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Research Article

The Existence and Multiplicity of Positive Solutions for Second-Order Periodic Boundary Value Problem

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The existence and multiplicity of positive solutions are established for second-order periodic boundary value problem. Our results are based on the theory of a fixed point index for A-proper semilinear operators defined on cones due to Cremins. Our approach is different in essence from other papers and the main results of this paper are also new.

1. Introduction

In the present paper, we discuss the existence of positive solutions of the periodic boundary value problem (PBVP) for second-order differential equation

$$\begin{gathered} x''(t) = f(t,x), \quad 0 < t < 1, \\ x(0) = x(1), \qquad x'(0) = x'(1), \end{gathered} \tag{1.1}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Our purpose here is to provide sufficient conditions for the existence of multiple positive solutions to the periodic boundary value problem (1.1). This will be done by applying the theory of a fixed point index for A-proper semilinear operators defined on cones obtained by Cremins [1].

We are interested in positive solutions of (1.1), because we have been motivated by a problem from the Theory of Nonlinear Elasticity modelling radial oscillations of an elastic spherical membrane made up of a neo-Hookean material and subjected to an internal pressure. Because of wide interests in physics and engineering, second-order periodic boundary value problems have been studied widely in the literature; we refer the reader to [2–30] and references therein. In [6, 7], by using Krasnoselskii's fixed point theorem, the existence and multiplicity of positive solutions are established to the periodic boundary value problem on

$$\begin{aligned} -x'' + \rho^2 x = f(t,x), &\quad 0 < t < 2\pi,\ \rho > 0, \\ x(0) = x(2\pi), &\quad x'(0) = x'(2\pi), \\ x'' + \rho^2 x = f(t,x), &\quad 0 < t < 2\pi,\ 0 < \rho < \frac{1}{2}, \\ x(0) = x(2\pi), &\quad x'(0) = x'(2\pi). \end{aligned} \tag{1.2}$$

Agarwal et al. [8] discussed the existence of positive solutions for the second-order differential equation

$$\begin{aligned} -x''(t) + b(t)x(t) = g(t)f(t,x(t)), &\quad 0 < t < \omega, \\ x(0) = x(\omega), &\quad x'(0) = x'(\omega), \end{aligned} \tag{1.3}$$

where $b(t)$ and $g(t)$ are continuous ω-periodic positive functions and $f \in C(\mathbb{R}\times[0,\infty),[0,\infty))$. By employing fixed point index theory in cones, they found sufficient conditions for the existence of at least one positive solution. Recently, Torres [9] and Yao [10] obtained some results on the existence of positive solutions of a general periodic boundary value problem

$$\begin{aligned} x''(t) = f(t,x(t)), &\quad 0 < t < 2\pi, \\ x(0) = x(2\pi), &\quad x'(0) = x'(2\pi). \end{aligned} \tag{1.4}$$

In this case, the problem (1.4) has no Green function. In order to overcome this difficulty, their main technique is to rewrite the original PBVP (1.4) as an equivalent one, so that the Krasnoselskii fixed point theorem on compression and expansion of cones can be applied. Inspired by the above work, the aim of this paper is to consider the existence and multiplicity of positive solutions for the periodic boundary value problem (1.1). The method we used here is different in essence from other papers and the main results of this paper are also new.

This paper is organized as follows. In Section 2, we give some preliminaries and establish several lemmas, and the main theorems are formulated and proved in Section 3. Finally, in Section 4, we give two examples to illustrate our results.

2. Notation and Preliminaries

We start by introducing some basic notation relative to theory of the fixed point index for A-proper semilinear operators defined on cones established by Cremins (see [1]).

Let X and Y be Banach spaces, D a linear subspace of X, $\{X_n\} \subset D$, and $\{Y_n\} \subset Y$ sequences of oriented finite-dimensional subspaces such that $Q_n y \to y$ in Y for every y and $\operatorname{dist}(x, X_n) \to 0$ for every $x \in D$, where $Q_n : Y \to Y_n$ and $P_n : X \to X_n$ are sequences of continuous linear projections. The projection scheme $\Gamma = \{X_n, Y_n, P_n, Q_n\}$ is then said to be admissible for maps from $D \subset X$ to Y.

