}, \nabla f(v) \rangle \\ &\geq \langle v - v_{n_i}, \nabla f(v) \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} + \nabla f(z_{n_i}) \right\rangle \\ &= \langle v - v_{n_i}, \nabla f(v) - \nabla f(v_{n_i}) \rangle + \langle v - v_{n_i}, \nabla f(v_{n_i}) - \nabla f(z_{n_i}) \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - v_{n_i}, \nabla f(v_{n_i}) - \nabla f(z_{n_i}) \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - z_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned} \quad (3.42)$$

Hence, we obtain  $\langle v - u, w \rangle \geq 0$  as  $i \rightarrow \infty$ . Since  $T$  is maximal monotone, we have  $u \in T^{-1}0$  and hence  $u \in \text{VI}(C, \nabla f) (= \Omega)$ .

Secondly, let us show  $u \in \text{Fix}(S)$ . Since  $x_n - y_n \rightarrow 0$  and  $x_{n_i} \rightharpoonup u$ , we have  $y_{n_i} \rightharpoonup u$ . Also, since  $y_n - Sy_n \rightarrow 0$ , it follows that  $y_{n_i} - Sy_{n_i} \rightarrow 0$  as  $i \rightarrow \infty$ . So, in terms of Proposition 2.5(ii) we obtain  $u \in \text{Fix}(S)$ .

Next, let us show  $u \in \text{GMEP}$ . From  $z_n = T_{r_n}^{(\Theta, \varphi)}(x_n - r_n Fx_n)$ , we know that

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.43)$$

From (H2), it follows that

$$\varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq \Theta(y, z_n), \quad \forall y \in C. \quad (3.44)$$

Replacing  $n$  by  $n_i$ , we have

$$\varphi(y) - \varphi(z_{n_i}) + \langle Fx_{n_i}, y - z_{n_i} \rangle + \left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \Theta(y, z_{n_i}), \quad \forall y \in C. \quad (3.45)$$

Put  $z_s = sy + (1-s)u$  for all  $s \in (0, 1]$  and  $y \in C$ . Then, we have  $z_s \in C$ . So, from (3.45), we have

$$\begin{aligned} \langle z_s - z_{n_i}, Fz_s \rangle &\geq \langle z_s - z_{n_i}, Fz_s \rangle - \varphi(z_s) + \varphi(z_{n_i}) - \langle z_s - z_{n_i}, Fx_{n_i} \rangle \\ &\quad - \left\langle z_s - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \Theta(z_s, z_{n_i}) \\ &= \langle z_s - z_{n_i}, Fz_s - Fz_{n_i} \rangle + \langle z_s - z_{n_i}, Fz_{n_i} - Fx_{n_i} \rangle - \varphi(z_s) + \varphi(z_{n_i}) \\ &\quad - \left\langle z_s - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \Theta(z_s, z_{n_i}). \end{aligned} \quad (3.46)$$

Since  $\|z_{n_i} - x_{n_i}\| \rightarrow 0$ , we have  $\|Fz_{n_i} - Fx_{n_i}\| \rightarrow 0$ . Further, from the monotonicity of  $F$ , we have  $\langle z_s - z_{n_i}, Fz_s - Fz_{n_i} \rangle \geq 0$ . So, from (H4), the weakly lower semicontinuity of  $\varphi$ ,  $(z_{n_i} - x_{n_i})/r_{n_i} \rightarrow 0$  and  $z_{n_i} \rightharpoonup u$ , we have

$$\langle z_s - z_{n_i}, Fz_s \rangle \geq -\varphi(z_s) + \varphi(u) + \Theta(z_s, u), \quad (3.47)$$

as  $i \rightarrow \infty$ . From (H1), (H4), and (3.47), we also have

$$\begin{aligned} 0 &= \Theta(z_s, z_s) + \varphi(z_s) + \varphi(z_s) \\ &\leq s\Theta(z_s, y) + (1-s)\Theta(z_s, u) + s\varphi(y) + (1-s)\varphi(u) - \varphi(z_s) \\ &= s[\Theta(z_s, y) + \varphi(y) - \varphi(z_s)] + (1-s)[\Theta(z_s, u) + \varphi(u) - \varphi(z_s)] \\ &\leq s[\Theta(z_s, y) + \varphi(y) - \varphi(z_s)] + (1-s)\langle z_s - u, Fz_s \rangle \\ &= s[\Theta(z_s, y) + \varphi(y) - \varphi(z_s)] + (1-s)s\langle y - u, Fz_s \rangle, \end{aligned} \quad (3.48)$$

and hence

$$0 \leq \Theta(z_s, y) + \varphi(y) - \varphi(z_s) + (1-s)\langle y - u, Fz_s \rangle. \quad (3.49)$$

