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Research Article

Besov-Schatten Spaces

A. N. Marcoci,[1] L. G. Marcoci,[1] and L. E. Persson[2]

[1] *Department of Mathematics and Computer Science, Technical University of Civil Engineering Bucharest, 020396 Bucharest, Romania*
[2] *Department of Mathematics, Luleå University of Technology, 97 187 Luleå, Sweden*

Correspondence should be addressed to L. G. Marcoci, liviu.marcoci@gmail.com

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We introduce the Besov-Schatten spaces $B_p(\ell^2)$, a matrix version af analytic Besov space, and we compute the dual of this space showing that it coincides with the matricial Bloch space introduced previously in Popa (2007). Finally we compute the space of all Schur multipliers on $B_1(\ell^2)$.

1. Introduction

Analytic Besov spaces first found its direct application in operator theory in Peller's paper [1]. A comprehensive account of the theory of Besov spaces is given in Peetre's book [2]. In what follows we consider the Besov-Schatten spaces in the framework of matrices, for example, infinite matrix-valued functions. The extension to the matriceal framework is based on the fact that there is a natural correspondence between Toeplitz matrices and formal series associated to 2π-periodic functions (see, e.g., [3–6]). We use the powerful device Schur multipliers and its characterizations in the case of Toeplitz matrices to prove some of the main results.

The *Schur product* (or Hadamard product) of matrices $A = (a_{jk})_{j,k\geq 0}$ and $B = (b_{jk})_{j,k\geq 0}$ is defined as the matrix $A * B$ whose entries are the products of the entries of A and B:

$$A * B = \left(a_{jk} b_{jk}\right)_{j,k\geq 0}. \tag{1.1}$$

If X and Y are two Banach spaces of matrices we define Schur multipliers from X to Y as the space

$$M(X,Y) = \left\{M : M * A \in Y \text{ for every } A \in X\right\}, \tag{1.2}$$

equipped with the natural norm

$$\|M\| = \sup_{\|A\|_X \leq 1} \|M * A\|_Y. \tag{1.3}$$

In the case $X = Y = B(\ell^2)$, where $B(\ell^2)$ is the space of all linear and bounded operators on ℓ^2, the space $M(B(\ell^2), B(\ell^2))$ will be denoted $M(\ell^2)$ and a matrix $A \in M(\ell^2)$ will be called Schur multiplier. We mention here an important result due to Bennett [7], which will be often used in this paper.

Theorem 1.1. *The Toeplitz matrix $M = (c_{j-k})_{j,k}$, where $(c_n)_{n\in\mathbb{Z}}$ is a sequence of complex numbers, is a Schur multiplier if and only if there exists a bounded and complex Borel measure μ on (the circle group) $\mathbb{T}$ with*

$$\widehat{\mu}(n) = c_n, \quad \text{for } n = 0, \pm 1, \pm 2, \ldots. \tag{1.4}$$

Moreover, one then has that

$$\|M\| = \|\mu\|. \tag{1.5}$$

We will denote by C_p, $0 < p < \infty$, the Schatten class operators (see, e.g., [8]). Let us summarize briefly some well-known properties of classes $M(C_p)$ which will be very often used in what follows. If $1 < p < \infty$, then $M(C_p) = M(C_{p'})$, where $1/p + 1/p' = 1$ and $M(\ell^2) = M(C_1)$. Next, interpolating between the classes C_p, we can easily see that $M(C_{p_1}) \subset M(C_{p_2})$ if $0 < p_1 \leq p_2 \leq 2$ (see, e.g., [9]). We will denote by A_k, the kth-*diagonal matrix* associated to A (see [4]). For an infinite matrix $A = (a_{ij})$ and an integer k we denote by A_k the matrix whose entries a'_{ij} are given by

$$a'_{ij} = \begin{cases} a_{ij} & \text{if } j - i = k, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

In what follows we will recall some definitions from [10] (see also [11]), which we will use in this paper. We consider on the interval $[0, 1)$ the Lebesgue measurable infinite matrix-valued functions $A(r)$. These functions may be regarded as infinite matrix-valued functions defined on the unit disc D using the correspondence

$$A(r) \longrightarrow f_A(r, t) = \sum_{k=-\infty}^{\infty} A_k(r) e^{ikt}, \tag{1.7}$$

where $A_k(r)$ is the kth-diagonal of the matrix $A(r)$, the preceding sum is a formal one, and t belongs to the torus $\mathbb{T}$. This matrix $A(r)$ is called *analytic matrix* if there exists an upper triangular infinite matrix A such that, for all $r \in [0, 1)$, we have $A_k(r) = A_k r^k$, for all $k \in \mathbb{Z}$. In what follows we identify the analytic matrices $A(r)$ with their corresponding upper triangular matrices A and we call them also analytic matrices.

