Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2012, Article ID 682960, 28 pages
doi:10.1155/2012/682960

Research Article

Lebesgue's Differentiation Theorems in R.I. Quasi-Banach Spaces and Lorentz Spaces $\Gamma_{p,w}$

Maciej Ciesielski[1] and Anna Kamińska[2]

[1] *Institute of Mathematics, Poznań University of Technology, Piotrowo 3a, 60-965 Poznań, Poland*
[2] *Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA*

Correspondence should be addressed to Anna Kamińska, kaminska@memphis.edu

Received 10 October 2011; Accepted 20 January 2012

Academic Editor: Lech Maligranda

The paper is devoted to investigation of new Lebesgue's type differentiation theorems (LDT) in rearrangement invariant (r.i.) quasi-Banach spaces E and in particular on Lorentz spaces $\Gamma_{p,w} = \{f : \int (f^{**})^p w < \infty\}$ for any $0 < p < \infty$ and a nonnegative locally integrable weight function w, where f^{**} is a maximal function of the decreasing rearrangement f^* for any measurable function f on $(0, \alpha)$, with $0 < \alpha \leq \infty$. The first type of LDT in the spirit of Stein (1970), characterizes the convergence of quasinorm averages of $f \in E$, where E is an order continuous r.i. quasi-Banach space. The second type of LDT establishes conditions for pointwise convergence of the best or extended best constant approximants f_ϵ of $f \in \Gamma_{p,w}$ or $f \in \Gamma_{p-1,w}$, $1 < p < \infty$, respectively. In the last section it is shown that the extended best constant approximant operator assumes a unique constant value for any function $f \in \Gamma_{p-1,w}$, $1 < p < \infty$.

1. Introduction

The present paper is devoted to investigation of maximal inequalities and Lebesgue's type differentiation theorems for best local approximations in r.i. quasi-Banach spaces and Lorentz spaces $\Gamma_{p,w}$ for $0 < p < \infty$. In 1910, Henry Lebesgue has proved one of the most famous differentiation theorem, which establishes a convergence of an integral average of any locally integrable function f on the ball $B(v, \epsilon) \subset \mathbb{R}^n$ to this function f, that is, for a.a. $v \in \mathbb{R}^n$,

$$\frac{1}{\mu(B(v,\epsilon))} \int_{B(v,\epsilon)} f(t)dt \longrightarrow f(v) \quad \text{as } \epsilon \longrightarrow 0. \tag{1.1}$$

In fact, Lebesgue's integral average coincides with a best constant approximant on the space $L^2(\mathbb{R}^n)$ [1]. The *Lebesgue Differentiation Theorem* (LDT) can be proved as a consequence of the *weak maximal inequality*

$$t(M_H f)^*(t) \leq 4^n \|f\|_{L^1} \tag{1.2}$$

for the Hardy-Littlewood maximal function $M_H f$ where $t \in \mathbb{R}^n$ and $f \in L^1(\mathbb{R}^n)$ [2]. The interesting exploration of LDT was initiated by Stein in [3], who introduced the maximal functions on $L^p(\mathbb{R}^n)$, associated with integral average, and applied it to obtain differentiation theorem in the notation of the norm in $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. In the spirit of this idea many authors developed new techniques of recovering functions in quasi-Banach function spaces. The first results in this subject were obtained by Bastero et al. [4] in 1999, who have investigated Hardy-Littlewood maximal functions and weak maximal inequalities in rearrangement invariant quasi-Banach function spaces. The next paper was published by Mazzone and Cuenya in 2001 [1], about generalizations of the classical Lebesgue differentiation theorem for the best local approximation by constants over balls in $L^p + L^\infty$ for $0 \leq p < \infty$. They evaluated maximal inequalities for the maximal function related to best constant approximants and proved convergence theorem for best constant approximants. In 2008 [5] Levis et al. extended the best constant approximant operator from Orlicz-Lorentz spaces $\Lambda_{w,\phi}$ to the spaces $\Lambda_{w,\phi'}$ and showed monotonicity of the extended operator. In view of this result, in 2009 [6] Levis established maximal inequalities for the maximal function associated with the best constant approximation and proved Lebesgue's type differentiation theorem for best constant approximants and for integral averages expressed in terms of the modular corresponding to these spaces. Recently, the authors have characterized properties of an expansion of the best constant approximant operator from Lorentz spaces $\Gamma_{p,w}$ to the spaces $\Gamma_{p-1,w}$. The present paper is a continuation of the previous results and devoted to investigation of maximal inequalities and Lebesgue's type differentiation theorems for local approximation in r.i. quasi-Banach space E and in particular in $\Gamma_{p,w}$.

The paper consists of three sections and is organized as follows.

In the preliminaries, Section 2, we establish some basic notations and definitions and also recall some auxiliary results, which will be used later.

Sections 3 and 4 consist of the main results of the paper.

We start Section 3 proving measurability of the maximal function $M_E^{(r)}(f)$ for $f \in E$, that corresponds to the quasi-norm average of f, in r.i. quasi-Banach function spaces E. Next we establish two types of generalization of LDT in r.i. quasi-Banach function spaces E and in $\Gamma_{p,w}$. In both types of LDT we employ the assumption of upper and lower ϕ-estimates of $\Gamma_{p,w}$. The first main result, in the spirit of Stein [3], has been proved for any order continuous r.i. quasi-Banach function space. The statement is expressed in terms of quasi-norm averages. In order to show it we first prove the inequality for maximal function $M_E^{(r)} f$, which corresponds to a quasi-norm average of $f \in E$. In the same spirit we also provide some conditions when the LDT does not hold in E or in $\Gamma_{p,w}$. Next we continue our discussion with another type of LDT. In order to complete the second main result in this section we characterize conditions for which Lorentz space $\Gamma_{p,w}$ satisfies a lower (resp., an upper) ϕ-estimate, where ϕ is the fundamental function of $\Gamma_{p,w}$. In view of this characterization we investigate pointwise convergence of the best constant approximants f_ϵ to f as $\epsilon \to 0$ whenever $f \in \Gamma_{p,w}$ and $1 \leq p < \infty$, as well as the convergence of the extended best constant approximants f_ϵ for any $f \in \Gamma_{p-1,w}$ and $1 < p < 2$. We also present examples showing that this assumption is

fulfilled by a large class of the spaces $\Gamma_{q,p}$. Finally, we investigate relations between maximal functions and the K-functional of Banach couple $(\Gamma_{q,p}, L^\infty)$ in the spirit of the inequalities stated in [4]. We finish Section 3 with an example showing that $M^{(r)}_{\Gamma_{p,w}}(f)$ is not equivalent to the K-functional of the pair $(\Gamma_{p,w}, L^\infty)$.

It is well known that the extension of the best constant approximant operator from $\Gamma_{1,w}$ to L^0, or from L_1 to L^0, is a set valued function [1, 7]. Contrary to this in Theorem 4.5 we prove that the extended best constant approximant operator assumes a unique value for any $f \in \Gamma_{p-1,w}$ and $1 < p < \infty$. To show the uniqueness we need to consider strict monotonicity of the right-hand Gâteaux derivative of the norm in $\Gamma_{p,w}$ at $(f-u)\chi_A$ in the direction χ_A for any $f \in \Gamma_{p-1,w}$ and $u \in \mathbb{R}$.

2. Preliminaries

Let $\mathbb{R}$ and $\mathbb{N}$ be the set of real and natural numbers, respectively. For any $A \subset [0,\alpha)$ denote $A^c = [0,\alpha) \setminus A$. Let $0 < \alpha \le \infty$ and μ be the Lebesgue measure on $\mathbb{R}$. We denote by L^0 the space of all extended real-valued μ-measurable and finite functions a.e. on $[0,\alpha)$. Denote the outer measure on $\mathbb{R}$ by μ_*, the support of $f \in L^0$ by $\mathbb{S}(f) = \operatorname{supp}(f)$, and the restriction of f to the set $A \subset [0,\alpha)$ by $f|_A$. By a simple (resp., step) function we mean a measurable function $f \in L^0$ with a finite measure support, which attains a finite number of values (resp., a finite number of values on a finite number of disjoint intervals). The *distribution function* d_f of a function $f \in L^0$ is given by $d_f(\lambda) = \mu(s \in [0,\alpha) : |f|(s) > \lambda)$ for all $\lambda \ge 0$. Two functions $f, g \in L^0$ are called *equimeasurable*, if $d_f(\lambda) = d_g(\lambda)$ for all $\lambda \ge 0$. We define the *decreasing rearrangement* for any $f \in L^0$ by $f^*(t) = \inf\{s > 0 : d_f(s) \le t\}$, $t \ge 0$. For given $f \in L^0$ we denote the *maximal function* of f^* by $f^{**}(t) = (1/t)\int_0^t f^*(s)ds$. It is well known that $f^* \le f^{**}$ and f^{**} is decreasing and subadditive, that is, $(f+g)^{**} \le f^{**} + g^{**}$ for any $f, g \in L^0$. For the properties of d_f, f^*, and f^{**} see [2, 8]. A subspace $E \subset L^0$ equipped with a quasinorm $\|\cdot\|_E$ is called a *quasinormed function space*, if the following conditions are satisfied.

(1) If $f \in L^0$, $g \in E$, and $|f| \le |g|$ a.e., then $f \in E$ and $\|f\|_E \le \|g\|_E$.

(2) There exists a strictly positive $f \in E$.

If E is complete, then it is said to be a *quasi-Banach function space*. We say that a quasi-Banach function space E is *rearrangement invariant* (r.i. for short), if whenever $f \in L^0$ and $g \in E$ with $d_f = d_g$, then $f \in E$ and $\|f\|_E = \|g\|_E$ (see [2]). Throughout the paper we use the notation $A \approx B$, which means that the expressions A and B are equivalent; that is, A/B is bounded from both sides. Let $0 < p < \infty$ and $w \in L^0$ be a nonnegative weight function. Lorentz space $\Gamma_{p,w}$ is a subspace of L^0 such that

$$\|f\| = \|f\|_{\Gamma_{p,w}} := \left(\int_0^\alpha f^{**p} w\right)^{1/p} = \left(\int_0^\alpha f^{**p}(t) w(t) dt\right)^{1/p} < \infty. \tag{2.1}$$

Given a measurable set $A \subset [0,\alpha)$ by $\Gamma_{p,w}(A)$ we denote the set of $f \in L^0$ restricted to A and satisfying the above inequality. Unless we say otherwise, throughout the paper we

assume that w belongs to the class D_p (in short $w \in D_p$), whenever it satisfies the following conditions:

$$W(s) := \int_0^s w < \infty, \qquad W_p(s) := s^p \int_s^\alpha t^{-p} w(t)\, dt < \infty \tag{2.2}$$

for all $0 < s \leq \alpha$ if $\alpha < \infty$ and for all $0 < s < \infty$ otherwise. These two conditions guarantee that $\Gamma_{p,w} \neq \{0\}$. We also assume that

$$W_p(s) > 0 \quad \text{for } 0 < s < \alpha, \qquad W(\infty) = \int_0^\infty w = \infty \quad \text{if } \alpha = \infty. \tag{2.3}$$

Under these assumptions $(\Gamma_{p,w}, \|\cdot\|)$ is a rearrangement invariant (r.i. for short) quasi-Banach function space such that it has the Fatou property and the order continuous norm. Letting $0 < p,\ q < \infty$ and $w(t) = t^{p/q-1}$, $t \in (0, \alpha)$, the space $\Gamma_{p,w}$ will be denoted by $\Gamma_{q,p}$.

Unless we say otherwise, throughout this paper we assume that ϕ is the fundamental function of $\Gamma_{p,w}$ defined as $\phi(t) = \|\chi_{(0,t)}\|$, $t \in (0, \alpha)$, and $\phi(0) = 0$. It is easy to show that the fundamental function ϕ is strictly increasing and continuous on $[0, \alpha)$, $\lim_{t\to\infty}\phi(t) = \infty$ and $\lim_{t\to\infty} f^*(t) = 0$ for $f \in \Gamma_{p,w}$. For more details about the properties of $\Gamma_{p,w}$ see [9].