Letting  $s \rightarrow 0$ , we have, for each  $y \in C$ ,

$$0 \leq \Theta(u, y) + \varphi(y) - \varphi(u) + \langle y - u, Fu \rangle. \quad (3.50)$$

This shows that  $u \in \text{GMEP}$ . Therefore,  $u \in \text{Fix}(S) \cap \Omega \cap \text{GMEP}$ . Hence, it follows from (2.4) that

$$\limsup_{n \rightarrow \infty} \langle Qx^* - x^*, x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle Qx^* - x^*, x_{n_i} - x^* \rangle = \langle Qx^* - x^*, u - x^* \rangle \leq 0. \quad (3.51)$$

*Step 5.* We claim that  $\lim_{n \rightarrow \infty} \|x_n - x^*\| = 0$ .

Indeed, from (3.2) and the convexity of  $\|\cdot\|$ , we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &= \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*) + \gamma_n\alpha_n(P_C(z_n - \lambda_n \nabla f(z_n)) - Qx_n)\|^2 \\ &\leq \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\|^2 \\ &\quad + 2\gamma_n\alpha_n \langle P_C(z_n - \lambda_n \nabla f(z_n)) - Qx_n, x_{n+1} - x^* \rangle \\ &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left\| \frac{1}{1 - \beta_n} [\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)] \right\|^2 \\ &\quad + 2\gamma_n\alpha_n \langle P_C(z_n - \lambda_n \nabla f(z_n)) - x^*, x_{n+1} - x^* \rangle + 2\gamma_n\alpha_n \langle x^* - Qx_n, x_{n+1} - x^* \rangle. \end{aligned} \quad (3.52)$$

Utilizing Lemma 3.1, we get from (3.52)

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \|y_n - x^*\|^2 + 2\gamma_n\alpha_n \|P_C(z_n - \lambda_n \nabla f(z_n)) - x^*\| \|x_{n+1} - x^*\| \\ &\quad + 2\gamma_n\alpha_n \langle x^* - Qx_n, x_{n+1} - x^* \rangle \\ &\leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n) \left[ (1 - \alpha_n) \|z_n - x^*\|^2 + 2\alpha_n \langle Qx_n - x^*, y_n - x^* \rangle \right] \\ &\quad + 2\gamma_n\alpha_n \|z_n - x^*\| \|x_{n+1} - x^*\| + 2\gamma_n\alpha_n \langle x^* - Qx_n, x_{n+1} - x^* \rangle. \end{aligned} \quad (3.53)$$

From (3.17), we note that  $\|z_n - x^*\| \leq \|x_n - x^*\|$ . Hence we have

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
& \leq \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)(1 - \alpha_n) \|x_n - x^*\|^2 + 2\alpha_n(1 - \beta_n) \langle Qx_n - x^*, y_n - x^* \rangle \\
& \quad + 2\gamma_n \alpha_n \|x_n - x^*\| \|x_{n+1} - x^*\| + 2\gamma_n \alpha_n \langle x^* - Qx_n, x_{n+1} - x^* \rangle \\
& \leq [1 - (1 - \beta_n)\alpha_n] \|x_n - x^*\|^2 + 2\alpha_n \gamma_n \langle Qx_n - x^*, y_n - x_{n+1} \rangle \\
& \quad + 2\alpha_n \delta_n \langle Qx_n - x^*, y_n - x^* \rangle + 2\alpha_n \gamma_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\
& \leq [1 - (1 - \beta_n)\alpha_n] \|x_n - x^*\|^2 + 2\alpha_n \gamma_n \|Qx_n - x^*\| \|y_n - x_{n+1}\| \\
& \quad + 2\alpha_n \delta_n \langle Qx_n - x^*, x_n - x^* \rangle + 2\alpha_n \delta_n \langle Qx_n - x^*, y_n - x_n \rangle + 2\alpha_n \gamma_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\
& \leq [1 - (1 - \beta_n)\alpha_n] \|x_n - x^*\|^2 + 2\alpha_n \gamma_n \|Qx_n - x^*\| \|y_n - x_{n+1}\| \\
& \quad + 2\alpha_n \delta_n \rho \|x_n - x^*\|^2 + 2\alpha_n \delta_n \langle Qx^* - x^*, x_n - x^* \rangle \\
& \quad + 2\alpha_n \delta_n \|Qx_n - x^*\| \|y_n - x_n\| + 2\alpha_n \gamma_n \|x_n - x^*\| \|x_{n+1} - x^*\| \\
& \leq [1 - (1 - \beta_n)\alpha_n] \|x_n - x^*\|^2 + 2\alpha_n \gamma_n \|Qx_n - x^*\| \|y_n - x_{n+1}\| \\
& \quad + 2\alpha_n \delta_n \rho \|x_n - x^*\|^2 + 2\alpha_n \delta_n \langle Qx^* - x^*, x_n - x^* \rangle \\
& \quad + 2\alpha_n \delta_n \|Qx_n - x^*\| \|y_n - x_n\| + \alpha_n \gamma_n (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2),
\end{aligned} \tag{3.54}$$