Definition 2.1 (see [1]). A map $T : D \subset X \to Y$ is called approximation-proper (A-proper) at a point $y \in Y$ with respect to Γ if $T_n \equiv P_n T|_{D\cap X_n}$ is continuous for each $n \in \mathbb{N}$ and whenever

$\{x_{n_j} : x_{n_j} \in D \cap X_{n_j}\}$ is bounded with $T_{n_j} x_{n_j} \to y$, then there exists a subsequence $\{x_{n_{j_k}}\}$ such that $x_{n_{j_k}} \to x \in D$, and $Tx = y$. T is said to be A-proper on a set Ω if it is A-proper at all points of Ω.

Let K be a cone in a finite-dimensional Banach space X, and let $\Omega \subset X$ be open and bounded with $\Omega \cap K = \Omega_K \neq \emptyset$. Let $T : \overline{\Omega}_K \to K$ be continuous such that $Tx \neq x$ on $\partial\Omega_K = \partial\Omega \cap K$, where $\overline{\Omega}_K$ and $\partial\Omega_K$ denote the closure and boundary, respectively, of Ω_K relative K. Let $\rho : X \to K$ be an arbitrary retraction.

The following definition of finite-dimensional index forms the basis of generalized index for A-proper maps $I - T$.

Definition 2.2 (see [1]). One defines

$$i_K(T, \Omega) = \deg_B\left(I - T\rho, \rho^{-1}(\Omega) \cap B_R, 0\right), \tag{2.1}$$

where the degree is the Brouwer degree and B_R is a ball containing Ω_K.

Now let K be a cone in an infinite-dimensional Banach space X with projection scheme Γ such that $Q_n(K) \subseteq K$ for every $n \in \mathbb{N}$. Let $\rho : X \to K$ be an arbitrary retraction and $\Omega \subset X$ an open bounded set such that $\Omega_K = \Omega \cap K \neq \emptyset$. Let $T : \overline{\Omega}_K \to K$ be such that $I - T$ is A-proper at 0. Write $K_n = K \cap X_n = Q_n K$ and $\Omega_n = \Omega_K \cap X_n$. Then $Q_n \rho : X_n \to K_n$ is a finite dimensional retraction.

Definition 2.3 (see [1]). If $Tx \neq x$ on $\partial\Omega_K$, then one defines

$$\operatorname{ind}_K(T, \Omega) = \left\{k \in \mathbb{Z} \cup \{\pm\infty\} : i_{K_{n_j}}\left(Q_{n_j} T, \Omega_{n_j}\right) \longrightarrow k \text{ for some } n_j \longrightarrow \infty\right\}, \tag{2.2}$$

that is, the index is the set of limit points of $i_{K_{n_j}}(Q_{n_j} T, \Omega_{n_j})$, where the finite dimensional index is that defined above.

Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm map of index zero, and let $P : X \to X$, $Q : Y \to Y$ be continuous projectors such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. The restriction of L to $\operatorname{dom} L \cap \operatorname{Ker} P$, denoted L_1, is a bijection onto $\operatorname{Im} L$ with continuous inverse $L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$. Since $\dim \operatorname{Im} Q = \dim \operatorname{Ker} L$, there exists a continuous bijection $J : \operatorname{Im} Q \to \operatorname{Ker} L$. Let K be a cone in an infinite-dimensional Banach space X with projection scheme Γ. If we let $H = L + J^{-1} P$, then $H : \operatorname{dom} L \subset X \to Y$ is a linear bijection with bounded inverse. Thus $K_1 = H(K \cap \operatorname{dom} L)$ is a cone in the Banach space Y.

Let $\Omega \subset X$ be open and bounded with $\Omega_K \cap \operatorname{dom} L \neq \emptyset$, $L : \operatorname{dom} L \subset X \to Y$ a bounded Fredholm operator of index zero, and $N : \overline{\Omega}_K \cap \operatorname{dom} L \to Y$ a bounded continuous nonlinear operator such that $L - N$ is A-proper at 0.

We can now extend the definition of the index to A-proper maps of the form $L - N$ acting on cones.

Definition 2.4 (see [1]). Let ρ_1 be a retraction from Y to K_1, and assume $Q_n K_1 \subset K_1$, $P + JQN + L_1^{-1}(I - Q)N$ maps $K \cap \operatorname{dom} L$ to $K \cap \operatorname{dom} L$ and $Lx \neq Nx$ on $\partial\Omega_K$. One defines the fixed point index of $L - N$ over Ω_K as

$$\operatorname{ind}_K([L, N], \Omega) = \operatorname{ind}_{K_1}(T, U), \tag{2.3}$$

where $U = H(\Omega_K)$, $T : Y \to Y$ is defined as $Ty = (N + J^{-1}P)H^{-1}y$ for each $y \in Y$, and the index on the right is that of Definition 2.3.