We also recall the definition of the matriceal Bloch space and the so-called little Bloch space of matrices (see [11]). The *matriceal Bloch space* $\mathcal{B}(D,\ell^2)$ is the space of all analytic matrices A with $A(r) \in B(\ell^2)$, $0 \le r < 1$, such that

$$\|A\|_{\mathcal{B}(D,\ell^2)} = \sup_{0\le r<1}\left(1-r^2\right)\left\|A'(r)\right\|_{B(\ell^2)} + \|A_0\|_{B(\ell^2)} < \infty, \tag{1.8}$$

where $B(\ell^2)$ is the usual operator norm of the matrix A on the sequence space ℓ^2 and $A'(r) = \sum_{k=0}^{\infty} A_k k r^{k-1}$.

The space $\mathcal{B}_0(D,\ell^2)$ is the space of all upper triangular infinite matrices A such that $\lim_{r\to 1^-}(1-r^2)\|(A * C(r))'\|_{B(\ell^2)} = 0$, where $C(r)$ is the Toeplitz matrix associated with the Cauchy kernel $1/(1-r)$, for $0 \le r < 1$.

An important tool in this paper is the *Bergman projection*. It is known (see, e.g., [10]) that for all strong measurable C_p-valued functions $r \to A(r)$ defined on $[0,1)$ with $\int_0^1 \|A(r)\|_{C_p}^p 2r dr < \infty$ and for all $i,j \in \mathbb{N}$ we have that

$$[P(A(\cdot))](r)(i,j) = \begin{cases} 2(j-i+1)r^{j-i}\displaystyle\int_0^1 a_{ij}(s)\cdot s^{j-i+1}ds, & \text{if } i \le j, \\ 0, & \text{otherwise.} \end{cases} \tag{1.9}$$

Now we consider a modified version of Bergman projection.

Let $\alpha > -1$. Then

$$[P_\alpha A(\cdot)](r) = \begin{cases} \dfrac{(\alpha+1)\Gamma(j-i+2+\alpha)}{(j-i)!\Gamma(\alpha+2)} r^{j-i}\left(2\displaystyle\int_0^1 a_{ij}(s)s^{j-i+1}\left(1-s^2\right)^{\alpha}ds\right) & \text{if } j \ge i \\ 0 & \text{if } j < i. \end{cases} \tag{1.10}$$

We remark that, for $\alpha = 0$, it follows that $P_\alpha = P$.

We recall now a lemma from [11] that we will use in the following.

Lemma 1.2. *Let $V = (P_2)^*$, that is,*

$$(P_2A(\cdot))^*(r)(i,j) = \begin{cases} \dfrac{(j-i+3)(j-i+2)(j-i+1)}{2} r^{j-i}\left(1-r^2\right)^2\displaystyle\int_0^1 a_{ij}(s)s^{j-i}(2sds) & \textit{if } j-i \ge 0 \\ 0 & \textit{otherwise.} \end{cases} \tag{1.11}$$

Then V is an isomorphic embedding of $\mathcal{B}_0(D,\ell^2)$ in $\mathcal{C}_0(D,\ell^2)$, where $\mathcal{C}_0(D,\ell^2)$ is the space of all continuous $B(\ell^2)$-valued functions $B(r)$ on $[0,1)$ such that $\lim_{r\to 1}B(r) = 0$ in the norm of $B(\ell^2)$.

The paper is organized as follows. In Section 2 we give a characterization of matrices in the Besov-Schatten space $B_p(\ell^2)$ using the Bergman projection. The main result in Section 3 is a new duality result (see Theorem 3.2).

2. Besov-Schatten Spaces

Now we introduce a new space of matrices the so-called *Besov-Schatten space*.