Recall that for given $0 < p < \infty$, classical Lorentz space $\Lambda_{p,w}$ is a subspace of L^0 such that

$$\|f\|_{\Lambda_{p,w}} := \left(\int_0^\alpha f^{*p} w\right)^{1/p} = \left(\int_0^\alpha f^{*p}(t) w(t) dt\right)^{1/p} < \infty. \tag{2.4}$$

In case when W satisfies the Δ_2-condition, that is $W(2s) \leq CW(s)$ for all $s > 0$ and some $C > 0$, as well as $W(\infty) = \infty$, the space $\Lambda_{p,w}$ is a separable r.i. order continuous quasi-Banach function space [9]. The space $\Lambda_{p,w}$ is a r.i. Banach function space, whenever the weight w is decreasing and $1 \leq p < \infty$ [10]. Since $f^* \leq f^{**}$, we have the natural inclusion $\Gamma_{p,w} \subset \Lambda_{p,w}$. Moreover, $\Gamma_{p,w} = \Lambda_{p,w}$ if and only if w satisfies condition B_p, ($w \in B_p$ for short) which means that there is $A > 0$ such that for all $s > 0$ we have $W_p(s) \leq AW(s)$ [11–13].

Let (Ω_1, μ_1) and (Ω_2, μ_2) be σ-finite measure spaces. A map γ from Ω_1 into Ω_2 is said to be a *measure-preserving transformation*, if whenever E is a μ_2-measurable subset of Ω_2, the set $\gamma^{-1}(E) = \{u \in \Omega_1 : \gamma(u) \in E\}$ is a μ_1-measurable subset of Ω_1 and $\mu_1(\gamma^{-1}(E)) = \mu_2(E)$. For given subsets $A, B \subset \mathbb{R}_+$ such that $\mu(A) = \mu(B)$, there exists a measure-preserving transformation $\delta : A \to B$ [14, Theorem 17, page 410].

Definition 2.1 (see [15]). Let $f, h \in L^0$. Denote

$$\begin{aligned}\tau_{(f,h)}(t) = d_f(|f|(t)) &+ \mu(u : |f|(u) = |f|(t),\ h(u)\ \operatorname{sign}(f(u)) > h(t)\operatorname{sign}(f(t))) \\ &+ \mu(u : |f|(u) = |f|(t),\ h(u)\ \operatorname{sign}(f(u)) = h(t)\ \operatorname{sign}(f(t)),\ u \leq t)\end{aligned} \tag{2.5}$$

for all $t \in [0, \alpha)$.

In 1970, Ryff proved in [16] that $\tau_{(f|_{[0,1]},0)} : [0,1] \to [0,1]$ is a measure-preserving transformation for any $f \in L^0$ and $|f| = f^* \circ \tau_{(|f|_{[0,1]},0)}$ a.e. on $[0,1]$. In 1993, Carothers et al.

established in [15] that $\tau_{(f,h)}$ is a measure preserving transformation from $\mathbb{S}(f)$ onto $\mathbb{S}(f^*)$ such that $|f| = f^* \circ \tau_{(f,h)}$ a.e. on $\mathbb{S}(f)$ for any $f \in L^0$ with $d_f(\lambda) < \infty$ for all $\lambda > 0$ and any $h \in L^0$. Notice that for any $f \in L^0$ with $d_f(\lambda) < \infty$ for any $\lambda > 0$ and $h \in L^0$, if $\mu(u : |f|(u) = v) = 0$ for every $v > 0$ we have that $\tau_{(f,h)}(t) = d_f(|f|(t))$ and it is the unique measure-preserving transformation up to measure zero satisfying $|f| = f^* \circ \tau_{(f,h)}$ a.e. on $\mathbb{S}(f)$.

Definition 2.2. Let $f, g \in \Gamma_{p,w}$ and let $\tau_{(f,g)}, \tau_{(g|_{\mathbb{S}(g)\setminus\mathbb{S}(f)},0)}$ be measure-preserving transformations given by Definition 2.1. Denote

$$\rho_{(f,g)}(s) = \begin{cases} \tau_{(f,g)}(s) & \text{if } s \in \mathbb{S}(f), \\ d_f(0) + \tau_{(g|_{\mathbb{S}(g)\setminus\mathbb{S}(f)},0)}(s) & \text{if } s \in \mathbb{S}(g) \setminus \mathbb{S}(f). \end{cases} \tag{2.6}$$

Definition 2.3. Let $f \in \Gamma_{p,w}$ and $A \subset [0, \alpha)$ with $\mu(A) < \infty$. Denote

$$\begin{aligned} K_{(f,A)}(u,t) &= \frac{1}{t}\int_A \left(1 - 2\chi_{\{f<u\}}\right)\chi_{(0,t)}\left(\rho_{((f-u)\chi_A,\chi_A)}\right), \\ S^p_{(f,A)}(u) &= \int_0^\alpha K_{(f,A)}(u,t)\left((f-u)\chi_A\right)^{**(p-1)}(t)w(t)dt \end{aligned} \tag{2.7}$$

for any $u \in \mathbb{R}$ and $t \in (0, \alpha)$.

Let $(X, \|\cdot\|_X)$ be a real normed space. Denote by B_X (resp., S_X) the *closed unit ball* (resp., the *unit sphere*) of X. Assume that Y is a subset of X and x is an element of X. An element $\tilde{x} \in Y$ is called *best approximant* to x from Y if

$$\|x - \tilde{x}\|_X = \inf_{y \in Y} \|x - y\|_X. \tag{2.8}$$

A nonempty subset Y of X is a *set of uniqueness* if for any element $x \in X$ there is no more than one element $\tilde{x} \in Y$ satisfying (2.8). The set Y is a *set of existence* if for every element $x \in X$ there is at least one element $\tilde{x} \in Y$ for which condition (2.8) holds. The set Y is a *Chebyshev set* if for every element $x \in X$ there exists exactly one element $\tilde{x} \in Y$ satisfying (2.8), that is, if Y is both a set of uniqueness and a set of existence (for more details see [17]). Let 2^Y be a collection of all subsets of Y. A set value map $T_Y : X \to 2^Y$ is said to be *best approximant operator*, if it assumes for any $x \in X$ a set of all best approximant elements to x from Y, that is,

$$T_Y(x) = \left\{\tilde{x} \in Y : \|x - \tilde{x}\|_X = \inf_{y \in Y}\|x - y\|_X\right\} = \left\{\tilde{x} \in Y : \|x - \tilde{x}\|_X \le \|x - y\|_X,\ \forall y \in Y\right\}. \tag{2.9}$$

In case when X is a norm function space and Y is a family of constant functions, then T_Y is called *best constant approximant operator,* and each element $\tilde{x} \in T_Y(x)$ is called *best constant approximant* to $x \in X$ from Y. Let $A \subset [0, \alpha)$ with $0 < \mu(A) < \infty$ and $\mathbb{K}(A) = \{c\chi_A : c \in \mathbb{R}\}$. It is well known that the set $T_{\mathbb{K}(A)}(f)$ is convex, compact, and a set of existence for all $f \in \Gamma_{p,w}$ [18, 19]. Let's recall some characterizations of best constant approximants over Lorentz spaces $\Gamma_{p,w}$.

Theorem 2.4 (see [20, Theorem 7.5]). *Let $1 \le p < \infty$ and let $f \in \Gamma_{p,w} \setminus \mathbb{K}(A)$. Then $u \in \mathbb{K}(A)$ is the best constant approximant of f if and only if*

$$S^p_{(f,A)}(u) \ge 0, \qquad S^p_{(-f,A)}(-u) \ge 0. \tag{2.10}$$

Corollary 2.5 (see [20, Corollary 7.3]). *For any $1 < p < \infty$ and w positive, we have that $T_{\mathbb{K}(A)}(f)$ is Chebyshev set for any $f \in \Gamma_{p,w}$; that is, there is an unique best constant approximant $u \in \mathbb{K}(A)$ to f.*

Recently, in [7] it has been developed an existence of extension of the best constant approximant operators from Lorentz space $\Gamma_{1,w}$ to L^0, if $w \in D_1$, and from $\Gamma_{p,w}$ to $\Gamma_{p-1,w}$, if $1 < p < \infty$ and $w \in D_{p-1}$. Now we recall definition of the extended operator $T_{(p,A)}$ on L^0, if $p = 1$, and on $\Gamma_{p-1,w}$, if $1 < p < \infty$.

Definition 2.6 (see [7]). Let $A \subset (0,\alpha)$ with $0 < \mu(A) < \infty$ and let $w \in D_{p-1}$ if $1 < p < \infty$, and $w \in D_1$ if $p = 1$. Assume that $f \in \Gamma_{p-1,w}$ if $p > 1$, and $f \in L^0$ if $p = 1$. Denote

$$\underline{f}_{(p,A)} = \min\left\{u : S^p_{(-f,A)}(-u) \ge 0\right\}, \qquad \overline{f}_{(p,A)} = \max\left\{u : S^p_{(f,A)}(u) \ge 0\right\}. \tag{2.11}$$

Then the extended best constant approximant operator is given by

$$T_{(p,A)}(f) = \left[\underline{f}_{(p,A)}\chi_A, \overline{f}_{(p,A)}\chi_A\right]. \tag{2.12}$$

In fact, any $u \in T_{(p,A)}(f)$ is called an *extended best constant approximant* of f. Notice that in view of Theorem 2.4, if $f \in \Gamma_{p,w}$ for $1 \le p < \infty$, then any $u \in T_{(p,A)}(f)$ is a classical best constant approximant of f.

Definition 2.7 (see [4]). Let $0 < r < \infty$, $(E, \|\cdot\|_E)$ be a r.i. quasi-Banach function space, and let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing bijection. E is said to satisfy an upper (resp., a lower) Φ-estimate for $\|\cdot\|_E^r$, if there exists $C > 0$ such that for all $n \in \mathbb{N}$ and $(f_i)_{i=1}^n \subset E$ with pairwise disjoint supports we have

$$\left\|\sum_{i=1}^n f_i\right\|_E^r \le C\Phi\left(\sum_{i=1}^n \Phi^{-1}(\|f_i\|_E^r)\right), \tag{2.13}$$

respectively

$$\Phi\left(\sum_{i=1}^n \Phi^{-1}(\|f_i\|_E^r)\right) \le C\left\|\sum_{i=1}^n f_i\right\|_E^r. \tag{2.14}$$

In the case when $\Phi(t) = t^{1/p}$ and $r = 1$, this definition covers the notions of the upper (resp., lower) p-estimate [21].

Let $0 < p < \infty$ and X be a quasi-Banach function space. We denote by $X^{(p)} = \{f : |f|^p \in X\}$ the *p-convexification* of X equipped with the quasinorm $\|\cdot\|_{X^{(p)}} = \||\cdot|^p\|_X^{1/p}$. Now we recall

the definition of the maximal function for any r.i. quasi-Banach function space, that plays a crucial role in process of generalization of Lebesgue's Differentiation Theorem in $\Gamma_{p,w}$.

Definition 2.8. Let $0 < r < \infty$, $(E, \|\cdot\|_E)$ be a r.i. quasi-Banach function space. For any $f \in E$ we denote

$$f_\epsilon^{(r)}(t) = \frac{\left\|f\chi_{B(t,\epsilon)}\right\|_E^r}{\left\|\chi_{B(t,\epsilon)}\right\|_E} \tag{2.15}$$

for all $\epsilon > 0$ and $t \in (0,\alpha)$. The maximal function $M_E^{(r)} f : (0,\alpha) \to \mathbb{R}$ is given by

$$M_E^{(r)} f(t) = \sup\left\{ f_\epsilon^{(r)}(t) : \epsilon > 0,\ B(t,\epsilon) \subset (0,\alpha) \right\} \tag{2.16}$$

for any $t \in (0,\alpha)$, where $B(t,\epsilon) = (t-\epsilon,\ t+\epsilon)$.

We finish the preliminaries with the following proposition needed further. It is a generalization of the well-known result, which in particular can be found in [22] for special case when $\varphi(t) = W(t)$ on $(0,\infty)$. The proof of the proposition is quite standard and is provided for the sake of completeness.

Proposition 2.9. *Let $\alpha = \infty$, $f \in L^0$, and let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing continuous function. If $\lim_{t\to\infty}\varphi(t) = \infty$ or $d_f(\lambda) < \infty$ for all $\lambda > 0$, then*

$$\sup_{t>0}\left\{\varphi(t) f^*(t)\right\} = \sup_{t>0}\left\{t\varphi\left(d_f(t)\right)\right\}. \tag{2.17}$$

Proof. Notice first if $\lim_{t\to\infty}\varphi(t) = \infty$ and there exists $t > 0$ such that $d_f(t) = \infty$, then $\lim_{s\to\infty}\varphi(s) f^*(s) = \infty$, and the conclusion follows.