For convenience, we recall some properties of ind_K.

Proposition 2.5 (see [1]). *Let $L : \mathrm{dom}\, L \to Y$ be Fredholm of index zero, and let $\Omega \subset X$ be open and bounded. Assume that $P + JQN + L_1^{-1}(I - Q)N$ maps K to K, and $Lx \neq Nx$ on $\partial\Omega_K$. Then one has*

- (P_1) *(existence property) if $\mathrm{ind}_K([L, N], \Omega) \neq \{0\}$, then there exists $x \in \Omega_K$ such that $Lx = Nx$;*
- (P_2) *(normality property) if $x_0 \in \Omega_K$, then $\mathrm{ind}_K([L, -J^{-1}P + \widehat{y}_0], \Omega) = \{1\}$, where $\widehat{y}_0 = Hx_0$ and $\widehat{y}_0(y) = y_0$ for every $y \in H(\Omega_K)$;*
- (P_3) *(additivity property) if $Lx \neq Nx$ for $x \in \overline{\Omega}_K \setminus (\Omega_1 \cup \Omega_2)$, where Ω_1 and Ω_2 are disjoint relatively open subsets of Ω_K, then*

$$\mathrm{ind}_K([L, N], \Omega) \subseteq \mathrm{ind}_K([L, N], \Omega_1) + \mathrm{ind}_K([L, N], \Omega_2) \tag{2.4}$$

with equality if either of indices on the right is a singleton;

- (P_4) *(homotopy invariance property) if $L - N(\lambda, x)$ is an A-proper homotopy on Ω_K for $\lambda \in [0, 1]$ and $(N(\lambda, x) + J^{-1}P)H^{-1} : K_1 \to K_1$ and $\theta \notin (L - N(\lambda, x))(\partial\Omega_K)$ for $\lambda \in [0, 1]$, then $\mathrm{ind}_K([L, N(\lambda, x)], \Omega) = \mathrm{ind}_{K_1}(T_\lambda, U)$ is independent of $\lambda \in [0, 1]$, where $T_\lambda = (N(\lambda, x) + J^{-1}P)H^{-1}$.*

The following two lemmas will be used in this paper.

Lemma 2.6. *If $L : \mathrm{dom}\, L \to Y$ is Fredholm of index zero, Ω is an open bounded set, and $\Omega_K \cap \mathrm{dom}\, L \neq \emptyset$, and let $L - \lambda N$ be A-proper for $\lambda \in [0, 1]$. Assume that N is bounded and $P + JQN + L_1^{-1}(I - Q)N$ maps K to K. If there exists $e \in K_1 \setminus \{\theta\}$, such that*

$$Lx - Nx \neq \mu e, \tag{2.5}$$

for every $x \in \partial\Omega_K$ and all $\mu \geq 0$, then $\mathrm{ind}_K([L, N], \Omega) = \{0\}$.

Proof. Choose a real number l such that

$$l > \sup_{x \in \Omega} \frac{\|Lx - Nx\|}{\|e\|}, \tag{2.6}$$

and define $N(\mu, x) : [0, 1] \times \overline{\Omega}_K \to Y$ by

$$N(\mu, x) = Nx + l\mu e. \tag{2.7}$$

Trivially, $(N(\mu, x) + J^{-1}P)H^{-1} : K_1 \to K_1$ and from (2.5) we obtain

$$Nx + l\mu e \neq Lx, \quad \text{for any } (\mu, x) \in [0, 1] \times \partial\Omega_K. \tag{2.8}$$

Again, by homotopy invariance property in Proposition 2.5, we have

$$\operatorname{ind}_K([L, N(0,x)], \Omega) = \operatorname{ind}_K([L, N], \Omega) = \operatorname{ind}_K([L, N(1,x)], \Omega). \tag{2.9}$$

However,

$$\operatorname{ind}_K([L, N(1,x)], \Omega) = \{0\}. \tag{2.10}$$

In fact, if $\operatorname{ind}_K([L, N(1,x)], \Omega) \neq \{0\}$, the existence property in Proposition 2.5 implies that there exists $x_0 \in \Omega_K$ such that

$$Lx_0 = Nx_0 + le. \tag{2.11}$$

Then

$$l = \frac{\|Lx_0 - Nx_0\|}{\|e\|}, \tag{2.12}$$

which contradicts (2.6). So

$$\operatorname{ind}_K([L, N], \Omega) = \{0\}. \tag{2.13}$$

□

Remark 2.7. The original condition of [1, Theorem 5] was given with $\theta \neq e \in L(K \cap \operatorname{dom} L)$ instead of $e \in K_1 \setminus \{\theta\}$. The modification is necessary since otherwise it cannot guarantee that $(N + \mu e + J^{-1}P)H^{-1} : K_1 \to K_1$.