Definition 2.1. Let $1 \leq p < \infty$ and a positive measure on $[0,1)$ given by

$$d\lambda(r) = \frac{2r\,dr}{(1-r^2)^2}. \tag{2.1}$$

The Besov-Schatten matrix space $B_p(\ell^2)$ is defined to be the space of all upper triangular infinite matrices A such that

$$\|A\|_{B_p(\ell^2)} = \left[\int_0^1 \left(1-r^2\right)^{2p} \|A''(r)\|_{C_p}^p \, d\lambda(r)\right]^{1/p} < \infty. \tag{2.2}$$

On $B_p(\ell^2)$ we introduce the norm

$$\|A\| = \|A_0\|_{C_1} + \|A\|_{B_p(\ell^2)}. \tag{2.3}$$

We introduce the notation $L^p(D, d\lambda, \ell^2)$ for the space of all strongly measurable functions $r \to A(r)$ defined on the measurable space $([0,1), d\lambda)$ with C_p values such that

$$\|A\|_{L^p(D,d\lambda,\ell^2)} = \left(\int_0^1 \|A(r)\|_{C_p}^p \, d\lambda(r)\right)^{1/p} < \infty. \tag{2.4}$$

We need the following interesting lemma in what follows (see [8, page 53]).

Lemma 2.2. *Let $z \in D$, c is real, $t > -1$, and*

$$I_{c,t} = \int_D \frac{\left(1-|w|^2\right)^t}{|1-z\overline{w}|^{2+t+c}} \, dA(w). \tag{2.5}$$

Then,

(1) *if $c < 0$, then $I_{c,t}(z)$ is bounded in z;*

(2) *if $c > 0$, then*

$$I_{c,t}(z) \sim \frac{1}{\left(1-|z|^2\right)^c} \quad \left(|z| \longrightarrow 1^-\right); \tag{2.6}$$

(3) *if* $c = 0$, *then*

$$I_{0,c}(z) \sim \log \frac{1}{1-|z|^2} \quad (|z| \longrightarrow 1^-). \tag{2.7}$$

The next theorem expresses a natural relation between the Bergman projection and the Besov-Schatten spaces. More precisely our main result of this section is the following equivalence theorem.

Theorem 2.3. *Let* $1 \le p < \infty$ *and* A *be an upper triangular matrix such that the* C_p*-valued function* $r \to A''(r)$ *is continuous on* $[0, r_0)$ *for some* $1 > r_0 > 0$. *Then the following assertions are equivalent:*

(1) $A \in B_p(\ell^2)$;

(2) $(1-r^2)^2 A''(r) \in L^p(D, d\lambda, \ell^2)$;

(3) $A \in PL^p(D, d\lambda, \ell^2)$, *where* P *is the Bergman projection.*

Proof. It is obvious that (1) is equivalent to (2). We observe that the Bergman projection may be described as follows:

$$P(A(\cdot)) = \sum_{k=0}^{\infty}(k+1)\int_0^1 [A(s)]_k s^k (2s ds), \tag{2.8}$$

where $A(\cdot) \in L^p(D, \ell^2)$. Then

$$P\left(\left(1-r^2\right)^2 A_k r^k\right)(s) = \frac{2s^k A_k}{(k+2)(k+3)}, \tag{2.9}$$

for all $k \ge 0$, and all $A_k \in C_p$.

It follows that each matriceal polynomial is in $PL^p(D, d\lambda, \ell^2)$ for all $1 \le p < \infty$.

Suppose that A is an upper triangular matrix with $A_k \in C_p$ for all $k \ge 0$. We write

$$A = \sum_{k=0}^{4} A_k + A^1, \tag{2.10}$$

where $A^1 := \sum_{k=5}^{\infty} A_k$.

If $(1-r^2)^2 A''(r) \in L^p(D, d\lambda, \ell^2)$, then we have that

$$\Phi(r) := \sum_{k=0}^{4} \frac{(k+2)(k+3)}{2}\left(1-r^2\right)^2 A_k r^k + \frac{\left(1-r^2\right)^2 \left(A^1\right)''(r)}{2! r^2} \tag{2.11}$$

is in $L^p(D, d\lambda, \ell^2)$ and moreover that $A = P\Phi$.

Indeed, for $0 < r < r_0$, $r \to (A^1)''(r)$ is a continuous function and, therefore

$$\int_0^{r_0} \frac{\left\|(A^1)''(s)\right\|_{C_p}^p}{s^{2p}} ds < \infty. \tag{2.12}$$

Consequently $\Phi \in L^p(D, d\lambda, \ell^2)$.