that is,

$$\begin{aligned}
& \|x_{n+1} - x^*\|^2 \\
& \leq \left[ 1 - \frac{(1 - 2\rho)\delta_n - \gamma_n}{1 - \alpha_n \gamma_n} \alpha_n \right] \|x_n - x^*\|^2 + \frac{[(1 - 2\rho)\delta_n - \gamma_n] \alpha_n}{1 - \alpha_n \gamma_n} \\
& \quad \times \left\{ \frac{2\gamma_n}{(1 - 2\rho)\delta_n - \gamma_n} \|Qx_n - x^*\| \|y_n - x_{n+1}\| + \frac{2\delta_n}{(1 - 2\rho)\delta_n - \gamma_n} \|Qx_n - x^*\| \|y_n - x_n\| \right. \\
& \quad \left. + \frac{2\delta_n}{(1 - 2\rho)\delta_n - \gamma_n} \langle Qx^* - x^*, x_n - x^* \rangle \right\}.
\end{aligned} \tag{3.55}$$

Note that  $\liminf_{n \rightarrow \infty} ((1 - 2\rho)\delta_n - \gamma_n) / (1 - \alpha_n \gamma_n) > 0$ . It follows that  $(\sum_{n=0}^{\infty} ((1 - 2\rho)\delta_n - \gamma_n) / (1 - \alpha_n \gamma_n)) \alpha_n = \infty$ . It is clear that

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} \left\{ \frac{2\gamma_n}{(1 - 2\rho)\delta_n - \gamma_n} \|Qx_n - x^*\| \|y_n - x_{n+1}\| + \frac{2\delta_n}{(1 - 2\rho)\delta_n - \gamma_n} \|Qx_n - x^*\| \|y_n - x_n\| \right. \\
& \quad \left. + \frac{2\delta_n}{(1 - 2\rho)\delta_n - \gamma_n} \langle Qx^* - x^*, x_n - x^* \rangle \right\} \leq 0.
\end{aligned} \tag{3.56}$$

Therefore, all conditions of Lemma 2.7 are satisfied. This immediately implies that  $x_n \rightarrow x^*$ . It is readily seen that both  $\{y_n\}$  and  $\{z_n\}$  converge strongly to the same point  $x^*$ . The proof is complete.  $\square$

Utilizing Theorem 3.2, we establish the following corollaries.

**Corollary 3.3.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  be a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $F : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ . For fixed  $u \in C$  and given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  be generated iteratively by*

$$\begin{aligned} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \alpha_n u + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.57)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$ , and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that:

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 2\alpha$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \delta_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then,  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  converge strongly to the same point  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} u$ .

**Corollary 3.4.** *Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  be a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $F : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ . Let  $Q : C \rightarrow C$  be a  $\rho$ -contraction with  $\rho \in [0, 1/2)$ . For given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  be generated iteratively by*

$$\begin{aligned} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.58)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$  and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 2\alpha$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $\gamma_n < (1 - 2\rho)\delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to the same point  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} Qx^*$ .

**Corollary 3.5.** Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $F : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a nonexpansive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ . For fixed  $u \in C$  and given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated iteratively by

$$\begin{aligned} \Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \alpha_n u + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.59)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$ , and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that:

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 2\alpha$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $\gamma_n < \delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \gamma_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then,  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  converge strongly to the same point  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} u$ .

**Corollary 3.6.** Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $A : C \rightarrow H$  is an  $\alpha$ -inverse strongly monotone mapping. Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{VI}(C, A) \neq \emptyset$ . For fixed  $u \in C$  and given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated iteratively by

$$\begin{aligned} z_n &= P_C(x_n - r_n A x_n), \\ y_n &= \alpha_n u + (1 - \alpha_n) P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.60)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 2\alpha]$ , and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that:

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 2\alpha$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \delta_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then,  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  converge strongly to the same point  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{VI}(C, A)} u$ .