We assume that there is a continuous bilinear form $[y, x]$ on $Y \times X$ such that $y \in \operatorname{Im} L$ if and only if $[y, x] = 0$ for each $x \in \operatorname{Ker} L$. This condition implies that if $\{x_1, x_2, \dots, x_n\}$ is a basis in $\operatorname{Ker} L$, then the linear map $J : \operatorname{Im} Q \to \operatorname{Ker} L$ defined by $Jy = \beta \sum_{i=1}^{n} [y, x_i] x_i$, $\beta \in \mathbb{R}^+$ is an isomorphism and that if $y = \sum_{i=1}^{n} y_i x_i$, then $[J^{-1}y, x_i] = y_i/\beta$ for $1 \leq i \leq n$ and $[J^{-1}x_0, x_0] > 0$ for $x_0 \in \operatorname{Ker} L$.

In [1], Cremins extended a continuation theorem related to that of Mawhin [31] and Petryshyn [32] for semilinear equations to cones; refer to [1, Corollary 1] for the details. By Lemma 2.6 and [1, Corollary 1], we obtain the following existence theorem of *positive* solutions to a semilinear equation in cones. It is worth mentioning that the positive or nonnegative solutions of an operator equation $Lx = Nx$ were also discussed by a recent paper of O'Regan and Zima [33] and the earlier papers [34–38].

Lemma 2.8. *If $L : \operatorname{dom} L \to Y$ is Fredholm of index zero, $K \subset X$ is a cone, Ω_1 and Ω_2 are open bounded sets such that $\theta \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$ and $\Omega_2 \cap K \cap \operatorname{dom} L \neq \emptyset$. Suppose that $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ with $N : \overline{\Omega_2 \cap K} \to Y$ bounded. Assume that*

(C_1) *$(P + JQN)(K) \subset K$ and $(P + JQN + L_1^{-1}(I - Q)N)(K) \subset K$,*

(C_2) *$Lx \neq \lambda Nx$ for $x \in \partial\Omega_2 \cap K$, $\lambda \in (0,1]$,*

(C_3) *$QNx \neq 0$ for $x \in \partial\Omega_2 \cap K \cap \operatorname{Ker} L$,*

(C_4) $[QNx, x] \leq 0$, *for all* $x \in \partial\Omega_2 \cap K \cap \operatorname{Ker} L$,

(C_5) *there exists* $e \in K_1 \setminus \{\theta\}$, *such that*

$$Lx - Nx \neq \mu e, \quad \text{for every } \mu \geq 0, \; x \in \partial\Omega_1 \cap K. \tag{2.14}$$

Then there exists $x \in \operatorname{dom} L \cap K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ *such that* $Lx = Nx$.

Corollary 2.9. *Assume all conditions of Lemma 2.8 hold except* (C_2) *and assume* $(C_2)'$ $\|Lx - Nx\|^2 \geq \|Nx\|^2 - \|Lx\|^2$ *for each* $x \in \partial\Omega_2 \cap K$. *Then the same conclusion holds.*

Proof. We show that $(C_2)'$ implies (C_2), that is, $Lx \neq \lambda Nx$, for each $x \in \partial\Omega_2 \cap K$, $\lambda \in (0,1]$. Here $\lambda \in [0,1)$. Otherwise, the proof is finished. If $x \in \operatorname{Ker} L \cap \partial\Omega_2 \cap K$, then it follows from $Lx = \lambda Nx = \theta$ that $Lx = Nx$ has a solution in $\operatorname{dom} L \cap K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, and Corollary 2.9 is proved. If $x \in \operatorname{dom} L \setminus \operatorname{Ker} L \cap \partial\Omega_2 \cap K$ and $Lx = \lambda Nx$ for some $\lambda \in (0,1)$, then $Nx = \lambda^{-1} Lx$ and

$$(\lambda - 1)^2 \|Nx\|^2 = \|Lx - Nx\|^2 \geq \|Nx\|^2 - \|Lx\|^2 = \left(1 - \lambda^2\right)\|Nx\|^2, \tag{2.15}$$

by condition $(C_2)'$; that is, $(\lambda - 1)^2 \geq 1 - \lambda^2$, contradicting the fact that $\lambda \in (0,1)$. This completes the proof of Corollary 2.9. □

The following lemma can be found by (a) of [32, Lemma 2].