Moreover $A = P\Phi$ since

$$\begin{aligned} &\sum_{k=5}^{\infty} \int_0^1 \frac{k(k-1)A_k s^{k-2}(1-s^2)^2}{s^2}(k+1)s^{k+1} ds \\ &\quad = \sum_{k=5}^{\infty}(k-1)k(k+1)A_k \int_0^1 s^{2k-3}\left(1-s^2\right)^2 ds = \sum_{k=5}^{\infty} A_k. \end{aligned} \tag{2.13}$$

Thus we have proved that (2) implies (3).

It remains to prove that (3) implies (2). Suppose that (3) holds, and let $A = P\Phi$ for some $\Phi(\cdot) \in L^p(D, d\lambda, \ell^2)$. Then we have that

$$\left(1-r^2\right)^2 A''(r) = \left(1-r^2\right)^2 \int_0^1 \left[\Phi(s) * \frac{6s^2}{(1-rs)^4}\right] 2s ds. \tag{2.14}$$

Using Fubini's theorem and Lemma 2.2 we obtain that

$$\begin{aligned} \int_0^1 \left(1-r^2\right)^2 \left\|A''(r)\right\|_{C_1} \frac{2r dr}{(1-r^2)^2} &\le \int_0^1 \left[\int_0^1 \|\Phi(s)\|_{C_1} \int_0^{2\pi} \frac{6s^2 d\theta}{\left|1-rse^{i\theta}\right|^4} 2s ds\right] 2r dr \\ &= \int_0^1 6s^2 \|\Phi(s)\|_{C_1} \left[\int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{\left|1-rse^{i\theta}\right|^4} 2r dr\right] 2s ds \\ &\sim \int_0^1 \|\Phi(s)\|_{C_1} \frac{12s^3}{\left(1-s^2\right)^2} ds \le 6 \int_0^1 \|\Phi(s)\|_{C_1} d\lambda(s) < \infty. \end{aligned} \tag{2.15}$$

Consequently, $A \in L^1(D, d\lambda, \ell^2)$ and this proves that (3) implies (2) in the case $p = 1$. The proof in the case $1 < p < \infty$ is similar to the classical case of functions (see, e.g., [8, Theorem 5.3.3.]). Let $T(rs) = ((t_{ij})(rs))_{i,j=1}^{\infty}$ be the Toeplitz matrix with

$$t_{ij}(rs) = t_{j-i}(rs) = \begin{cases} s^2(rs)^{j-i}(j-i+3)(j-i+2)(j-i+1) & \text{if } j \ge i \\ 0 & \text{otherwise.} \end{cases} \tag{2.16}$$

Since $T(rs)$ is a Schur multiplier with $\|T(rs)\|_{M(\ell^2)} = \|T(rs)\|_{L^1(\mathbb{T})} = \|6s^2/(1-rse^{i\theta})^4\|_{L^1(\mathbb{T})}$ and $M(\ell^2) = M(C_1) \subset M(C_p)$, $1 \le p < \infty$ we get that

$$\begin{aligned}
\left(1-r^2\right)^2 \|A''(r)\|_{C_p} &= \left(1-r^2\right)^2 \left\| \int_0^1 \left[\phi(s) * \frac{6s^2}{(1-rs)^4} \right] (2sds) \right\|_{C_p} \\
&\le \left(1-r^2\right)^2 \int_0^1 \|\phi(s)\|_{C_p} \left\| \frac{6s^2}{\left(1-rse^{i\theta}\right)^4} \right\|_{L^1(\mathbb{T})} (2sds) \\
&= \left(1-r^2\right)^2 \int_0^1 \|\phi(s)\|_{C_p} \left(1-s^2\right)^2 \left\| \frac{6s^2}{\left(1-rse^{i\theta}\right)^4} \right\|_{L^1(\mathbb{T})} d\lambda(s) := S_\phi(r).
\end{aligned} \tag{2.17}$$

From Schur's theorem (see, e.g., [8]) it follows that $S_\phi(r)$ is bounded on $L^p([0,1), d\lambda)$ which in its turn implies that

$$\left(1-r^2\right)^2 A''(r) \in L^p\left(D, d\lambda, \ell^2\right) \tag{2.18}$$

for $1 < p < \infty$. Thus also the implication (3)⇒(2) is proved and the proof is complete. □

3. The Dual of Besov-Schatten Spaces

Our aim in this section is to characterize the Banach dual spaces of Besov-Schatten spaces.