Now assume that $f = \sum_{i=1}^n a_i \chi_{E_i}$ is a nonnegative simple function, where $a_i > a_{i+1}$ for any $1 \le i \le n-1$, $a_{n+1} = 0$, and $E_i \cap E_j = \emptyset$ whenever $i \ne j$. Then $f^*(t) = \sum_{i=1}^n a_i \chi_{[m_{i-1},m_i)}(t)$ for any $t > 0$, where $m_0 = 0$ and $m_i = \sum_{j=1}^i \mu(E_j)$ for $1 \le i \le n$. We claim that

$$\sup_{t>0}\left\{\varphi(t) f^*(t)\right\} = \max_{1\le i\le n}\left\{a_i\varphi(m_i)\right\}. \tag{2.18}$$

By monotonicity and continuity of φ we obtain $\sup_{t>0}\{\varphi(t) f^*(t)\} \le \max_{1\le i\le n}\{a_i\varphi(m_i)\}$. On the other hand we have $\sup_{t>0}\{\varphi(t) f^*(t)\} \ge \sup_{t>0}\{\varphi(t) a_i \chi_{[m_{i-1},m_i)}(t)\} = a_i\varphi(m_i)$. Therefore $\sup_{t>0}\{\varphi(t) f^*(t)\} \ge \max_{1\le i\le n}\{a_i\varphi(m_i)\}$, which implies condition (2.18). Now we will show

$$\sup_{t>0}\left\{t\varphi\left(d_f(t)\right)\right\} = \max_{1\le i\le n}\left\{a_i\varphi(m_i)\right\}. \tag{2.19}$$

For $t > 0$ we have $t\varphi(d_f(t)) = t\sum_{i=1}^n \varphi(m_i)\chi_{[a_{i+1},a_i)}(t) \le \max_{1\le i\le n}\{a_i\varphi(m_i)\}$, and so $\sup_{t>0}\{t\varphi(d_f(t))\} \le \max_{1\le i\le n}\{a_i\varphi(m_i)\}$. On the other hand, $\sup_{t>0}\{t\varphi(d_f(t))\} \ge a_i\varphi(m_i)$ for every $1 \le i \le n$, and consequently, $\sup_{t>0}\{t\varphi(d_f(t))\} \ge \max_{1\le i\le n}\{a_i\varphi(m_i)\}$, which provides (2.19). Both (2.18) and (2.19) show (2.17) for any nonnegative simple function f.

Now suppose that f is a measurable function and $d_f(t) < \infty$ for any $t > 0$. Then, by standard arguments of existing a sequence of nonnegative simple functions (f_n) such that $d_{f_n} \uparrow d_f$ and $f_n^* \uparrow f^*$ as $n \to \infty$ we can show that

$$\sup_{t>0}\{\varphi(t)f^*(t)\} = \lim_{n\to\infty}\sup_{t>0}\{\varphi(t)f_n^*(t)\} = \lim_{n\to\infty}\sup_{t>0}\{t\varphi(d_{f_n}(t))\} = \sup_{t>0}\{t\varphi(d_f(t))\} \tag{2.20}$$

and conclude the proof. □

In fact, Proposition 2.9 describes the largest family of increasing and continuous functions φ, for which (2.17) is satisfied. Indeed, let $\alpha = \infty$, $f \equiv 1$, and $\varphi(t) = 2 - (1/(1+t))$ for any $t \in [0,\infty)$. Then $f^* \equiv f$, $d_f(t) = 0$ for all $t \geq 1$ and $d_f(t) = \infty$ for any $t \in [0,1)$. Clearly, φ is increasing and continuous and also $\lim_{t\to\infty}\varphi(t) = 2$. Therefore, $\sup_{t>0}\{\varphi(t)f^*(t)\} = 2$ and $\sup_{t>0}\{t\varphi(d_f(t))\} = \infty$, which implies that condition (2.17) does not hold.

3. Lebesgue's Differentiation Theorems

The intention of this section is to establish generalizations of LDT in r.i. quasi-Banach function spaces E in terms of the formulas expressed by quasinorm averages. We also focus on convergence of the best and the extended best constant approximant of $f \in E$ to f, which is another type of LDT. First we introduce the notion of the differentiation property for a quasi-Banach function space E.

Definition 3.1. Let $(E, \|\cdot\|_E)$ be a quasi-Banach function space on $[0,\alpha)$. We say that E has the Lebesgue differentiation property (LDP), whenever for any $f \in E$ and for a.a. $t \in (0,\alpha)$ we have

$$\lim_{\epsilon\to 0}\frac{\|(f - f(t))\chi_{B(t,\epsilon)}\|_E}{\|\chi_{B(t,\epsilon)}\|_E} = 0. \tag{3.1}$$

Observe that letting $(E, \|\cdot\|_E)$ be a quasi-Banach function space on $[0,\alpha)$ with LDP, by the Aoki-Rolewicz theorem [23] there exist $0 < r \leq 1$ and an equivalent r-norm $|||\cdot|||_E$ to $\|\cdot\|_E$ such that for any $f \in E$ and for a.a. $t \in (0,\alpha)$ we get

$$\lim_{\epsilon\to 0}\frac{|||f\chi_{B(t,\epsilon)}|||_E}{|||\chi_{B(t,\epsilon)}|||_E} = |f(t)|. \tag{3.2}$$

If $(E, \|\cdot\|_E)$ is a normed space, then the quasinorm $|||\cdot|||_E$ can be replaced by $\|\cdot\|_E$.

In the next proposition we establish measurability of the maximal function $M_E^{(r)}f$.

Proposition 3.2. *Let $0 < r < \infty$ and let E be a r.i. order continuous quasi-Banach function space. If $f \in E$, then the maximal function $M_E^{(r)}f$ is measurable on $(0,\alpha)$.*

Proof. Let $\epsilon > 0$ and $t \in (0,\alpha)$. We first observe that $f_\epsilon^{(r)}(t)$ is continuous on $(0,\alpha)$. In fact, for any $t_n \to t$, $\lim_{n\to\infty}\chi_{B(t_n,\epsilon)} = \chi_{B(t,\epsilon)}$, and by order continuity of E we obtain that

$\lim_{n\to\infty}\|f\chi_{B(t_n,\epsilon)}\|_E^r = \|f\chi_{B(t,\epsilon)}\|_E^r$ and $\lim_{n\to\infty}\|\chi_{B(t_n,\epsilon)}\|_E = \|\chi_{B(t,\epsilon)}\|_E$. Now by Fatou's property of E we have

$$\lim_{\delta\to\epsilon^-}\|f\chi_{B(t,\delta)}\|_E = \|f\chi_{B(t,\epsilon)}\|_E, \qquad \lim_{\delta\to\epsilon^-}\|\chi_{B(t,\delta)}\|_E = \|\chi_{B(t,\epsilon)}\|_E. \tag{3.3}$$

Hence we have

$$M_E^{(r)}f(t) = \sup\left\{f_\epsilon^{(r)}(t) : \epsilon > 0,\ \epsilon\in\mathbb{Q},\ B(t,\epsilon)\subset(0,\alpha)\right\}, \tag{3.4}$$

and thus $M_E^{(r)}f$ is measurable. □

Remark 3.3. If W satisfies Δ_2 condition, then we obtain measurability of the maximal function $M_{\Lambda_{p,w}}^{(r)}f$ for any $f\in\Lambda_{p,w}$, analogously as in case of the maximal function $M_{\Gamma_{p,w}}^{(r)}g$ for any $g\in\Gamma_{p,w}$ when $\alpha=\infty$ and $W(\infty)=\infty$.

In view of Theorem 1 in [4], we investigate the so-called weak inequality for the maximal function $M_E^{(r)}$ whenever E is a r.i. quasi-Banach function space.

Theorem 3.4. *Let $0<r<\infty$. If a r.i. quasi-Banach function space E satisfies a lower ϕ-estimate for $\|\cdot\|_E^r$, then there exists $C>0$ such that for all $f\in E$ and $\lambda>0$ we have*

$$\lambda\phi\left(d_{M_E^{(r)}f}\right)(\lambda) \le C\|f\|_E^r. \tag{3.5}$$

Proof. Assume that $\lambda>0$. Denote

$$\Omega_\lambda = \left\{t\in(0,\alpha) : M_E^{(r)}f(t) > \lambda\right\}. \tag{3.6}$$

Clearly, by Proposition 3.2 we get that Ω_λ is measurable for all $\lambda>0$. Letting $t\in\Omega_\lambda$, there exists $\epsilon_t>0$ such that $B(t,\epsilon_t)\subset(0,\alpha)$ and

$$\lambda < f_{\epsilon_t}^{(r)}(t). \tag{3.7}$$

Let $c<\mu(\Omega_\lambda)$ and denote $B=\bigcup_{t\in\Omega_\lambda}B(t,\epsilon_t)$. Since $\Omega_\lambda\subset B$, we get $c<\mu(B)$. Hence, by regularity of the Lebesgue measure μ there is a compact set $K\subset B$ such that $c<\mu(K)$. By the fact that a collection $D=\{B(t,\epsilon_t) : t\in\Omega_\lambda\}$ is an open covering of the set K and by the Vitali covering lemma [2, Lemma 3.2], there exists a pairwise disjoint finite collection $\{B(t_k,\epsilon_{t_k}) : 1\le k\le n\}\subset D$ such that $\mu(K)\le 4\sum_{k=1}^n\mu(B(t_k,\epsilon_{t_k}))$. Therefore, by $\phi(4s)\le 4\phi(s)$ and by (3.7) we get

$$\phi(c)\le\phi(\mu(K))\le 4\phi\left(\sum_{k=1}^n\mu(B(t_k,\epsilon_{t_k}))\right)\le 4\phi\left(\sum_{k=1}^n\phi^{-1}\left(\lambda^{-1}\left\|f\chi_{B(t_k,\epsilon_{t_k})}\right\|_E^r\right)\right). \tag{3.8}$$

Hence, by assumption that E satisfies a lower ϕ-estimate for $\|\cdot\|_E^r$, there is $C > 0$ such that for any $1 \le k \le n$ we get

$$\phi(c) < 4\phi\left(\sum_{k=1}^{n}\phi^{-1}\left(\lambda^{-1}\left\|f\chi_{B(t_k,\epsilon_{t_k})}\right\|_E^r\right)\right) \le \frac{4C}{\lambda}\left\|\sum_{k=1}^{n}f\chi_{B(t_k,\epsilon_{t_k})}\right\|_E^r \le \frac{4C}{\lambda}\|f\|_E^r. \tag{3.9}$$

Since $c < \mu(\Omega_\lambda)$ is arbitrary, we obtain $\phi(\mu(\Omega_\lambda)) \le (4C/\lambda)\|f\|_E^r$, which finishes the proof. □

In the next theorem we present Lebesgue's differentiation property in the space E.

Theorem 3.5. *Let $\alpha = \infty$ and let E be a r.i. order continuous quasi-Banach function space E. If E satisfies a lower ϕ-estimate for $\|\cdot\|_E$, then E has LDP, that is,*

$$\lim_{\epsilon\to 0}\frac{\left\|(f - f(t))\chi_{B(t,\epsilon)}\right\|_E}{\left\|\chi_{B(t,\epsilon)}\right\|_E} = 0 \tag{3.10}$$

for all $f \in E$ and for a.a. $t \in (0,\infty)$. If in addition E is normable, then

$$\lim_{\epsilon\to 0}\frac{\left\|f\chi_{B(t,\epsilon)}\right\|_E}{\left\|\chi_{B(t,\epsilon)}\right\|_E} = |f(t)|. \tag{3.11}$$

Proof. Observe first that the set of step functions with supports of finite measure is dense in E. The proof of this observation is standard, by density of the simple functions, which is equivalent to order continuity of E and regularity of the Lebesgue measure μ on $\mathbb{R}$ (cf. [8]).