Lemma 2.10. *Suppose either* N *or* L_1^{-1} *is compact, then* $L - \lambda N$ *is A-proper for* $\lambda \in [0,1]$.

3. Main Results

The goal of this section is to apply Lemma 2.8 to discuss the existence and multiplicity of positive solutions for the PBVP (1.1).

Let $X = \{x \in C[0,1] : x'' \in C[0,1],\ x(0) = x(1),\ x'(0) = x'(1)\}$ endowed with the norm $\|x\|_X = \max_{t\in[0,1]}|x(t)|$, and let $Y = C[0,1]$ with the norm $\|y\|_Y = \max_{t\in[0,1]}|y(t)|$ and $K = \{x \in X : x(t) \geq 0,\ t \in [0,1]\}$, then K is a cone of X.

We define

$$\begin{gathered} \operatorname{dom} L = X, \\ L : \operatorname{dom} L \longrightarrow Y, \qquad Lx(t) = -x''(t), \\ N : X \longrightarrow Y, \qquad Nx(t) = -f(t, x(t)), \end{gathered} \tag{3.1}$$

then PBVP (1.1) can be written as

$$Lx = Nx, \quad x \in K. \tag{3.2}$$

It is easy to check that

$$\begin{aligned}
\operatorname{Ker} L &= \{x \in \operatorname{dom} L : x(t) \equiv c \ \text{ on } [0,1],\ c \in \mathbb{R}\},\\
\operatorname{Im} L &= \left\{ y \in Y : \int_0^1 y(s)ds = 0 \right\},\\
\dim \operatorname{Ker} L &= \operatorname{codim} \operatorname{Im} L = 1,
\end{aligned} \tag{3.3}$$

so that L is a Fredholm operator of index zero.

Next, define the projections $P : X \to X$ by

$$Px = \int_0^1 x(s)ds \tag{3.4}$$

and $Q : Y \to Y$ by

$$Qy = \int_0^1 y(s)ds. \tag{3.5}$$

Furthermore, we define the isomorphism $J : \operatorname{Im} Q \to \operatorname{Im} P$ as $Jy = \beta y$, where $\beta = 1/24$. It is easy to verify that the inverse operator $L_1^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ of $L|_{\operatorname{dom} L \cap \operatorname{Ker} P} : \operatorname{dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ as $(L_1^{-1}y)(t) = \int_0^1 G(s,t)y(s)ds$, where

$$G(s,t) = \begin{cases} \dfrac{s}{2}(1-2t+s), & 0 \le s < t \le 1, \\[2mm] \dfrac{1}{2}(1-s)(2t-s), & 0 \le t \le s \le 1. \end{cases} \tag{3.6}$$

For notational convenience, we set $H(s,t) = 1/24 + G(s,t) - \int_0^1 G(s,t)ds$ or

$$H(s,t) = \begin{cases} \dfrac{1}{24} + \dfrac{s}{2}(1-2t+s) + \dfrac{t^2}{2} - \dfrac{t}{2} + \dfrac{1}{12}, & 0 \le s < t \le 1, \\[2mm] \dfrac{1}{24} + \dfrac{1}{2}(1-s)(2t-s) + \dfrac{t^2}{2} + \dfrac{t}{2} + \dfrac{1}{12}, & 0 \le t \le s \le 1. \end{cases} \tag{3.7}$$

By routine methods of advanced calculus, we get $\max_{s,t\in[0,1]} H(s,t) = 1/8$.

Now we can state and prove our main results.

Theorem 3.1. *Assume that there exist two positive numbers a, b such that*

(H_1) $f(t,x) \le x,\ for\ all\ t \in [0,1],\ x \in [\min\{a,b\}, \max\{a,b\}]$

(H_2) *if one of the two conditions*

(i) $\max_{t\in[0,1]} f(t,a) < 0,\ \min_{t\in[0,1]} f(t,b) > 0$,

(ii) $\min_{t\in[0,1]} f(t,a) > 0,\ \max_{t\in[0,1]} f(t,b) < 0$

is satisfied, then the PBVP (1.1) *has at least one positive solution* $x^* \in K$ *satisfying* $\min\{a,b\} \leq \|x^*\|_X \leq \max\{a,b\}$.