First we prove the following lemma of independent interest.

Lemma 3.1. *Let $V = (P_2)^*$, that is,*

$$\begin{aligned}
&[V(A(\cdot))](r)(i,j) \\
&\quad = \begin{cases} \dfrac{(j-i+3)(j-i+2)(j-i+1)}{2} r^{j-i}\left(1-r^2\right)^2 \displaystyle\int_0^1 a_{ij}(s) s^{j-i}(2sds) & \text{if } j-i \ge 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.1}$$

Then V is an embedding from $B_p(\ell^2)$ into $L^p(D, d\lambda, \ell^2)$ for all $p \ge 1$, if $B_p(\ell^2) = PL^p(D, d\lambda, \ell^2)$ is equipped with the quotient norm.

Proof. Suppose that $A \in B_p(\ell^2)$ and $B(\cdot) \in L^p(D, d\lambda, \ell^2)$ with $A = PB(\cdot)$. Since

$$P(B(\cdot))(r)(i,j) = \begin{cases} 2(j-i+1) r^{j-i} \displaystyle\int_0^1 b_{ij}(s) s^{j-i+1} ds & \text{if } j-i \ge 0 \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

it is easy to see that

$$PV = P, \qquad VP = V \tag{3.3}$$

on $L^p(D, d\lambda, \ell^2)$. Therefore $V(A) = V(B(\cdot))$ for all $A \in B_p(\ell^2)$ and $B(\cdot) \in L^p(D, d\lambda, \ell^2)$.

We will now prove that V is a bounded operator on $L^p(D, d\lambda, \ell^2)$. We first prove this fact for $p = 1$. By Fubini's theorem we have that

$$\begin{aligned}
\|V(A(\cdot))\|_{L^1(D,d\lambda,\ell^2)} &= \int_0^1 \|[V(A(\cdot))]\|_{C_1} d\lambda(r) \\
&= \int_0^1 \left\| \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k \left(1-r^2\right)^2 \int_0^1 A_k(s) s^k (2s ds) \right\|_{C_1} d\lambda(r) \\
&\le \iint_0^1 \left\| \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k (1-r^2)^2 A_k(s) s^{k+1} \right\|_{C_1} (2ds) d\lambda(r) \\
&= \int_0^1 \left[\int_0^1 \left\| \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k A_k(s) s^k \left(1-s^2\right)^2 \right\|_{C_1} (2rdr) \right] d\lambda(s) \\
&= \iint_0^1 \|A(s) * C(rs)\|_{C_1} (2rdr) d\lambda(s),
\end{aligned} \tag{3.4}$$

where $C(rs) = (c_{ij}(rs))_{i,l=1}^{\infty}$ means the Toeplitz matrix given by

$$\begin{aligned}
c_{ij}(rs) &= c_{j-i}(rs) \\
&= \begin{cases} (rs)^{j-i} s^2 \left(1-s^2\right)^2 \dfrac{(j-i+3)(j-i+2)(j-i+1)}{2} & \text{if } j \ge i, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.5}$$

Since the Toeplitz matrix $C(rs)$ is a Schur multiplier with

$$\|C(rs)\|_{M(\ell^2)} = \left\| \frac{6s^2\left(1-s^2\right)^2}{\left(1-rse^{i\theta}\right)^4} \right\|_{L^1(\mathbb{T})}, \tag{3.6}$$

then, according to Lemma 2.2, it follows that

$$\begin{aligned}
\iint_0^1 \|A(s) * C(rs)\|_{C_1} (2rdr) d\lambda(s) &\le \int_0^1 \|A(s)\|_{C_1} \int_0^1 \|C(rs)\|_{M(\ell^2)} (2rdr) d\lambda(s) \\
&\sim \int_0^1 \|A(s)\|_{C_1} d\lambda(s).
\end{aligned} \tag{3.7}$$