Define an operator $L : (0,\infty) \to \mathbb{R}$ by

$$Lh(t) = \limsup_{\epsilon\to 0}\left\{h_\epsilon^{(1)}(t)\right\} = \limsup_{\epsilon\to 0}\left\{\frac{\left\|h\chi_{B(t,\epsilon)}\right\|_E}{\left\|\chi_{B(t,\epsilon)}\right\|_E}\right\} \tag{3.12}$$

for any $h \in E$ and $t \in (0,\infty)$. Assume that $\epsilon > 0$ and $B(t,\epsilon) \subset (0,\infty)$ and also $f \in E$, $c \in \mathbb{R}$. Let $g = b\chi_A$ be a characteristic function of an open interval A. Notice that for a.a. $t \in (0,\infty)$ there exist $\delta_t > 0$ such that for all $0 < \epsilon < \delta_t$ we have either $B(t,\epsilon) \subset A$ or $B(t,\epsilon) \subset A^c$ and consequently $((g-c)\chi_{B(t,\epsilon)})^* = |g(t) - c|\chi_{(0,2\epsilon)}$. Therefore,

$$(g-c)_\epsilon^{(1)}(t) = \frac{\left\|(g-c)\chi_{B(t,\epsilon)}\right\|_E}{\left\|\chi_{B(t,\epsilon)}\right\|_E} = \frac{\left\|(g(t)-c)\chi_{(0,2\epsilon)}\right\|_E}{\left\|\chi_{(0,2\epsilon)}\right\|_E} = |g(t) - c| \tag{3.13}$$

for a.a. $t \in (0,\infty)$ and for any $0 < \epsilon < \delta_t$. Hence

$$L(g-c)(t) = \limsup_{\epsilon\to 0}\left\{(g-c)_\epsilon^{(1)}(t)\right\} = |g(t) - c| \tag{3.14}$$

for a.a. $t \in (0,\infty)$. Observe that the above equation can be proved analogously for any step function g with support of finite measure. Let ξ be a constant in the triangle inequality of the quasinorm $\|\cdot\|_E$. Thus

$$L(f-c)(t) \le \xi(L(f-g)(t) + L(g-c)(t)) = \xi(L(f-g)(t) + |g(t)-c|) \tag{3.15}$$

for a.a. $t \in (0,\infty)$. Clearly $L(f-g)(t) \le M_E^{(1)}(f-g)(t)$, whence

$$L(f-c)(t) \le \xi\left(M_E^{(1)}(f-g)(t) + |g(t)-c|\right) \tag{3.16}$$

for a.a. $t \in (0,\infty)$. Now replacing c by $f(t)$ we get

$$L(f-f(t))(t) \le \xi\left(M_E^{(1)}(f-g)(t) + |f(t)-g(t)|\right) \tag{3.17}$$

for a.a. $t \in (0,\infty)$. Define

$$\Omega_s = \{t \in (0,\infty) : L(f-f(t))(t) > s\} \tag{3.18}$$

for any $s > 0$. By Proposition 3.2, we have that $M_E^{(1)}(f-g)$ is μ-measurable. Recall [2, 8] that $d_{f_1+f_2}(s) \le d_{f_1}(s/2) + d_{f_2}(s/2)$ for any $f_1, f_2 \in L^0$. Thus in view of (3.17) we obtain for $s > 0$,

$$\begin{aligned}\mu_*(\Omega_s) &\le \mu\left(t \in (0,\infty) : M_E^{(1)}(f-g)(t) + |f-g|(t) > \frac{s}{\xi}\right)\\ &\le d_{M_E^{(1)}(f-g)}\left(\frac{s}{2\xi}\right) + d_{f-g}\left(\frac{s}{2\xi}\right).\end{aligned} \tag{3.19}$$

Now since ϕ satisfies the triangle inequality with constant ξ, we get

$$\phi(\mu_*(\Omega_s)) \le \xi\phi\left(d_{M_E^{(1)}(f-g)}\left(\frac{s}{2\xi}\right)\right) + \xi\phi\left(d_{f-g}\left(\frac{s}{2\xi}\right)\right) \tag{3.20}$$

for every $s > 0$. Observe that for any $h \in E$, $t \in (0,\alpha)$, we have $h^*(t)\phi(t) \le \|h^*\chi_{(0,t)}\|_E \le \|h\|_E$. Thus, by Proposition 2.9 we have

$$\phi\left(d_{f-g}\left(\frac{s}{2\xi}\right)\right) \le \frac{2\xi}{s}\sup_{t>0}\{\phi(t)(f-g)^*(t)\} \le \frac{2\xi}{s}\|f-g\|_E. \tag{3.21}$$

Furthermore, by Theorem 3.4 there exists $C > 0$ such that

$$\phi\left(d_{M_E^{(1)}(f-g)}\left(\frac{s}{2\xi}\right)\right) \le \frac{2\xi C}{s}\|f-g\|_E. \tag{3.22}$$

Therefore, for any step function g and for all $s > 0$,

$$\phi(\mu_*(\Omega_s)) \leq \frac{2\xi^2(C+1)}{s}\|f-g\|_E. \tag{3.23}$$

Hence we have

$$\phi(\mu_*(\Omega_s)) = \phi(\mu(\Omega_s)) = 0 \tag{3.24}$$

for all $s > 0$. So $L(f - f(t))(t) = 0$ for a.a. $t \in (0,\infty)$, which shows the first formula.

The second formula results from the first one since $\|\cdot\|_E$ is a norm in E. □

Now we characterize the lower and upper ϕ-estimate of $\Gamma_{q,p}$ on $(0,\infty)$, for $0 < p,\ q < \infty$. Clearly in this case $w(t) = t^{p/q-1}$ satisfies B_p condition. Thus $\Lambda_{p,w} = \Gamma_{p,w}$ and $\|\cdot\|_{\Lambda_{p,w}} \approx \|\cdot\|_{\Gamma_{p,w}}$. Hence by Theorems 3 and 7 in [9] and by Hölder's inequality we obtain the following corollary.

Corollary 3.6. *Let $\alpha = \infty$, $0 < p,\ r < \infty$ and, $1 < q < \infty$.*

(i) *If $p \leq q$, then $\Gamma_{q,p}$ satisfies a lower ϕ-estimate for the functional $\|\cdot\|^r_{\Gamma_{q,p}}$, where $1 \leq r$, and an upper ϕ-estimate for $\|\cdot\|^r_{\Gamma_{q,p}}$, where $r \leq p/q$.*

(ii) *If $p \geq q$, then $\Gamma_{q,p}$ satisfies an upper ϕ-estimate for the functional $\|\cdot\|^r_{\Gamma_{q,p}}$, where $r \leq 1$, and a lower ϕ-estimate for the functional $\|\cdot\|^r_{\Gamma_{q,p}}$, where $r \geq p$.*

The immediate consequence of Theorem 3.5 is the next result, which describes under what conditions $\Gamma_{p,w}$ has LDP. The second part follows from the previous corollary.

Proposition 3.7. *Let $0 < p < \infty$ and $\alpha = \infty$. If the Lorentz space $\Gamma_{p,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|$, then $\Gamma_{p,w}$ has LDP. Consequently, for $0 < p \leq q < \infty$ and $1 < q$, Lorentz spaces $\Gamma_{q,p}$ on $(0,\infty)$ have LDP.*

We will show next which spaces $(E, \|\cdot\|_E)$ do not have LDP with respect to $\|\cdot\|^r_E$. Notice that the ratio $\|f\|^{r_1}_E/\|g\|^{r_2}_E$ can be reduced easily to the quotient $\|f\|^r_E/\|g\|_E$ for any $f, g \in E$, where $g \neq 0$. In view of this fact, we state the following theorem.

Theorem 3.8. *Let E be a r.i. quasi-Banach function space.* (i) *Let $1 < r < \infty$. If order continuous E satisfies a lower ϕ-estimate for $\|\cdot\|^r_E$, then for any $f \in E$ and for a.a. $t \in (0,\alpha)$ we have*

$$\lim_{\epsilon\to 0}\frac{\|f\chi_{B(t,\epsilon)}\|^r_E}{\|\chi_{B(t,\epsilon)}\|_E} = 0. \tag{3.25}$$

(ii) *Let $0 < r < 1$. Then for any $f \in E$ and for a.a. $t \in \mathbb{S}(f)$ we have*

$$\lim_{\epsilon\to 0}\frac{\|f\chi_{B(t,\epsilon)}\|^r_E}{\|\chi_{B(t,\epsilon)}\|_E} = \infty. \tag{3.26}$$

Proof. Let $L : (0,\alpha) \to \mathbb{R}$ be the operator on E given by $Lf(t) = \limsup_{\epsilon\to 0}\{f_\epsilon^{(r)}(t)\}$, $f \in E$.

(i) Let g be a step function with a finite measure support. Notice that for a.a. $t \in (0,\alpha)$ and for small enough $\epsilon > 0$ we have

$$g_\epsilon^{(r)}(t) = \frac{\|g\chi_{B(t,\epsilon)}\|_E^r}{\|\chi_{B(t,\epsilon)}\|_E} = |g(t)|^r \|\chi_{B(t,\epsilon)}\|_E^{r-1}. \tag{3.27}$$

Therefore $Lg(t) = \limsup_{\epsilon\to 0}\{g_\epsilon^{(r)}(t)\} = 0$ and thus

$$Lf(t) \le \xi\big(L(f-g)(t) + Lg(t)\big) = \xi L(f-g)(t) \le \xi M_E^{(r)}(f-g)(t) \tag{3.28}$$

for a.a. $t \in (0,\alpha)$. Denoting $\Omega_s = \{t \in (0,\alpha) : Lf(t) > s\}$, $s > 0$, by Proposition 3.2, $M_E^{(r)}(f-g)$ is μ-measurable. Now by Theorem 3.4 there exists $C > 0$ such that for all $s > 0$,

$$\phi(\mu_*(\Omega_s)) \le \phi\left(d_{M_E^{(r)}(f-g)}\left(\frac{s}{\xi}\right)\right) \le \frac{\xi C}{s}\|f-g\|_E^r, \tag{3.29}$$

which shows that $\mu(\Omega_s) = 0$ and thus $Lf(t) = 0$ a.e.

(ii) Let $f_n \uparrow |f|$ a.e., where $0 \le f_n$ are step functions. Therefore, for a.a. $t \in \mathbb{S}(f)$ there is $n \in \mathbb{N}$ such that $0 < f_n(t) \le |f(t)|$ and for small enough $\epsilon > 0$ we have

$$(f_n)_\epsilon^{(r)}(t) = \frac{\|f_n\chi_{B(t,\epsilon)}\|_E^r}{\|\chi_{B(t,\epsilon)}\|_E} = |f_n(t)|^r \|\chi_{B(t,\epsilon)}\|_E^{r-1}. \tag{3.30}$$

Thus, by the assumption that $0 < r < 1$ we get

$$Lf(t) \ge L(f_n)(t) = \liminf_{\epsilon\to 0}\left\{(f_n)_\epsilon^{(r)}(t)\right\} = \infty \tag{3.31}$$

for a.a. $t \in \mathbb{S}(f)$, and the proof is finished. □

The next corollary follows directly from Theorem 3.8.

Corollary 3.9. (i) *Let $0 < p < \infty$, $1 < r < \infty$. If Lorentz space $\Gamma_{p,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|^r$, then for any $f \in \Gamma_{p,w}$ and for a.a. $t \in (0,\alpha)$ we have*

$$\lim_{\epsilon\to 0}\frac{\|f\chi_{B(t,\epsilon)}\|^r}{\|\chi_{B(t,\epsilon)}\|} = 0. \tag{3.32}$$

(ii) *Let $0 < p < \infty$, $0 < r < 1$. Then for any $f \in \Gamma_{p,w}$ and for a.a. $t \in \mathbb{S}(f)$ we have*

$$\lim_{\epsilon\to 0}\frac{\|f\chi_{B(t,\epsilon)}\|^r}{\|\chi_{B(t,\epsilon)}\|} = \infty. \tag{3.33}$$

The next result needed for further applications states conditions, which guarantee that $\Gamma_{p,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|^r$, where $0 < r < \infty$. We omit the proof of the following proposition.

Proposition 3.10. *Assume that $0 < p,\ r < \infty$, and Lorentz space $\Gamma_{p,w}$ satisfies a lower r-estimate. If ϕ is concave, then $\Gamma_{p,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|^r$.*

The following example shows that Theorem 3.8 is not an empty statement in case of $\Gamma_{p,w}$.

Example 3.11. Let $\alpha = \infty$, $p = 2$, and $w(s) = \ln(s+1) + 1$ for all $s \in [0, \infty)$. Then for any $f \in \Gamma_{2,w}$ and for a.a $t \in (0,\infty)$ we have

$$\lim_{\epsilon \to 0} \frac{\|f\chi_{B(t,\epsilon)}\|^2_{\Gamma_{2,w}}}{\|\chi_{B(t,\epsilon)}\|_{\Gamma_{2,w}}} = 0. \tag{3.34}$$

Proof. Notice that for any $t \in (0,\infty)$,

$$W(t) = (t+1)\ln(t+1), \qquad W_2(t) = t(t+1)\ln(t+1) - t^2\ln(t) + t. \tag{3.35}$$

Consequently,

$$\phi(t) = \|\chi_{(0,t)}\|_{\Gamma_{2,w}} = \left((t+1)^2\ln(t+1) - t^2\ln(t) + t\right)^{1/2} \tag{3.36}$$

for any $t \in (0,\infty)$. Since w is increasing, w satisfies RB_2 condition. Moreover, we have

$$\int_0^1 \frac{w(t)}{t^2}\,dt = \int_1^\infty w(t)\,dt = \infty, \qquad \frac{d}{dt}\left(\frac{W_2(t)}{t}\right) = \ln\left(1 + \frac{1}{t}\right) > 0 \tag{3.37}$$

for any $t \in (0,\infty)$, which concludes that $W_2(t)/t$ is increasing. Hence, by Theorem 3.3 [24] we obtain that $\Gamma_{2,w}$ satisfies a lower 2-estimate. Now we claim that ϕ is concave. By simple calculations we get the second derivative on $(0,\infty)$,

$$\phi''(t) = \frac{(t - \ln(t+1))\ln(t) - (t+2)\ln(t+1)}{t^2\ln(1+1/t) \ + (2t+1)\ \ln(t+1) + t}. \tag{3.38}$$

We observe that

$$t^2\ln\left(1 + \frac{1}{t}\right) + (2t+1)\ln(t+1) + t > 0, \qquad (t - \ln(t+1))\ln(t) < (t+2)\ln(t+1). \tag{3.39}$$

Therefore, $\phi''(t) < 0$ for all $t \in (0,\infty)$, which implies concavity of ϕ. Hence, by Proposition 3.10 we obtain that $\Gamma_{2,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|^2_{\Gamma_{2,w}}$. Finally, in view of Corollary 3.9 we finish the proof. □

The last part of this section is devoted to a pointwise convergence of the best and the extended best constant approximant of f to f, that is, another type of LDT.