*Proof.* In Theorem 3.2, putting  $\Theta = 0$ ,  $\varphi = 0$  and  $F = A$ , the following relation

$$\Theta(z_n, y) + \varphi(y) - \varphi(z_n) + \langle Fx_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \quad (3.61)$$

is reduced to

$$\langle Ax_n, y - z_n \rangle + \frac{1}{r_n} \langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C. \quad (3.62)$$

This implies that

$$\langle y - z_n, x_n - r_n Ax_n - z_n \rangle \leq 0, \quad \forall y \in C. \quad (3.63)$$

So, it follows that  $z_n = P_C(x_n - r_n Ax_n)$  for all  $n \geq 0$ . Then, by Theorem 3.2, we obtain the desired result.  $\square$

Let  $T : C \rightarrow C$  be a  $\tilde{k}$ -strictly pseudocontractive mapping. For recent convergence result for strictly pseudocontractive mappings, we refer to Zeng et al. [38]. Putting  $F = I - T$ , we know that

$$\|(I - F)x - (I - F)y\|^2 \leq \|x - y\|^2 + \tilde{k} \|Fx - Fy\|^2, \quad \forall x, y \in C. \quad (3.64)$$

Note that

$$\|(I - F)x - (I - F)y\|^2 = \|x - y\|^2 + \|Fx - Fy\|^2 - 2\langle x - y, Fx - Fy \rangle. \quad (3.65)$$

Hence

$$\langle x - y, Fx - Fy \rangle \geq \frac{1 - \tilde{k}}{2} \|Fx - Fy\|^2, \quad \forall x, y \in C. \quad (3.66)$$

This implies that the mapping  $F = I - T$  is  $((1 - \tilde{k})/2)$ -inverse-strongly monotone.

**Corollary 3.7.** Let  $C$  be a nonempty bounded closed convex subset of a real Hilbert space  $H$ . Let  $\Theta : C \times C \rightarrow \mathbf{R}$  be a bifunction satisfying conditions (H1)–(H4) and  $\varphi : C \rightarrow \mathbf{R}$  a lower semicontinuous and convex function with assumptions (A1) or (A2). Suppose the minimization (1.5) is consistent and let  $\Omega$  denote its solution set. Assume the gradient  $\nabla f$  is  $L$ -Lipschitzian with constant  $L > 0$  and  $T : C \rightarrow C$  is a  $\tilde{k}$ -strictly pseudocontractive mapping. Let  $S : C \rightarrow C$  be a  $k$ -strictly pseudocontractive mapping such that  $\text{Fix}(S) \cap \Omega \cap \text{GMEP} \neq \emptyset$ , where  $F = I - T$ . For fixed  $u \in C$  and given  $x_0 \in C$  arbitrarily, let the sequences  $\{x_n\}$ ,  $\{y_n\}$ , and  $\{z_n\}$  be generated iteratively by

$$\begin{aligned} z_n &= T_{r_n}^{(\Theta, \varphi)}((1 - r_n)x_n + r_nTx_n), \\ y_n &= \alpha_n u + (1 - \alpha_n)P_C(z_n - \lambda_n \nabla f(z_n)), \\ x_{n+1} &= \beta_n x_n + \gamma_n P_C(z_n - \lambda_n \nabla f(z_n)) + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \quad (3.67)$$

where  $\{\lambda_n\} \subset (0, 2/L]$ ,  $\{r_n\} \subset (0, 1 - \tilde{k}]$  and  $\{\alpha_n\}$ ,  $\{\beta_n\}$ ,  $\{\gamma_n\}$ ,  $\{\delta_n\}$  are four sequences in  $[0, 1]$  such that

- (i)  $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < 2/L$  and  $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1}) = 0$ ;
- (ii)  $0 < \liminf_{n \rightarrow \infty} r_n \leq \limsup_{n \rightarrow \infty} r_n < 1 - \tilde{k}$  and  $\lim_{n \rightarrow \infty} (r_n - r_{n+1}) = 0$ ;
- (iii)  $\beta_n + \gamma_n + \delta_n = 1$  and  $(\gamma_n + \delta_n)k \leq \gamma_n < \delta_n$  for all  $n \geq 0$ ;
- (iv)  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (v)  $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$  and  $\liminf_{n \rightarrow \infty} \delta_n > 0$ ;
- (vi)  $\lim_{n \rightarrow \infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$ .

Then,  $\{x_n\}$ ,  $\{y_n\}$  and  $\{z_n\}$  converge strongly to the same point  $x^* = P_{\text{Fix}(S) \cap \Omega \cap \text{GMEP}} u$ .

*Proof.* Since  $T$  is a  $\tilde{k}$ -strictly pseudocontractive mapping, the mapping  $F = I - T$  is  $(1 - \tilde{k})/2$ -inverse-strongly monotone. In this case, put  $\alpha = (1 - \tilde{k})/2$ . Then, we conclude that

$$z_n = T_{r_n}^{(\Theta, \varphi)}(x_n - r_n F x_n) = T_{r_n}^{(\Theta, \varphi)}(x_n - r_n(I - T)x_n) = T_{r_n}^{(\Theta, \varphi)}((1 - r_n)x_n + r_n T x_n). \quad (3.68)$$

So, by Theorem 3.2, we obtain the desired result.  $\square$

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