Consequently V is bounded on $L^1(D, d\lambda, \ell^2)$. For $1 < p < \infty$ we have that

$$\begin{aligned}\|VA(\cdot)(r)\|_{C_p} &\leq \int_0^1 \left\| \sum_{k=0}^{\infty} \frac{(k+3)(k+2)(k+1)}{2} r^k s^k \left(1-r^2\right)^2 \left(1-s^2\right)^2 A_k(s) \right\|_{C_p} d\lambda(s) \\ &= \int_0^1 \|A(s) * T(rs)\|_{C_p} d\lambda(s),\end{aligned} \tag{3.8}$$

where $T(rs) = (t_{j-i}(rs))_{i,j}$ is a Toeplitz matrix and

$$t_{j-i}(rs) = \begin{cases} (rs)^{j-i}\left(1-s^2\right)^2\left(1-r^2\right)^2 \dfrac{(j-i+3)(j-i+2)(j-i+1)}{2} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases} \tag{3.9}$$

$T(rs)$ is a Schur multiplier, therefore

$$\int_0^1 \|A(s) * T(rs)\|_{C_p} d\lambda(s) \leq \int_0^1 \|A(s)\|_{C_p} \left(1-r^2\right)^2 \left(1-s^2\right)^2 \left\| \frac{6}{\left(1-rse^{i\theta}\right)^4} \right\|_{L^1(\mathbb{T})} := S_A(r). \tag{3.10}$$

From Schur's theorem, (see, e.g., [8]) we obtain that $S_A(r)$ is bounded on $L^p([0,1), d\lambda)$. Hence V is bounded on $L^p(D, d\lambda, \ell^2)$, $1 \leq p < \infty$, and there is a constant $C > 0$ such that

$$\|V(A(\cdot))\|_{L^p(D,d\lambda,\ell^2)} \leq C\|B(\cdot)\|_{L^p(D,d\lambda,\ell^2)} \tag{3.11}$$

for all $A = PB(\cdot)$. Taking the infimum over B, we get that

$$\|V(A)\|_{L^p(D,d\lambda,\ell^2)} \leq C\|A\|_{B_p(\ell^2)}. \tag{3.12}$$

Thus $V : B_p(\ell^2) \to L^p(D, d\lambda, \ell^2)$ is bounded.

On the other hand, since $PV = P$ and $VP = V$ on $L^p(D, d\lambda, \ell^2)$ we get easily that $A = PV(A)$ for all $A \in B_p(\ell^2)$. Thus

$$\|A\|_{B_p(\ell^2)} = \inf\left\{ \|B(\cdot)\|_{L^p(D,d\lambda,\ell^2)} : \ A = PB \right\} \leq \|VA\|_{L^p(D,d\lambda,\ell^2)}, \tag{3.13}$$

and hence $V : B_p(\ell^2) \to L^p(D, d\lambda, \ell^2)$ is an embedding. The proof is complete. □

We denote by $\mathcal{B}_{0,c}(D, \ell^2)$ the closed Banach subspace of $\mathcal{B}_0(D, \ell^2)$ consisting of all upper triangular matrices whose diagonals are compact operators. Now we can formulate and prove the duality of Besov-Schatten spaces.

Theorem 3.2. *Under the pairing*

$$\langle A, B\rangle = \int_0^1 \operatorname{tr}\big(V(A)[V(B)]^*\big)d\lambda(r) \tag{3.14}$$

One has the following dualities:

(1) $B_p(\ell^2)^* \approx B_q(\ell^2)$ *if* $1 < p < \infty$ *and* $1/p + 1/q = 1$;

(2) $\mathcal{B}_{0,c}(D, \ell^2)^* \approx B_1(\ell^2)$ *and* $B_1(\ell^2)^* \approx \mathcal{B}(D, \ell^2)$.

Proof. Since V is an embedding from $B_p(\ell^2)$ into $L^p(D, d\lambda, \ell^2)$ for all $1 \le p < \infty$, Hölder's inequality shows that $B_q(\ell^2) \subset B_p(\ell^2)^*$ for $1 \le p < \infty$ and $B_1(\ell^2) \subset \mathcal{B}^*_{0,c}(D, \ell^2)$.