The first result is a corollary of Proposition 3.7. In fact, let $t \in (0,\infty)$, $\epsilon > 0$ be such that $B(t,\epsilon) \subset (0,\infty)$. Assume that $f_\epsilon(t) \in T_{(p,B(t,\epsilon))}(f)$ for $f \in \Gamma_{p,w}$. By definition of the best constant approximant we get

$$|f(t) - f_\epsilon(t)| \leq \frac{\|(f(t)-f)\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}}{\|\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}} + \frac{\|(f_\epsilon(t)-f)\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}}{\|\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}} \leq 2\frac{\|(f(t)-f)\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}}{\|\chi_{B(t,\epsilon)}\|_{\Gamma_{p,w}}}. \tag{3.40}$$

Now applying Proposition 3.7 we get that $f_\epsilon(t) \to f(t)$ as $\epsilon \to 0$. Therefore we get the following theorem.

Theorem 3.12. *Let $\alpha = \infty$ and $1 \leq p < \infty$. If the space $\Gamma_{p,w}$ satisfies a lower ϕ-estimate for $\|\cdot\|_{\Gamma_{p,w}}$, then for every $f \in \Gamma_{p,w}$ we have*

$$f_\epsilon(t) \longrightarrow f(t) \quad \text{as } \epsilon \longrightarrow 0 \tag{3.41}$$

for a.a. $t \in (0,\infty)$, where $f_\epsilon(t) \in T_{(p,B(t,\epsilon))}(f)$ is a best constant approximant of f.

In order to prove the next approximation theorem let us first establish an inequality related to extended best constant approximants of any function $f \in \Gamma_{p-1,w}$ for $1 < p < 2$.

Lemma 3.13. *Let $1 < p < 2$, $A \subset [0,\alpha)$, $0 < \mu(A) < \infty$, and let $f \in \Gamma_{p-1,w}$, $f \geq 0$. Then for any $u\chi_A \in T_{(p,A)}(f)$ extended best constant approximant of f and for a.a. $s \in (0,\alpha)$ we have*

$$|u - f(s)|^{p-1}\|\chi_A\|^p_{\Gamma_{p,w}} \leq 5\|(f - f(s))\chi_A\|^{p-1}_{\Gamma_{p-1,w}}. \tag{3.42}$$

Proof. Let $s \in (0,\alpha)$ and $|f(s)| < \infty$. Since $(\chi_A)^{**} \leq 1$, by subadditivity of the maximal function and the power function v^{p-1} for $1 < p < 2$ we get

$$\begin{aligned} |u - f(s)|^{p-1}\|\chi_A\|^p_{\Gamma_{p,w}} &= \int_0^\alpha (\chi_A)^{**}(t)((u - f(s))\chi_A)^{**(p-1)}(t)w(t)dt \\ &\leq \int_0^\alpha (\chi_A)^{**}(t)((f - u)\chi_A)^{**(p-1)}(t)w(t)dt \\ &\quad + \|(f - f(s))\chi_A\|^{p-1}_{\Gamma_{p-1,w}}. \end{aligned} \tag{3.43}$$

We will finish the proof under assumption that $f(s) \geq u$. In the other case the proof is similar. Since $u\chi_A \in T_{(p,A)}(f) = [\underline{f}_{(p,A)}\chi_A, \overline{f}_{(p,A)}\chi_A]$, by Corollary 2.11 [7] and by Hardy-Littlewood inequality we obtain

$$\begin{aligned}
&\int_0^\alpha (\chi_A)^{**}(t)((f-u)\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq 2\int_0^\alpha \frac{1}{t}\int_{A\cap\{f\leq u\}} \chi_{(0,t)}(\rho_{((u-f)\chi_A,\chi_A)})d\mu((f-u)\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq 2\int_0^\alpha (\chi_{A\cap\{f\leq u\}})^{**}(t)((f-u)\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq 2\|(f-f(s))\chi_A\|^{p-1}_{\Gamma_{p-1,w}} + 2\int_0^\alpha (\chi_{A\cap\{f\leq u\}})^{**}(t)((f(s)-u)\chi_A)^{**(p-1)}(t)w(t)\,dt.
\end{aligned} \tag{3.44}$$

Moreover, by assumption that $1 < p < 2$ we have

$$\begin{aligned}
(\chi_{A\cap\{f\leq u\}})^{**}(t) &\leq \chi_{(0,\mu(A\cap\{f\leq u\}))}(t) + \left(\frac{\mu(A\cap\{f\leq u\})}{t}\right)^{p-1}\chi_{[\mu(A\cap\{f\leq u\}),\alpha)}(t) \\
&= (\chi_{A\cap\{f\leq u\}})^{**(p-1)}(t)
\end{aligned} \tag{3.45}$$

for any $t \in (0,\alpha)$. Consequently, by the fact that $f(s) \geq u$ we obtain

$$\begin{aligned}
&\int_0^\alpha (\chi_{A\cap\{f\leq u\}})^{**}(t)((f(s)-u)\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq \int_0^\alpha ((f(s)-u)\chi_{A\cap\{f\leq u\}})^{**(p-1)}(t)(\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq \int_0^\alpha ((f(s)-f)\chi_{A\cap\{f\leq u\}})^{**(p-1)}(t)(\chi_A)^{**(p-1)}(t)w(t)\,dt \\
&\quad \leq \|(f-f(s))\chi_A\|^{p-1}_{\Gamma_{p-1,w}}.
\end{aligned} \tag{3.46}$$

Hence, by conditions (3.43) and (3.44) we finish the proof. □

Corollary 3.14. *Let $1 < p < \infty$ and $w \in D_{p-1}$. Assume that ϕ and ψ are the fundamental functions of $\Gamma_{p,w}$ and $\Gamma_{p-1,w}$, respectively. If one of the following conditions*

(i) *there exists $A > 0$ such that $W_{p-1}(t) \leq AW_p(t)$ for all $t \in (0,\alpha)$,*

(ii) *$w \in B_{p-1}$,*

holds, then $\phi^p \approx \psi^{p-1}$.

Proof. Notice that for all $t \in (0,\alpha)$,

$$\phi(t)^p = W(t) + t^p\int_t^\alpha s^{-p}w(s)\,ds \leq W(t) + t^p\int_t^\alpha s^{-p+1}t^{-1}w(s)\,ds = \psi(t)^{p-1}. \tag{3.47}$$

Suppose now that there is $A > 0$ such that $W_{p-1}(t) \le AW_p(t)$ for all $t \in (0,\alpha)$. Therefore,

$$\varphi(t)^{p-1} = W(t) + W_{p-1}(t) \le W(t) + AW_p(t) \le A\phi(t)^p \tag{3.48}$$

for any $t \in (0,\alpha)$, and so $\phi^p \approx \varphi^{p-1}$.

Now assume that $w \in B_{p-1}$. Then there exists $B > 0$ such that for all $f \in \Gamma_{p-1,w}$ we have $\|f\|^{p-1}_{\Gamma_{p-1,w}} \le B\|f\|^{p-1}_{\Lambda_{p-1,w}}$. Consequently,

$$\varphi(t)^{p-1} = \left\|\chi_{(0,t)}\right\|^{p-1}_{\Gamma_{p-1,w}} \le B\left\|\chi_{(0,t)}\right\|^{p-1}_{\Lambda_{p-1,w}} = BW(t) \le B\phi(t)^p \tag{3.49}$$

for any $t \in (0,\alpha)$, and the proof is completed. □

Theorem 3.15. *Let $\alpha = \infty$, $1 < p < 2$, $w \in D_{p-1}$ and $f \in \Gamma_{p-1,w}$, $f \ge 0$. Assume that ϕ and φ are the fundamental functions of $\Gamma_{p,w}$ and $\Gamma_{p-1,w}$, respectively. If Lorentz space $\Gamma_{p-1,w}$ satisfies a lower φ-estimate for $\|\cdot\|_{\Gamma_{p-1,w}}$ and $\phi^p \approx \varphi^{p-1}$, then*

$$f_\epsilon(t) \longrightarrow f(t) \quad \text{as } \epsilon \longrightarrow 0 \tag{3.50}$$

for a.a. $t \in (0,\infty)$, where $f_\epsilon(t) \in T_{(p,B(t,\epsilon))}(f)$ is an extended best constant approximant of f.

Proof. Let $\epsilon > 0$ and $t \in (0,\infty)$, $|f(t)| < \infty$. By Lemma 3.13 for any $f_\epsilon(t) \in T_{(p,B(t,\epsilon))}(f)$ we get

$$\left|f_\epsilon(t) - f(t)\right| \le 5^{1/(p-1)} \frac{\left\|\left(f - f(t)\right)\chi_{B(t,\epsilon)}\right\|_{\Gamma_{p-1,w}}}{\left\|\chi_{B(t,\epsilon)}\right\|^{p/(p-1)}_{\Gamma_{p,w}}}. \tag{3.51}$$

Since $\phi^p \approx \varphi^{p-1}$, there is $B > 0$ such that for all $\delta > 0$ we have $\|\chi_{B(t,\delta)}\|_{\Gamma_{p-1,w}} \le B\|\chi_{B(t,\delta)}\|^{p/(p-1)}_{\Gamma_{p,w}}$. Consequently,

$$\left|f_\epsilon(t) - f(t)\right| \le 5^{1/(p-1)} B \frac{\left\|\left(f - f(t)\right)\chi_{B(t,\epsilon)}\right\|_{\Gamma_{p-1,w}}}{\left\|\chi_{B(t,\epsilon)}\right\|_{\Gamma_{p-1,w}}} \tag{3.52}$$

for any $t \in (0,\infty)$ and $\epsilon > 0$. Hence, by Proposition 3.7 and by assumption that $\Gamma_{p-1,w}$ satisfies a lower φ-estimate for $\|\cdot\|_{\Gamma_{p-1,w}}$ we obtain that $f_\epsilon(t) \to f(t)$ as $\epsilon \to 0$ for a.a. $t \in (0,\infty)$. □

Now we present a specific family of Lorentz spaces $\Gamma_{p,w}$ for which Theorems 3.12 and 3.15 are fulfilled.

Corollary 3.16. *Let $\alpha = \infty$ and $1 < q < \infty$. If one of the following conditions*

(i) *$1 \le p \le q$ and $f \in \Gamma_{q,p}$,*

(ii) *$1 < p < 2$ and $f \in \Gamma_{q,p-1}$, $f \ge 0$,*

is satisfied, then for a.a. $t \in (0,\infty)$ we have

$$f_\epsilon(t) \longrightarrow f(t) \quad \text{as } \epsilon \longrightarrow 0, \tag{3.53}$$

where $f_\epsilon(t) \in T_{(p,B(t,\epsilon))}(f)$ is the best constant approximant of $f \in \Gamma_{p,w}$ and the extended best constant approximant of $f \in \Gamma_{p-1,w}$.