Proof. It is easy to see $a \neq b$. Without loss of generality, let $a < b$.

First, we note that L, as so defined, is Fredholm of index zero, L_1^{-1} is compact by Arzela-Ascoli theorem, and thus $L - \lambda N$ is A-proper for $\lambda \in [0,1]$ by (a) of Lemma 2.10.

For each $x \in K$, then by condition (H_1)

$$\begin{aligned} Px + JQNx &= \int_0^1 x(s)ds - \frac{1}{24}\int_0^1 f(s,x(s))ds \geq \frac{23}{24}\int_0^1 x(s)ds \geq 0, \\ Px + JQNx + L_1^{-1}(I-Q)Nx &= \int_0^1 x(s)ds - \frac{1}{24}\int_0^1 f(s,x(s))ds \\ &\quad + \int_0^1 G(s,t)\left[-f(s,x(s)) + \int_0^1 f(s,x(s))ds\right]ds \\ &= \int_0^1 x(s)ds - \int_0^1 H(s,t)f(s,x(s))ds \\ &\geq \int_0^1 (1 - H(s,t))x(s)ds \geq 0. \end{aligned} \tag{3.8}$$

This implies that condition (C_1) of Lemma 2.8 is satisfied. To apply Lemma 2.8, we should define two open bounded subsets Ω_1, Ω_2 of X so that (C_2)–(C_5) of Lemma 2.8 hold.

We prove only Case (H_2)(i). In the same way, we can prove Case (H_2)(ii).

Let

$$\Omega_1 = \{x \in X : \|x\|_X < a\}, \qquad \Omega_2 = \{x \in X : \|x\|_X < b\}. \tag{3.9}$$

Clearly, Ω_1 and Ω_2 are bounded and open sets and

$$\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2. \tag{3.10}$$

Next we show that (H_2)(i) implies (C_2). For this purpose, suppose that there exist $x_0 \in K \cap \partial\Omega_2$ and $\lambda_0 \in (0,1]$ such that $Lx_0 = \lambda_0 N x_0$ then $x_0''(t) = \lambda_0 f(t, x_0(t))$ for all $t \in [0,1]$. Let $t_0 \in [0,1]$, such that $x_0(t_0) = \max_{t\in[0,1]} x_0(t) = b$. From boundary conditions, we have $t_0 \in [0,1)$. Then we have the following two cases.

Case 1 ($t_0 = 0$). In this case, $x_0'(0) \leq 0$, $x_0'(1) \geq 0$. Since boundary condition $x_0'(0) = x_0'(1)$, we have $x_0'(0) = x_0'(1) = 0$. So we have $x_0''(0) = \lambda_0 f(0, x_0(0)) = \lambda_0 f(0,b) > 0$. It follows from $x_0''(t)$ being continuous in $[0,1]$ that there exists $\delta \in (0,1)$, such that $x_0''(t) > 0$ when $t \in (0,\delta]$. Thus, $x_0'(t) = x_0'(0) + \int_0^t x_0''(s)ds > 0$. Hence,

$$x_0(t) = x_0(0) + \int_0^t x_0'(s)ds > x_0(0), \quad t \in (0,\delta], \tag{3.11}$$

and $x_0(0)$ is not the maximum on $[0,1]$, a contradiction.

Case 2 ($t_0 \in (0,1)$). In this case, $x_0'(t_0) = 0$, $x_0''(t_0) \le 0$. This gives

$$0 \ge x_0''(t_0) = \lambda_0 f(t_0, x(t_0)) = \lambda_0 f(t_0, b) > 0, \tag{3.12}$$

which contradicts (H_2)(i). So for each $x \in \partial\Omega_2 \cap K$ and $\lambda \in (0,1]$, we have $Lx \neq \lambda Nx$. Thus, (C_2) of Lemma 2.8 is satisfied.