Suppose that F is a bounded linear functional on the Besov-Schatten space $B_p(\ell^2)$ with $1 \le p < \infty$. Then $F \circ V^{-1} : VB_p(\ell^2) \to \mathbb{C}$ extends to a bounded linear functional on $L^p(D, d\lambda, \ell^2)$. Thus there exists $C(\cdot) \in L^q(D, d\lambda, \ell^2)$ such that $\|C(\cdot)\|_{L^q(D,d\lambda,\ell^2)} = \|F \circ V^{-1}\|$ and

$$\left(F \circ V^{-1}\right)(B) = \int_0^1 \operatorname{tr}(B(r))[C(r)]^* d\lambda(r), \quad B(\cdot) \in L^p\left(D, d\lambda, \ell^2\right). \tag{3.15}$$

In particular, if $B(\cdot) = V(A)$ with $A \in B_p(\ell^2)$, then

$$F(A) = \int_0^1 \operatorname{tr}((VA)(r))[C(r)]^* d\lambda(r). \tag{3.16}$$

Let $B = P(C)$. Then $B \in B_q(\ell^2)$ and it is easy to check that

$$F(A) = \int_0^1 \operatorname{tr}\big((VA)(r)[(VB)(r)]^*\big)d\lambda(r), \quad A \in B_p\left(\ell^2\right), \tag{3.17}$$

with $\|B\|_{B_q(\ell^2)} \le \|C(\cdot)\|_{L^p(D,d\lambda,\ell^2)} = \|F \circ V^{-1}\| \le \|V^{-1}\|\|F\|$. This proves the duality $B_p(\ell^2)^* \approx B_q(\ell^2)$ for $1 \le p < \infty$.

It remains to prove the duality $\mathcal{B}^*_{0,c}(D, \ell^2) \approx B_1(\ell^2)$.

Let us assume that F is a bounded linear functional on $\mathcal{B}_{0,c}(D, \ell^2)$. Then we will prove that there is a matrix C from $B_1(\ell^2)$ such that

$$F(B) = \int_0^1 \operatorname{tr}\big[VB(r)(VC)^*(r)\big]d\lambda(r), \tag{3.18}$$

for B from a dense subset of $\mathcal{B}_0(D, \ell^2)$. By Lemma 1.2 it follows that $V : \mathcal{B}_0(D, \ell^2) \to \mathcal{C}_0(D, \ell^2)$ is an isomorphic embedding. Thus $X = V(\mathcal{B}_{0,c}(D, \ell^2))$ is a closed subspace in $\mathcal{C}_0(D, C_\infty)$ and $F \circ (V)^{-1} : X \to \mathbb{C}$ is a bounded linear functional on X, where $\mathbb{C}_0(D, C_\infty)$ is the subset in $\mathcal{C}_0(D, \ell^2)$ whose elements are C_∞-valued functions. By the Hahn-Banach theorem $F \circ (V)^{-1}$ can be extended to a bounded linear functional on $\mathcal{C}_0(D, C_\infty)$.

Let $\Phi : \mathcal{C}_0(D, C_\infty) \to \mathbb{C}$ denote this functional. It follows that $\mathcal{C}_0(D, C_\infty) = \mathcal{C}_0[0,1]\widehat{\otimes}_\epsilon C_\infty$ and, thus, Φ is a bilinear integral map, that is, there is a bounded Borel measure

μ on $[0,1]\times U_{C_1}$, where U_{C_1} is the unit ball of the space C_1 with the topology $\sigma(C_1, C_\infty)$, such that

$$\Phi(f\otimes A) = \int_{[0,1]\times U_{C_1}} f(r)\,\mathrm{tr}(AB^*)\,d\mu(r,B) \tag{3.19}$$

for every $f\in \mathcal{C}_0[0,1]$ and $A\in C_\infty$.

Thus, for the matrix $\sum_{k=0}^{n} A_k \in \mathcal{B}_{0,c}(D,\ell^2)$, identified with the analytic matrix $\sum_{k=0}^{n} A_k r^k$, we have that

$$\begin{aligned}
F\left(\sum_{k=0}^{n} A_k\right) &= F\left(\sum_{k=0}^{n} r^k A_k\right) = \left[F\circ (V)^{-1}\right]\left[V\left(\sum_{k=0}^{n} r^k A_k\right)\right] \\
&= \Phi\left(\sum_{k=0}^{n} \frac{(k+3)(k+2)}{2} r^k \left(1-r^2\right)^2 A_k\right) \\
&= \int_{[0,1]\times U_{C_1}} \sum_{k=0}^{n} \mathrm{tr}\left[\left(\frac{(k+3)(k+2)}{2} r^k \left(1-r^2\right)^2 A_k\right) B^*\right] d\mu(r,B) \\
&\overset{\text{def}}{=} \left\langle \mu(r,B), \mathrm{tr}\left(\sum_{k=0}^{n} \frac{(k+3)(k+2)}{2} r^k A_k\right) B^* \left(1-r^2\right)^2 \right\rangle.
\end{aligned} \tag{3.20}$$