Proof. Suppose that condition (i) is fulfilled. Immediately, by Theorem 3.12 and Corollary 3.6 we complete the first case. Now assume that the condition (ii) is satisfied. Define $w(t) = t^{(p-1)/q-1}$ for $t \in (0,\infty)$. Since $1 < p < 2$, we have that

$$W(t) = \frac{q}{p-1} t^{(p-1)/q}, \qquad W_{p-1}(t) = \frac{q}{(p-1)q-p+1} t^{(p-1)/q} \tag{3.54}$$

for all $t \in (0,\infty)$, which implies that $w \in D_{p-1} \cap B_{p-1}$. Consequently, by Definition 2.6 there is $T_{(p,B(t,\epsilon))}$ the extended best constant approximant operator on $\Gamma_{q,p-1}$. Moreover, by Corollary 3.14 the fundamental functions ϕ and ψ of the spaces $\Gamma_{q,p}$ and $\Gamma_{q,p-1}$ respectively, satisfy $\phi^p \approx \psi^{p-1}$. Hence, by Corollary 3.6 we get that $\Gamma_{q,p-1}$ satisfies a lower ψ-estimate for $\|\cdot\|_{\Gamma_{q,p-1}}$. Finally, by Theorem 3.15 we obtain that $f_\epsilon(t)$, an extended best constant approximant of f, converges to $f(t)$ as $\epsilon \to 0$ for a.a. $t \in (0,\infty)$. □

Observe that the K-functional of the couple $(\Gamma_{q,p}, L^\infty)$ can be expressed equivalently by $\|f^* \chi_{(0,t)}\|_{\Gamma_{q,p}}/\phi(t)$ for any $f \in \Gamma_{q,p}$ and $t \in (0,\infty)$ [25]. The next result on inequalities between maximal function $M^{(1)}_{q,p} f$ and the K-functional of $(\Gamma_{q,p}, L^\infty)$ follows immediately as a consequence of Theorems 1 and 2 in [4], Corollary 3.6, and Remark 3.3.

Corollary 3.17. *Let $\alpha = \infty$, $p \in [1,\infty)$, and $q \in (0,\infty)$.*

(i) *If $1 < p \le q < \infty$, then there exists $C > 0$ such that for all $f \in \Gamma_{q,p}$ and $t > 0$ we have*

$$\left(M^{(1)}_{\Gamma_{q,p}} f\right)^*(t) \le \frac{C}{\phi(t)} \|f^* \chi_{(0,t)}\|_{\Gamma_{q,p}}. \tag{3.55}$$

(ii) *If $1 < q \le p < \infty$, then there exists $C > 0$ such that for all $f \in \Gamma_{q,p}$ and $t > 0$ we get*

$$\left(M^{(1)}_{\Gamma_{q,p}} f\right)^*(t) \ge \frac{C}{\phi(t)} \|f^* \chi_{(0,t)}\|_{\Gamma_{q,p}}. \tag{3.56}$$

(iii) *If $p = 1 < q < \infty$, then there is $C > 0$ such that for any $f \in \Gamma_{q,p}$ and $t > 0$ we obtain the inequality in the condition* (i).

The decreasing rearrangement of the maximal function $(M^{(1)}_{\Gamma_{q,p}} f)^*$ and K-functional of $(\Gamma_{q,p}, L^\infty)$ are not equivalent; that is the opposite inequalities to the ones in Corollary 3.17 do not hold. It is similar as in spaces $\Lambda_{q,p}$, $1 \le p < q < \infty$, as we see in the next example [4].

Example 3.18. Let $1 < r < \infty$. There exists a function $f \in L^0$ such that for all $t > 0$,

$$\|f^* \chi_{(0,t)}\|_{\Gamma_{r,1}} = \int_0^\infty (f^* \chi_{(0,t)})^{**}(s) s^{(1/r)-1} ds = \infty, \qquad \left(M^{(1)}_{\Gamma_{r,1}} f\right)^*(t) < \infty. \tag{3.57}$$

Proof. Since $w(s) = s^{(1/r)-1}$ satisfies B_1 condition, there is $B > 0$ such that

$$\left\|h\chi_{B(t,\epsilon)}\right\|_{\Lambda_{r,1}} \le \left\|h\chi_{B(t,\epsilon)}\right\|_{\Gamma_{r,1}} \le B\left\|h\chi_{B(t,\epsilon)}\right\|_{\Lambda_{r,1}} \tag{3.58}$$

for all $h \in \Gamma_{r,1}$, $t \in (0,\infty)$ and suitable $\epsilon > 0$. Hence $M^{(1)}_{\Lambda_{r,1}} f \ge (1/B) M^{(1)}_{\Gamma_{r,1}} f$. Therefore, by Remark 3.3 we conclude that

$$\left(M^{(1)}_{\Lambda_{r,1}} f\right)^*(t) \ge \frac{1}{B}\left(M^{(1)}_{\Gamma_{r,1}} f\right)^*(t) \tag{3.59}$$

for all $t > 0$. Now, by Theorem 3 in [4] there exists $f \in L^0$ such that $\|f^*\chi_{(0,t)}\|_{\Lambda_{r,1}} = \infty$ and $(M^{(1)}_{\Lambda_{r,1}} f)^*(t) < \infty$ for all $t > 0$, which implies that $\|f^*\chi_{(0,t)}\|_{\Gamma_{r,1}} = \infty$ and $(M^{(1)}_{\Gamma_{r,1}} f)^*(t) < \infty$ for any $t > 0$. □

Remark 3.19. Example 3.18 can be expanded to the case when $1 \le p < q < \infty$. Indeed if $1 \le p < q < \infty$ and $r = q/p$ then we have $\Gamma_{q,p} = \Gamma^{(p)}_{r,1}$ and $\|\cdot\|_{\Gamma_{q,p}} \approx \||\cdot|^p\|^{1/p}_{\Gamma_{q,p}}$. As well as $(M^{(1)}_{\Gamma_{r,1}}|\cdot|^p)^{1/p} \approx M^{(1)}_{\Gamma_{q,p}}(\cdot)$ on $\{f \in \Gamma_{q,p}(C) : C \text{ is convex}, \mu(C) < \infty\}$. This can be obtained by simple observation that the weight function $w(t) = t^{1/r-1}$ satisfies condition B_s for any $1/r < s < \infty$, and then $\Gamma_{r,1} = \Lambda_{r,1}$ and $\Gamma_{q,p} = \Lambda_{q,p}$ as sets with equivalent norms.

4. Uniqueness of the Extended Best Constant APProximant on $\Gamma_{P,W}$

In this section we prove uniqueness of the expansion of the best constant approximant for any $f \in \Gamma_{p-1,w}$, if $1 < p < \infty$ and $w > 0$. Namely, we show that the extended best constant approximant operator $T_{(p,A)}$ becomes the point value operator. Throughout this section we assume that $A \subset [0,\alpha)$ is a set of positive and finite measure. Let the truncation of any function $f \in L^0$ be defined as $f^{(n)}(s) = f(s)$ if $|f(s)| < n$, and $f^{(n)}(s) = \operatorname{sign} f(s) n$ if $|f(s)| \ge n$.

Lemma 4.1. *Let $1 < p < \infty$ and $f \in \Gamma_{p-1,w}$. Then $f^{(n)} \in \Gamma_{p,w}$ for all $n \in \mathbb{N}$.*

Proof. Since $f \in \Gamma_{p-1,w}$, we get that $d_f(\lambda) < \infty$ for every $\lambda > 0$. Let $\epsilon \in (0,1)$. Then

$$c_n := \mu(s : |f(s)| \ge n) \le \mu(s : |f(s)| > n - \epsilon) = d_f(n-\epsilon) < \infty \tag{4.1}$$

for any $n \in \mathbb{N}$. Moreover, $(f^{(n)})^{**} \le f^{**}$ and $(f^{(n)})^{**} \le n$ for any $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\int_0^\alpha \left(f^{(n)}\right)^{**p}(t) w(t)\,dt &\le n^p W(c_n) + \int_{c_n}^\alpha \left(f^{(n)}\right)^{**(p-1)}(t)\left(f^{(n)}\right)^{**}(t) w(t)dt \\ &\le n^p W(c_n) + n\int_{c_n}^\alpha f^{**(p-1)}(t) w(t)dt < \infty\end{aligned} \tag{4.2}$$

for all $n \in \mathbb{N}$, which concludes the proof. □

We omit the standard proof of the next lemma.

Lemma 4.2. *Let $f \in L^0$ and $u \in \mathbb{R}$. Then for a.a. $s \in A$ there exists $N(u,s) \in \mathbb{N}$ such that for all $n \geq N(u,s)$ we have the following.*

(i) $\{v \in A : f^{(n)}(v) + f^{(n)}(s) < 2u\} = \{v \in A : f(v) + f(s) < 2u\}$.

(ii) $\{v \in A : f^{(n)}(v) = f^{(n)}(s),\ v \leq s\} = \{v \in A : f(v) = f(s),\ v \leq s\}$.

(iii) $\{v \in A : f^{(n)}(v) > f^{(n)}(s)\} = \{v \in A : f(v) > f(s)\}$.

Now we discuss a convergence of a sequence of functions $S^p_{(f^{(n)},A)}(u)$ to the function $S^p_{(f,A)}(u)$ as $n \to \infty$ for all $u \in \mathbb{R}$ and $1 < p < \infty$, whenever $f \in \Gamma_{p-1,w}$.

Theorem 4.3. *Let $1 < p < \infty$ and $f \in \Gamma_{p-1,w}$. Then for all $u \in \mathbb{R}$,*

$$\lim_{n\to\infty} S^p_{(f^{(n)},A)}(u) = S^p_{(f,A)}(u). \tag{4.3}$$

Proof. Denote $E = A \cap \{f \geq u\}$. Notice that there exists $N_u \in \mathbb{N}$ such that $N_u > |u|$ and for all $n \geq N_u$ we have $E = A \cap \{f^{(n)} \geq u\}$. Since $\rho_{((f-u)\chi_A,\chi_A)}$ is a measure-preserving transformation, by Definition 2.3 we get

$$\begin{aligned} S^p_{(f,A)}(u) = {} & 2\int_0^\alpha \frac{1}{t}\int_{A\cap\{f\geq u\}} \chi_{(0,t)}\left(\rho_{((f-u)\chi_A,\chi_A)}\right)\left((f-u)\chi_A\right)^{**(p-1)}(t)w(t)dt \\ & - \int_0^\alpha (\chi_A)^{**}(t)\left((f-u)\chi_A\right)^{**(p-1)}(t)w(t)dt. \end{aligned} \tag{4.4}$$

We claim that for a.a. $s \in E$ there exists $N(u,s) \in \mathbb{N}$ such that for all $n \geq N(u,s)$ we have

$$\rho_{((f^{(n)}-u)\chi_A,\chi_A)}(s) = \rho_{((f-u)\chi_A,\chi_A)}(s). \tag{4.5}$$

Let $s \in E$ and $|f(s)| < \infty$. If $f(s) > u$, then by Lemma 4.2 there exists $N_{(u,s)} \in \mathbb{N}$ such that for any $n \geq N_{(u,s)}$ we obtain

$$\begin{aligned} \rho_{((f-u)\chi_A,\chi_A)}(s) = {} & \mu\left(v \in A : f(v) > f(s)\right) + \mu\left(v \in A : f(v) + f(s) < 2u\right) \\ & + \mu\left(v \in A : f(v) = f(s),\ v \leq s\right) \\ = {} & \mu\left(v \in A : f^{(n)}(v) > f^{(n)}(s)\right) + \mu\left(v \in A : f^{(n)}(v) + f^{(n)}(s) < 2u\right) \\ & + \mu\left(v \in A : f^{(n)}(v) = f^{(n)}(s),\ v \leq s\right) = \rho_{((f^{(n)}-u)\chi_A,\chi_A)}(s). \end{aligned} \tag{4.6}$$

If $f(s) = u$, then again by Lemma 4.2 for $n \geq N_{(u,s)}$ we have

$$\begin{aligned} \rho_{((f-u)\chi_A,\chi_A)}(s) = {} & \mu\left(v \in A : f(v) > f(s)\right) + \mu\left(v \in A : f(v) + f(s) < 2u\right) \\ & + \mu(v \in A : v \leq s) - \mu\left(v \in A : f(v) = f(s),\ v \leq s\right) \\ = {} & \mu\left(v \in A : f^{(n)}(v) > f^{(n)}(s)\right) + \mu\left(v \in A : f^{(n)}(v) + f^{(n)}(s) < 2u\right) \\ & + \mu(v \in A : v \leq s) - \mu\left(v \in A : f^{(n)}(v) = f^{(n)}(s),\ v \leq s\right) = \rho_{((f^{(n)}-u)\chi_A,\chi_A)}(s). \end{aligned} \tag{4.7}$$