To prove (C_4) of Lemma 2.8, we define the bilinear form $[\cdot,\cdot] : Y \times X \to \mathbb{R}$ as

$$[y, x] = \int_0^1 y(t)x(t)dt. \tag{3.13}$$

It is clear that $[\cdot,\cdot]$ is continuous and satisfies $[y, x] = 0$ for every $x \in \operatorname{Ker} L$, $y \in \operatorname{Im} L$. In fact, for any $x \in \operatorname{Ker} L$ and $y \in \operatorname{Im} L$, we have $x \equiv c$, a constant, and there exists $x \in X$ such that $y(t) = -x''(t)$ for each $t \in [0,1]$. By $x'(0) = x'(1)$, we get

$$[y, x] = \int_0^1 y(t)x(t)dt = -c\int_0^1 x''(t)dt = 0. \tag{3.14}$$

Let $x \in \operatorname{Ker} L \cap \partial\Omega_2 \cap K$, then $x(t) \equiv b$, so we have by condition (H_2)(i)

$$\begin{aligned} QNx &= -\int_0^1 f(t, b)dt \neq 0, \\ [QNx, x] &= -\iint_0^1 f(t, b)dt \cdot b\, ds < 0. \end{aligned} \tag{3.15}$$

Thus, (C_3) and (C_4) of Lemma 2.8 are verified.

Finally, we prove (C_5) of Lemma 2.8 is satisfied. We may suppose that $Lx \neq Nx$, for all $x \in \partial\Omega_1 \cap K$. Otherwise, the proof is completed.

Let $e \equiv 1 \in K_1 \setminus \{\theta\}$. We claim that

$$Lx - Nx \neq \mu e, \qquad \forall x \in \partial\Omega_1 \cap K,\ \mu \ge 0. \tag{3.16}$$

In fact, if not, there exist $x_2 \in \partial\Omega_1 \cap K$, $\mu_1 > 0$, such that

$$Lx_2 - Nx_2 = \mu_1. \tag{3.17}$$

Since $QL = \theta$, operating on both sides of the latter equation by Q, we obtain

$$QNx_2 + Q\mu_1 = 0, \tag{3.18}$$

that is,

$$\int_0^1 (-f(t, x_2) + \mu_1)dt = 0. \tag{3.19}$$

For any $x_2 \in \partial\Omega_1 \cap K$, we have $\|x_2\|_X = a$. Then there exists $t_1 \in [0,1]$, such that $x_2(t_1) = a$. By condition (H_2)(i) and $\mu_1 > 0$,

$$\int_0^1 (-f(t, x_2(t_1)) + \mu_1)dt = \int_0^1 (-f(t, a) + \mu_1)dt > 0, \tag{3.20}$$

in contradiction to (3.19). So (3.16) holds; that is, (C_5) of Lemma 2.8 is verified.

Thus, all conditions of Lemma 2.8 are satisfied and there exists $x \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $Lx = Nx$ and the assertion follows. Thus, $x^* \in K$ and $a \le \|x^*\|_X \le b$. □

Let $[c]$ be the integer part of c. The following result concerns the existence of n positive solutions.

Theorem 3.2. *Assume that there exist $n+1$ positive numbers $a_1 < a_2 < \cdots < a_{n+1}$ such that*

$(H_1)'$ $f(t,x) \le x$, *for all* $t \in [0,1]$, $x \in [a_1, a_{n+1}]$,

$(H_2)'$ *if one of the two conditions*

(i) $\max_{t\in[0,1]} f(t, a_{2i-1}) < 0$, $i = 1,2,\ldots,[(n+2)/2]$, $\min_{t\in[0,1]} f(t, a_{2i}) > 0$, $i = 1,2,\ldots,[(n+1)/2]$

(ii) $\min_{t\in[0,1]} f(t, a_{2i-1}) > 0$, $i = 1,2,\ldots,[(n+2)/2]$, $\min_{t\in[0,1]} f(t, a_{2i}) < 0$, $i = 1,2,\ldots,[(n+1)/2]$

is satisfied, then the PBVP (1.1) *has at least n positive solutions $x_i^* \in K$, $i = 1,2,\ldots,n$ satisfying $a_i < \|x_i^*\|_X < a_{i+1}$.*

Proof. Modeling the proof of Theorem 3.1, we can prove that if there exist two positive numbers a, b such that $\max_{t\in[0,1]} f(t,a) < 0$, $\min_{t\in[0,1]} f(t,b) > 0$, then PBVP (1.1) has at least one positive solution $x^* \in K$ satisfying $\min\{a,b\} < \|x^*\|_X < \max\{a,b\}$.

By the claim, for every pair of positive numbers $\{a_i, a_{i+1}\}$, $i = 1,2,\ldots,n$, (1.1) has at least n positive solutions $x_i^* \in K$ satisfying $a_i < \|x_i^*\|_X < a_{i+1}$. □

We have the following existence result for two positive solutions.

