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Research Article

A Characterization of Some Function Classes

M. T. Karaev

Isparta Vocational School, Suleyman Demirel University, 32260 Isparta, Turkey

Correspondence should be addressed to M. T. Karaev, mubariztapdigoglu@sdu.edu.tr

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We give in terms of Berezin symbols a characterization of Hardy and Besov classes with a variable exponent.

1. Introduction and Notations

In his book [1, page 96], Pavlović proved the following characterization of functions belonging to the classical Hardy space:

$$H^1 = H^1(\mathbb{D}) := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \|f\|_1 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{it}\right) \right| dt < \infty \right\}, \tag{1.1}$$

where $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc of the complex plain $\mathbb{C}$.

Theorem A. *For a function f analytic in $\mathbb{D}$, the following assertions are equivalent:*

(a) $f \in H^1$;

(b) $\sup_n (1/a_n) \sum_{j=0}^{n} (1/(j+1)) \|s_j(f)\|_1 < \infty$;

(c) $\sup_n \|P_n f\|_1 < \infty$.

Here, $P_n f = (1/a_n) \sum_{j=0}^{n} (1/(j+1)) s_j(f)$, where $a_n = \sum_{j=0}^{n} (1/(j+1))$ $(n = 0, 1, 2, \ldots)$ and $s_j(f)$ are the partial sums of the Taylor series of f.

Recently, Popa [2] gave some generalization of this result by proving a similar characterization of upper triangular trace class matrices.

In the present paper, we give in terms of the so-called Berezin symbols a new characterization of analytic functions belonging to the Hardy class $H^{p(\cdot)}$ and Besov class $B_{p(\cdot)}$ with a variable exponent. Our results are new even for the usual Hardy and Besov spaces H^p and B_p.

Recall that the Hardy space $H^p = H^p(\mathbb{D})$ $(1 \le p < \infty)$ is the collection of holomorphic functions in $\mathbb{D}$ which satisfy the inequality

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{it}\right)\right|^p dt\right)^{1/p} < \infty. \tag{1.2}$$

Let $dA(z)$ be the area measure on $\mathbb{D}$ normalized so that the area of $\mathbb{D}$ is 1. In rectangular and polar coordinates,

$$dA(z) = \frac{1}{\pi} dx\, dy = \frac{1}{\pi} r\, dr\, d\theta. \tag{1.3}$$

For $1 < p < +\infty$, the Besov space $B_p = B_p(\mathbb{D})$ is defined to be the space of analytic functions f in such that

$$\|f\|_{B_p} := \left(\int_{\mathbb{D}} \left(1 - |z|^2\right)^p \left|f'(z)\right|^p d\lambda(z)\right)^{1/p} < \infty, \tag{1.4}$$

where

$$d\lambda(z) = \frac{dA(z)}{\left(1 - |z|^2\right)^2} \tag{1.5}$$

is the Möbius invariant measure on $\mathbb{D}$. We refer to Duren [3] and Zhu [4] for the theory of these spaces.

Let $\mathbb{T} = \partial\mathbb{D}$, and let $p = p(t)$, $t \in \mathbb{T}$, be a bounded, positive, measurable function defined on it. Following by Kokilashvili and Paatashvili [5, 6], we say that the analytic in the disc $\mathbb{D}$ function f belongs to the Hardy class $H^{p(\cdot)}$ if

$$\sup_{0<r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{p(\theta)} d\theta = C < +\infty, \tag{1.6}$$

where $p(\theta) = p(e^{i\theta})$, $\theta \in [0, 2\pi)$.

For $p(\theta) = p = \text{const} > 0$, the $H^{p(\cdot)}$ class coincides with the classical Hardy class H^p.

Analogously, we say that the analytic in $\mathbb{D}$ function f belongs to the Besov class $B_{p(\cdot)}$ with a variable exponent if

$$\int_0^{2\pi} \int_0^1 \left(1 - r^2\right)^{p(t)} \left|f'\left(re^{it}\right)\right|^{p(t)} r\, dr \frac{dt}{\pi} < +\infty, \tag{1.7}$$

where $p(t) = p(e^{it})$, $t \in [0, 2\pi)$.

For $p(t) = p = \text{const} > 0$, the $B_{p(\cdot)}$ class coincides with the Besov class B_p.