Hence we obtain our claim. It follows

$$\lim_{n\to\infty}\int_{A\cap\{f^{(n)}\geq u\}}\chi_{(0,t)}\left(\rho_{((f^{(n)}-u)\chi_A,\chi_A)}\right)=\int_{A\cap\{f\geq u\}}\chi_{(0,t)}\left(\rho_{((f-u)\chi_A,\chi_A)}\right) \tag{4.8}$$

for any $t\in(0,\alpha)$. Moreover $\lim_{n\to\infty}((f^{(n)}-f)\chi_A)^{**}(t)=0$ for all $t\in(0,\alpha)$, and consequently by triangle inequality for maximal function we obtain

$$\lim_{n\to\infty}\left(\left(f^{(n)}-u\right)\chi_A\right)^{**}(t)=((f-u)\chi_A)^{**}(t) \tag{4.9}$$

for all $t\in(0,\alpha)$. Since $\rho_{((f^{(n)}-u)\chi_A,\chi_A)}$ is the measure-preserving transformation for all $n\in\mathbb{N}$, by the fact that the power function v^{p-1} for $p>1$ is subadditive with a constant $C>0$, we have

$$\begin{aligned}&\frac{1}{t}\int_{A\cap\{f^{(n)}\geq u\}}\chi_{(0,t)}\left(\rho_{((f^{(n)}-u)\chi_A,\chi_A)}\right)\left(\left(f^{(n)}-u\right)\chi_A\right)^{**(p-1)}(t)\\&\quad\leq(\chi_A)^{**}(t)\left(\left(f^{(n)}-u\right)\chi_A\right)^{**(p-1)}(t)\leq C\left((f\chi_A)^{**(p-1)}(t)+(u\chi_A)^{**(p-1)}(t)\right)\end{aligned} \tag{4.10}$$

for every $t\in(0,\alpha)$ and $n\in\mathbb{N}$. Combining now (4.8), (4.9), and (4.10) we get

$$\begin{aligned}&\lim_{n\to\infty}\int_0^\alpha\frac{1}{t}\int_{A\cap\{f^{(n)}\geq u\}}\chi_{(0,t)}\left(\rho_{((f^{(n)}-u)\chi_A,\chi_A)}\right)\left(\left(f^{(n)}-u\right)\chi_A\right)^{**(p-1)}(t)w(t)dt\\&\quad=\int_0^\alpha\frac{1}{t}\int_{A\cap\{f\geq u\}}\chi_{(0,t)}\left(\rho_{((f-u)\chi_A,\chi_A)}\right)((f-u)\chi_A)^{**(p-1)}(t)w(t)dt,\\&\lim_{n\to\infty}\int_0^\alpha(\chi_A)^{**}(t)\left(\left(f^{(n)}-u\right)\chi_A\right)^{**(p-1)}(t)w(t)dt=\int_0^\alpha(\chi_A)^{**}(t)((f-u)\chi_A)^{**(p-1)}(t)w(t)dt.\end{aligned} \tag{4.11}$$

In view of (4.4) we complete the proof. □

The following characterization of the function $S^p_{(f,A)}(u)$ is an essential fact in the proof of the main result.

Proposition 4.4. *Let $1<p<\infty$, w be a positive weight function and let $f\in\Gamma_{p-1,w}$. Then the function $S^p_{(f,A)}(u)$ is strictly decreasing with respect to u.*

Proof. First we will show that for any $u,v\in\mathbb{R}$, $u<v$ there exist $a\in(u,v]$ and $B\subset[0,\alpha)$, $\mu(B)>0$ such that for all $t\in B$,

$$K_{(f,A)}(-u,t)((f+u)\chi_A)^{**(p-1)}(t)<K_{(f,A)}(-a,t)((f+a)\chi_A)^{**(p-1)}(t). \tag{4.12}$$

Suppose for a contrary that there exist $u<v$ such that for all $a\in(u,v]$ and for a.a. $t\in[0,\alpha)$ we have

$$K_{(f,A)}(-u,t)((f+u)\chi_A)^{**(p-1)}(t)\geq K_{(f,A)}(-a,t)((f+a)\chi_A)^{**(p-1)}(t). \tag{4.13}$$

Since $K_{(f,A)}(a,t)$ and $((f+a)\chi_A)^{**}(t)$ are continuous functions with respect to $t \in (0,\alpha)$ and $K_{(f,A)}(-a,t)$ is right-continuous with respect to $a \in \mathbb{R}$, we get that the above inequality is fulfilled for all $t \in (0,\alpha)$ and for any $a \in [u,v]$. By convexity of the function $a \to ((f+a)\chi_A)^{**p}(t)$ and by Proposition 4.2 [20] we get that $(d^+/da)((f+a)\chi_A)^{**p}(t) = pK_{(f,A)}(-a,t)((f+a)\chi_A)^{**(p-1)}(t)$ is increasing with respect to $a \in \mathbb{R}$ for any $t \in (0,\alpha)$, which implies that

$$K_{(f,A)}(-a,t)\left((f+a)\chi_A\right)^{**(p-1)}(t) \le K_{(f,A)}(-b,t)\left((f+b)\chi_A\right)^{**(p-1)}(t) \tag{4.14}$$

for any $a < b$ and for all $t \in (0,\alpha)$. Hence

$$K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t) = K_{(f,A)}(-a,t)\left((f+a)\chi_A\right)^{**(p-1)}(t) \tag{4.15}$$

for all $a \in [u,v]$ and $t \in (0,\alpha)$. Pick up $b \in (u,v]$. Denote $\delta = (b-u)/2$. Notice that

$$K_{(f,A)}(-a,t)\left((f+a)\chi_A\right)^{**(p-1)}(t) = K_{(f,A)}(-a-\delta,t)\left((f+a+\delta)\chi_A\right)^{**(p-1)}(t) \tag{4.16}$$

for all $a \in [u,u+\delta]$ and $t \in (0,\alpha)$, which yields that

$$\begin{aligned}\int_u^{u+\delta} \frac{d^+}{da}\left((f+a)\chi_A\right)^{**p}(t)\,da &= \int_u^{u+\delta} \frac{d^+}{da}\left((f+a+\delta)\chi_A\right)^{**p}(t)da \\ &= \int_{u+\delta}^{b} \frac{d^+}{da}\left((f+a)\chi_A\right)^{**p}(t)da\end{aligned} \tag{4.17}$$

for all $t \in (0,\alpha)$. Consequently,

$$\left(f\chi_A + \frac{u+b}{2}\chi_A\right)^{**p}(t) = \frac{1}{2}\left((f+u)\chi_A\right)^{**p}(t) + \frac{1}{2}\left((f+b)\chi_A\right)^{**p}(t) \tag{4.18}$$

for any $t \in (0,\alpha)$. Moreover, by subadditivity of the maximal function and by convexity of the power function s^p for $p > 1$ we get

$$\begin{aligned}\left(f\chi_A + \frac{u+b}{2}\chi_A\right)^{**p}(t) &\le \left(\frac{1}{2}\left((f+u)\chi_A\right)^{**}(t) + \frac{1}{2}\left((f+b)\chi_A\right)^{**}(t)\right)^p \\ &\le \frac{1}{2}\left((f+u)\chi_A\right)^{**p}(t) + \frac{1}{2}\left((f+b)\chi_A\right)^{**p}(t)\end{aligned} \tag{4.19}$$

for all $t \in (0,\alpha)$. Therefore,

$$\left((f+u)\chi_A + (f+b)\chi_A\right)^{**}(t) = \left((f+u)\chi_A\right)^{**}(t) + \left((f+b)\chi_A\right)^{**}(t), \tag{4.20}$$

which implies that

$$\int_0^t \left((f+u)\chi_A + (f+b)\chi_A\right)^{*}(s) - \left((f+u)\chi_A\right)^{*}(s) - \left((f+b)\chi_A\right)^{*}(s)\,ds = 0 \tag{4.21}$$

for any $t \in (0,\alpha)$. Hence

$$((f+u)\chi_A + (f+b)\chi_A)^*(s) = ((f+u)\chi_A)^*(s) + ((f+b)\chi_A)^*(s) \tag{4.22}$$

for a.a. $s \in (0,\alpha)$ and for any $b \in [u,v]$. By Corollary 9 [8, page 65] we obtain that the functions $(f+u)\chi_A$ and $(f+b)\chi_A$ for all $b \in [u,v]$ are of constant sign almost everywhere, that is,

$$\operatorname{sign}(f+u)(s) = \operatorname{sign}(f+b)(s) \tag{4.23}$$

for a.a. $s \in A$ and have a common system of sets $\{E_t : t \in (0,\alpha)\}$ such that $\mu(E_t) = t$ and

$$\int_0^t ((f+b)\chi_A)^*(s)ds = \int_{E_t\cap A} |f+b|(s)ds \tag{4.24}$$

for all $t \in (0,\alpha)$ and for all $b \in [u,v]$. We claim that there exist $b \in (u,v]$ and $B \subset (0,\alpha)$, $\mu(B) > 0$, such that we have either for all $t \in B$,

$$((f+u)\chi_A)^{**}(t) < ((f+b)\chi_A)^{**}(t), \tag{4.25}$$

or for all $t \in B$,

$$((f+u)\chi_A)^{**}(t) > ((f+b)\chi_A)^{**}(t). \tag{4.26}$$

Suppose that the claim does not hold. Then for any $b \in (u,v]$ and $t \in (0,\alpha)$ we get

$$\int_0^t ((f+u)\chi_A)^*(s)ds = \int_0^t ((f+b)\chi_A)^*(s)ds. \tag{4.27}$$

Thus, by the condition (4.24) we obtain

$$\int_{E_t\cap A} |f+u|(s) - |f+b|(s)\, ds = 0 \tag{4.28}$$

for every $b \in (u,v]$ and $t \in (0,\alpha)$. Since $E_r \subset E_t$ for any $r < t$ and $\mu(E_t) = t$ for any $t \in (0,\alpha)$, we get that $|f(s)+u| = |f(s)+b|$ for any $b \in (u,v]$ and for a.a. $s \in A$. Hence, by the fact (4.23) we get a contradiction. Now we will consider two cases.

Case 1. Assume that there exist $b \in (u,v]$ and $B \subset (0,\alpha)$, $\mu(B) > 0$ such that

$$((f+u)\chi_A)^{**}(t) < ((f+b)\chi_A)^{**}(t) \tag{4.29}$$

for all $t \in B$. Thus and by conditions (4.23) and (4.24) we obtain

$$0 < \int_0^t ((f+b)\chi_A)^*(s)\,ds - \int_0^t ((f+u)\chi_A)^*(s)\,ds = \int_{E_t\cap A} |f(s)+b| - |f(s)+u|\,ds$$
$$= (b-u)\int_{E_t\cap A} \operatorname{sign}(f(s)+u)ds \tag{4.30}$$

for all $t \in B$. Since $b-u>0$, we have

$$0 < \int_{E_t\cap A} \operatorname{sign}(f(s)+u)ds \tag{4.31}$$

for any $t \in B$. The function $a \to ((f+a)\chi_A)^{**}(t)$ is strictly increasing on $[u,v]$ and $t \in B$. In fact, let $\xi,\eta \in [0,v-u]$ and $\xi<\eta$. By conditions (4.23) and (4.24) we conclude that

$$\begin{aligned}
&((f+u+\xi)\chi_A)^{**}(t) - ((f+u+\eta)\chi_A)^{**}(t)\\
&\quad = \frac{1}{t}\int_{E_t\cap A} \operatorname{sign}(f(s)+u)(f(s)+u+\xi)\,ds - \frac{1}{t}\int_{E_t\cap A} \operatorname{sign}(f(s)+u)(f(s)+u+\eta)ds\\
&\quad = \frac{\xi-\eta}{t}\int_{E_t\cap A} \operatorname{sign}(f(s)+u)\,ds < 0
\end{aligned} \tag{4.32}$$

for any $t \in B$. Let $a \in (u,v)$ and $\delta_a = \min\{v-a, a-u\}/2$. Then, by condition (4.32) we get

$$((f+u)\chi_A)^{**}(t) < ((f+u+\epsilon)\chi_A)^{**}(t) < ((f+a)\chi_A)^{**}(t) < ((f+a+\epsilon)\chi_A)^{**}(t) \tag{4.33}$$

for every $0<\epsilon\le\delta_a$ and for all $t \in B$. Repeating calculations in condition (4.32) with a instead of u, in view of (4.23) and (4.24) we have

$$((f+u+\epsilon)\chi_A)^{**}(t) - ((f+u)\chi_A)^{**}(t) = ((f+a+\epsilon)\chi_A)^{**}(t) - ((f+a)\chi_A)^{**}(t) > 0 \tag{4.34}$$

for all $0<\epsilon\le\delta_a$ and $t \in B$. Since the functions $((f+u+\delta_a)\chi_A)^{**}(t)$ and $((f+a)\chi_A)^{**}(t)$ are continuous and decreasing with respect to t, there exists a compact interval $[c,d] \subset B$ such that