Corollary 3.3. *Assume that there exist three positive numbers $a_1 < a_2 < a_3$ such that*

$(H_1)''$ $f(t,x) \le x$, *for all* $t \in [0,1]$, $x \in [a_1, a_3]$,

$(H_2)''$ *if one of the two conditions*

(i) $\max_{t\in[0,1]} f(t, a_1) < 0$, $\min_{t\in[0,1]} f(t, a_2) > 0$, $\max_{t\in[0,1]} f(t, a_3) < 0$,

(ii) $\min_{t\in[0,1]} f(t, a_1) > 0$, $\max_{t\in[0,1]} f(t, a_2) < 0$, $\min_{t\in[0,1]} f(t, a_3) > 0$

is satisfied, then the PBVP (1.1) *has at least two positive solutions $x_1^*, x_2^* \in K$ satisfying $a_1 \le \|x_1^*\|_X < a_2 < \|x_2^*\|_X \le a_3$.*

We also have the following existence result for three positive solutions.

Corollary 3.4. *Assume that there exist four positive numbers $a_1 < a_2 < a_3 < a_4$ such that*

$(H_1)'''$ $f(t,x) \le x$, *for all* $t \in [0,1]$, $x \in [a_1, a_4]$,

$(H_2)'''$ *if one of the two conditions*

(i) $\max_{t\in[0,1]} f(t,a_1) < 0$, $\min_{t\in[0,1]} f(t,a_2) > 0$, $\max_{t\in[0,1]} f(t,a_3) < 0$, $\min_{t\in[0,1]} f(t,a_4) > 0$, *and*

(ii) $\min_{t\in[0,1]} f(t,a_1) > 0$, $\max_{t\in[0,1]} f(t,a_2) < 0$, $\min_{t\in[0,1]} f(t,a_3) > 0$, $\max_{t\in[0,1]} f(t,a_4) < 0$,

is satisfied, then the PBVP (1.1) *has at least three positive solutions* $x_1^*, x_2^*, x_3^* \in K$ *satisfying* $a_1 \le \|x_1^*\|_X < a_2 < \|x_2^*\|_X < a_3 < \|x_3^*\|_X \le a_4$.

Remark 3.5. From similar arguments and techniques, we can also deal with the following periodic boundary value problem (PBVP)

$$\begin{aligned} -x''(t) &= f(t,x), \qquad 0<t<1, \\ x(0) = x(1), &\qquad x'(0) = x'(1). \end{aligned} \tag{3.21}$$

We can also verify that the similar results presented in this paper are valid for PBVP (3.21); we omit the details here.

4. Some Examples

In this section, we give some examples to illustrate the main results of the paper.

Example 4.1. Consider the following second-order periodic boundary value problem (PBVP):

$$\begin{aligned} x''(t) &= \frac{4}{5}\left(t^2-t-1\right)\left(2x^3+3x^2-12x+6\right)x, \qquad 0<t<1, \\ x(0) &= x(1), \qquad x'(0) = x'(1), \end{aligned} \tag{4.1}$$

where $f(t,x(t)) = (t^2-t-1)(2x^3+3x^2-12x+6)x$. In this case, $f(t,x) \le x,\ x \ge 0,\ 0 \le t \le 1$.

Corresponding to the assumptions of Corollary 3.3, we set $a_1 = 1/2,\ a_2 = 1$, and $a_3 = 2$. It is easy to check that the other conditions of Corollary 3.3 are satisfied; hence, PBVP (4.1) has at least two positive solutions x_1^*, x_2^* satisfying $1/2 \le \|x_1^*\|_X < 1 < \|x_2^*\|_X \le 2$.

Example 4.2. Consider the periodic boundary value problem (PBVP)

$$\begin{aligned} x''(t) &= \frac{5t+2}{10}\sin x, \qquad 0<t<1, \\ x(0) = x(1), &\qquad x'(0) = x'(1). \end{aligned} \tag{4.2}$$

Now, let $f(t,x) = ((5t+2)/10)\sin x$; thus, $f(t,x) \le x,\ x \ge 0,\ 0 \le t \le 1$. Set $a_1 = \pi/2,\ a_2 = 3\pi/2,\ a_3 = 5\pi/2,\ a_4 = 7\pi/2$. Then Corollary 3.4 ensures that there exist at least three positive solutions $x_1^*,\ x_2^*,\ x_3^*$ satisfying $\pi/2 \le \|x_1^*\|_X < 3\pi/2 < \|x_2^*\|_X < 5\pi/2 < \|x_3^*\|_X \le 7\pi/2$.

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