On the other hand, we wish to have that

$$\begin{aligned}
F(A) &= \int_0^1 \mathrm{tr}\, V(A)(V(C))^* d\lambda(s) \\
&= \int_0^1 \mathrm{tr}\left(\sum_{k=0}^{n} \frac{(k+3)(k+2)}{2} s^k A_k\right)(V(C))^*(2s\,ds) \\
&= \int_0^1 \mathrm{tr}\left(\sum_{k=0}^{n} s^{2k} \frac{(k+3)^2(k+2)^2}{4} \left(1-s^2\right)^2 A_k C_k^*\right)(2s\,ds) \\
&= \sum_{k=0}^{n} \mathrm{tr}\, A_k\left(\frac{(k+3)(k+2)}{2(k+1)} C_k^*\right).
\end{aligned} \tag{3.21}$$

Therefore, letting $A = e_{i,i+k}$, denote the matrix having 1 as the single nonzero entry on the ith-row and the $(i+k)$th-column, for $i\geq 1$ and $j\geq 0$, we have that

$$C_k = \left\langle \overline{\mu}(r,B), (k+1) r^k \left(1-r^2\right)^2 B_k \right\rangle, \quad k = 0,1,2,\ldots. \tag{3.22}$$

Then, it yields that

$$\begin{aligned}
&\int_0^1 \|C''(s)\|_{C_1} 2sds \\
&\quad = \int_0^1 \left\| \int_{[0,1]\times U_{C_1}} \sum_{k=2}^{n} \frac{(k+1)!}{(k-2)!} s^{k-2} r^k \left(1-r^2\right)^2 B_k d\mu(r,B) \right\|_{C_1} (2sds) \\
&\quad \le \int_{[0,1]\times U_{C_1}} \left[\int_0^1 \left\| \sum_{k=2}^{n} \frac{(k+1)!}{(k-2)!} (rs)^{k-2} r^2 \left(1-r^2\right)^2 B_k \right\|_{C_1} (2sds) \right] d|\mu|(r,B) \\
&\quad \le \int_{[0,1]\times U_{C_1}} \left[\int_0^1 \left\| \sum_{k=2}^{n} \frac{(k+1)!}{(k-2)!} (rs)^{k-2} r^2 \left(1-r^2\right)^2 e^{ik(\cdot)} \right\|_{L^1(\mathbb{T})} \|B\|_{C_1} (2sds) \right] d|\mu|(r,B) \\
&\quad \le \int_{[0,1]\times U_{C_1}} \left(\int_0^1 \int_0^{2\pi} \frac{r^2\left(1-r^2\right)^2}{\left|1-rse^{i\theta}\right|^4} \frac{d\theta}{2\pi} (2sds) \right) d|\mu|(r,B) \\
&\quad \sim \int_{[0,1]\times U_{C_1}} r^2\left(1-r^2\right)^2 \frac{1}{\left(1-r^2\right)^2} d|\mu|(r,B) \le \|\mu\| < \infty.
\end{aligned} \tag{3.23}$$

Consequently, $C \in B_1(\ell^2)$ and we get the relation (3.18) by using the fact that the set of all matrices $\sum_{k=0}^{n} A_k$ is dense in $\mathcal{B}_{0,c}(D,\ell^2)$. □

As an application of the description of the dual space of Besov-Schatten space we give a characterization of the space of all Schur multipliers between Besov-Schatten spaces $B_1(\ell^2)$.

Theorem 3.3. *One has* $(B_1(\ell^2), B_1(\ell^2)) = H_1^{1,\infty,1}(\ell^2) \overset{\text{def}}{=} \{A : \sup_{r<1}(1-r)\|\sum_{k\ge 0} kA_k\|_{M(\ell^2)} < \infty\}$.

Proof. By Lemma 3.1 we have that $V(A*B) = V(A)*B$ for all $A \in B_1(\ell^2)$ and for all matrices B such that $A*B \in B_1(\ell^2)$. Consequently $(B_1(\ell^2), B_1(\ell^2)) = (\mathcal{B}(\ell^2), \mathcal{B}(\ell^2))$. Finally, by using [12, Theorem 6] we get the stated result. □

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