$$((f+u+\delta_a)\chi_A)^{**}(t) < ((f+a)\chi_A)^{**}(t) \tag{4.35}$$

for all $t \in [c,d]$. Define

$$F(t) = ((f+a)\chi_A)^{**}(t) - ((f+u+\delta_a)\chi_A)^{**}(t) \tag{4.36}$$

for any $t \in [c,d]$. Since F is positive and continuous on $[c,d]$, we get $\min_{t\in[c,d]}\{F(t)\} > 0$. Consequently, there is $\gamma_a \in (0,1)$ such that

$$\left((f+u+\delta_a)\chi_A\right)^{**}(t) < (1-\gamma_a)\left((f+a)\chi_A\right)^{**}(t) \tag{4.37}$$

for all $t \in [c,d]$. Hence, by condition (4.33) we get

$$\begin{aligned}\left((f+u)\chi_A\right)^{**}(t) &< \left((f+u+\epsilon)\chi_A\right)^{**}(t) < (1-\gamma_a)\left((f+a)\chi_A\right)^{**}(t)\\ &< (1-\gamma_a)\left((f+a+\epsilon)\chi_A\right)^{**}(t)\end{aligned} \tag{4.38}$$

for every $t \in [c,d]$ and $0 < \epsilon \le \delta_a$. Therefore, by (4.34) we obtain

$$\frac{\left((f+u+\epsilon)\chi_A\right)^{**p}(t)-\left((f+u)\chi_A\right)^{**p}(t)}{\epsilon} \le (1-\gamma_a)^{p-1}\frac{\left((f+a+\epsilon)\chi_A\right)^{**p}(t)-\left((f+a)\chi_A\right)^{**p}(t)}{\epsilon} \tag{4.39}$$

for any $t \in [c,d]$ and $0 < \epsilon \le \delta_a$. Now according to Proposition 4.2 [20] we conclude

$$K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t) < K_{(f,A)}(-a,t)\left((f+a)\chi_A\right)^{**(p-1)}(t) \tag{4.40}$$

for all $t \in [c,d]$, which contradicts condition (4.15) and finishes the first case.

Case 2. Suppose that there exist $b \in (u,v]$ and $B \subset (0,\alpha)$ with $\mu(B) > 0$ such that

$$\left((f+u)\chi_A\right)^{**}(t) > \left((f+b)\chi_A\right)^{**}(t) \tag{4.41}$$

for any $t \in B$. Analogously as in the previous case we obtain that

$$\int_{E_t\cap A} \operatorname{sign}(f(s)+u)\,ds < 0 \tag{4.42}$$

for all $t \in B$. We claim that the function $a \to \left((f+a)\chi_A\right)^{**}(t)$ is strictly decreasing for any $a \in [u,v]$. Let $\xi,\eta \in [0,v-u]$ and $\xi < \eta$. The conditions (4.23) and (4.24) imply

$$\begin{aligned}\left((f+u+\xi)\chi_A\right)^{**}(t) - \left((f+u+\eta)\chi_A\right)^{**}(t) &= \frac{1}{t}\int_{E_t\cap A} \left|f(s)+u+\xi\right| - \left|f(s)+u+\eta\right| ds\\ &= \frac{\xi-\eta}{t}\int_{E_t\cap A} \operatorname{sign}(f(s)+u)\,ds > 0\end{aligned} \tag{4.43}$$

for any $t \in B$. Therefore, for any $a \in (u,v)$, $0 < \epsilon < min\{v-a, a-u\}/2$ and for all $t \in B$ we have

$$\left((f+u)\chi_A\right)^{**}(t) > \left((f+u+\xi)\chi_A\right)^{**}(t) > \left((f+a)\chi_A\right)^{**}(t) > \left((f+a+\epsilon)\chi_A\right)^{**}(t). \tag{4.44}$$

Repeating calculation in condition (4.43) with a instead of u, in view of (4.23) and (4.24) we also get

$$\left((f+u)\chi_A\right)^{**}(t) - \left((f+u+\epsilon)\chi_A\right)^{**}(t) = \left((f+a)\chi_A\right)^{**}(t) - \left((f+a+\epsilon)\chi_A\right)^{**}(t) > 0 \quad (4.45)$$

for all $t \in B$. Similarly as in the previous case there exists $\gamma_a > 0$ such that

$$\frac{\left((f+u+\epsilon)\chi_A\right)^{**p}(t) - \left((f+u)\chi_A\right)^{**p}(t)}{\epsilon} \leq (1+\gamma_a)^{p-1} \frac{\left((f+a+\epsilon)\chi_A\right)^{**p}(t) - \left((f+a)\chi_A\right)^{**p}(t)}{\epsilon} \quad (4.46)$$

for any $t \in B$ and $0 < \epsilon < min\{a-u, v-a\}/2$. Consequently, by Proposition 4.2 [20] we get

$$K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t) < K_{(f,A)}(-a,t)\left((f+a)\chi_A\right)^{**(p-1)}(t) \quad (4.47)$$

for all $t \in B$, which gives us a contradiction with (4.15) and completes the proof of inequality (4.12).

Let now $a, b \in \mathbb{R}$ and $a < b$. Denote $v = -a$ and $u = -b$. Clearly $u < v$. By (4.12) there exist $c \in (u, v]$ and $B \subset [0, \alpha)$, $\mu(B) > 0$ such that

$$K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t) < K_{(f,A)}(-c,t)\left((f+c)\chi_A\right)^{**(p-1)}(t) \quad (4.48)$$

for all $t \in B$. Hence, by (4.14) and by Proposition 2.4 in [7] we get

$$\begin{aligned} S^p_{(f,A)}(b) = S^p_{(f,A)}(-u) &= \int_{B\cup B^c} K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t)w(t)\,dt \\ &< \int_B K_{(f,A)}(-c,t)\left((f+c)\chi_A\right)^{**(p-1)}(t)w(t)\,dt \\ &\quad + \int_{B^c} K_{(f,A)}(-u,t)\left((f+u)\chi_A\right)^{**(p-1)}(t)w(t)\,dt \\ &\leq S^p_{(f,A)}(-c) \leq S^p_{(f,A)}(-v) = S^p_{(f,A)}(a), \end{aligned} \quad (4.49)$$

and the proof is done. □

Now we establish the main theorem of this section.

Theorem 4.5. *Let $1 < p < \infty$ and w be a positive weight function. Then the extended best constant approximant operator $T_{(p,A)}$ assumes an unique value, that is, for any $f \in \Gamma_{p-1,w}$,*

$$T_{(p,A)}(f) = \underline{f}_{(p,A)}\chi_A = \overline{f}_{(p,A)}\chi_A. \quad (4.50)$$

Proof. Suppose that $\underline{f}_{(p,A)} < \overline{f}_{(p,A)}$. Then there exist $u, v \in (\underline{f}_{(p,A)}, \overline{f}_{(p,A)})$ such that $u < v$. By Theorem 2.9 [7] and by Proposition 4.4 we obtain

$$\begin{gathered} 0 \le S_{(f,A)}\left(\overline{f}_{(p,A)}\right) < S_{(f,A)}(v) < S_{(f,A)}(u), \\ 0 \le S_{(-f,A)}\left(-\underline{f}_{(p,A)}\right) < S_{(-f,A)}(-u) < S_{(f,A)}(-v). \end{gathered} \tag{4.51}$$

Let $(f^{(n)})$ be a sequence of truncations of f. By Theorem 4.3 there exists $N_0 \in \mathbb{N}$ such that for all $n \ge N_0$ we have

$$0 < S_{(f^{(n)},A)}(v), \qquad 0 < S_{(-f^{(n)},A)}(-v) \tag{4.52}$$

as well as

$$0 < S_{(-f^{(n)},A)}(-u), \qquad 0 < S_{(f^{(n)},A)}(u). \tag{4.53}$$

Choose $n \ge N_0$. By Lemma 4.1, $f^{(n)} \in \Gamma_{(p,w)}$ for all $n \in \mathbb{N}$. Now by conditions (4.52) and (4.53) and by Theorem 2.4 we get that $u, v \in T_{(p,A)}(f^{(n)})$ for all $n \ge N_0$. Finally, by Corollary 2.5 we obtain that $T_{(p,A)}(f^{(n)})$ is unique for $n \ge N_0$, which implies a contradiction and finishes the proof. □

References

[1] F. Mazzone and H. Cuenya, "Maximal inequalities and Lebesgue's differentiation theorem for best approximant by constant over balls," *Journal of Approximation Theory*, vol. 110, no. 2, pp. 171–179, 2001.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, vol. 129 of *Pure and Applied Mathematics*, Academic Press, Boston, Mass, USA, 1988.

[3] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, NJ, USA, 1970.

[4] J. Bastero, M. Milman, and F. J. Ruiz, "Rearrangement of Hardy-Littlewood maximal functions in Lorentz spaces," *Proceedings of the American Mathematical Society*, vol. 128, no. 1, pp. 65–74, 2000.

[5] F. E. Levis, H. H. Cuenya, and A. N. Priori, "Best constant approximants in Orlicz-Lorentz spaces," *Commentationes Mathematicae*, vol. 48, no. 1, pp. 59–73, 2008.

[6] F. E. Levis, "Weak inequalities for maximal functions in Orlicz-Lorentz spaces and applications," *Journal of Approximation Theory*, vol. 162, no. 2, pp. 239–251, 2010.

[7] M. Ciesielski and A. Kamińska, "The best constant approximant operators in Lorentz spaces $\Gamma_{p,w}$ and their applications," *Journal of Approximation Theory*, vol. 162, no. 9, pp. 1518–1544, 2010.

[8] S. G. Kreĭn, Yu. Ĭ. Petunīn, and E. M. Semënov, *Interpolation of Linear Operators*, vol. 54 of *Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, USA, 1982.

[9] A. Kamińska and L. Maligranda, "Order convexity and concavity of Lorentz spaces $\Lambda_{p,w}$, $0 < p < \infty$," *Studia Mathematica*, vol. 160, no. 3, pp. 267–286, 2004.

[10] G. G. Lorentz, "On the theory of spaces Λ," *Pacific Journal of Mathematics*, vol. 1, pp. 411–429, 1951.

[11] M. A. Ariño and B. Muckenhoupt, "Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions," *Transactions of the American Mathematical Society*, vol. 320, no. 2, pp. 727–735, 1990.

[12] E. Sawyer, "Boundedness of classical operators on classical Lorentz spaces," *Studia Mathematica*, vol. 96, no. 2, pp. 145–158, 1990.

[13] V. D. Stepanov, "The weighted Hardy's inequality for nonincreasing functions," *Transactions of the American Mathematical Society*, vol. 338, no. 1, pp. 173–186, 1993.

[14] H. L. Royden, *Real Analysis*, Macmillan Publishing Company, New York, NY, USA, 3rd edition, 1988.
[15] N. L. Carothers, R. Haydon, and P.-K. Lin, "On the isometries of the Lorentz function spaces," *Israel Journal of Mathematics*, vol. 84, no. 1-2, pp. 265–287, 1993.
[16] J. V. Ryff, "Measure preserving transformations and rearrangements," *Journal of Mathematical Analysis and Applications*, vol. 31, pp. 449–458, 1970.
[17] R. E. Megginson, *An Introduction to Banach Space Theory*, vol. 183 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1998.
[18] E. W. Cheney, *Introduction to Approximation Theory*, AMS Chelsea Publishing, Providence, RI, USA, 1998.
[19] A. M. Pinkus, *On L^1-Approximation*, vol. 93 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1989.
[20] M. Ciesielski, A. Kamińska, and R. Płuciennik, "Gateaux derivatives and their applications to approximation in Lorentz spaces $\Gamma_{p,w}$," *Mathematische Nachrichten*, vol. 282, no. 9, pp. 1242–1264, 2009.
[21] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. II. Function Spaces*, vol. 97 of *Results in Mathematics and Related Areas*, Springer, Berlin, Germany, 1979.
[22] M. J. Carro, J. A. Raposo, and J. Soria, "Recent developments in the theory of Lorentz spaces and weighted inequalities," *Memoirs of the American Mathematical Society*, vol. 187, no. 877, 2007.
[23] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An F-Space Sampler*, vol. 89 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, UK, 1984.
[24] A. Kamińska and L. Maligranda, "On Lorentz spaces $\Gamma_{p,w}$," *Israel Journal of Mathematics*, vol. 140, pp. 285–318, 2004.
[25] J. Bastero and F. J. Ruiz, "Elementary reverse Hölder type inequalities with application to operator interpolation theory," *Proceedings of the American Mathematical Society*, vol. 124, no. 10, pp. 3183–3192, 1996.