Suppose that $\underline{p} := \inf_{t\in\mathbb{T}} p(t)$, $\overline{p} := \sup_{t\in\mathbb{T}} p(t)$. If $\underline{p} > 0$, then it is obvious that

$$\begin{gathered} H^{\overline{p}} \subset H^{p(\cdot)} \subset H^{\underline{p}}, \\ B_{\overline{p}} \subset B_{p(\cdot)} \subset B_{\underline{p}}. \end{gathered} \tag{1.8}$$

Recall that for any bounded linear operator A acting in the functional Hilbert space $H = H(\Omega)$ over some set Ω with reproducing kernel $k_\lambda(z)$, its Berezin symbol $\widetilde{A}$ is defined by

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \quad (\lambda \in \Omega), \tag{1.9}$$

where $\widehat{k}_\lambda := k_\lambda / \|k_\lambda\|$ is the normalized reproducing kernel of H. (We mention [4, 7–11] as references for the Berezin symbols.)

2. On the Membership of Functions in Hardy and Besov Classes with a Variable Exponent

In this section, we characterize the function classes $H^{p(\cdot)}$ and $B_{p(\cdot)}$ in terms of Berezin symbols.

For any bounded sequence $\{a_n\}_{n\geq 0}$ of complex numbers a_n, let $D_{\{a_n\}}$ denote the associate diagonal operator acting in the Hardy space H^2 by the formula

$$D_{\{a_n\}} z^n = a_n z^n, \quad n = 0, 1, 2, \ldots. \tag{2.1}$$

It is known that the reproducing kernel of the Hardy space H^2 has the form $k_\lambda(z) = 1/(1 - \overline{\lambda} z)$ $(\lambda \in \mathbb{D})$. Then, it is easy to show that (see [11])

$$\widetilde{D}_{\{a_n\}}(\lambda) = \left(1 - |\lambda|^2\right) \sum_{k=0}^{\infty} a_n |\lambda|^{2k} \quad (\lambda \in \mathbb{D}), \tag{2.2}$$

that is, the Berezin symbol of the diagonal operator $D_{\{a_n\}}$ on the Hardy space H^2 is a radial function.

Note that the inequality $|\widehat{f}(n)| \leq \text{const}$, $n \geq 0$, is the necessary condition for the function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$ to be in the spaces H^p $(1 \leq p \leq \infty)$. Also note that if $f \in B_p$, then $\widehat{f}(n) = O(n^{-1/p})$ $(p \geq 1)$ (see, for instance, Duren [3] and Zhu [4]).

Our main result is the following theorem.

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n \in \mathrm{Hol}(\mathbb{D})$ be a function with the bounded sequence $\{\widehat{f}(n)\}_{n\geq 0}$ of Taylor coefficients $\widehat{f}(n) = f_{(0)}^{(n)}/n!$ $(n = 0, 1, 2, \ldots)$. Then, the following are true:*

(a) *$f \in H^{p(\cdot)}$ if and only if*

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|\widetilde{D}_{\{\widehat{f}(n)e^{\mathrm{int}}\}}(\sqrt{r})\right|^{p(t)} dt}{(1-r)^{p(t)}} < +\infty; \tag{2.3}$$

(b) *if, in addition,* $\widehat{f}(n) = O(n^{-1})$ *as* $n \to \infty$, *then* $f \in B_{p(\cdot)}$ *if and only if*

$$\int_0^{2\pi}\int_0^1 \left|\widetilde{D}_{\{(n+1)\widehat{f}(n+1)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)\right|^{p(t)} \frac{(1+r)^{p(t)-2}}{(1-r)^2} r\,dr\,dt < +\infty. \tag{2.4}$$

Proof. Indeed, by using the concept of Berezin symbols and formula (2.2), let us rewrite the function $f(z) = \sum_{n=0}^{\infty}\widehat{f}(n)z^n \in \mathrm{Hol}(\mathbb{D})$ as follows:

$$\begin{aligned} f(z) = f\left(re^{it}\right) &= \sum_{n=0}^{\infty}\widehat{f}(n)\left(re^{it}\right)^n = \sum_{n=0}^{\infty}\widehat{f}(n)e^{\mathrm{int}}r^n \\ &= \frac{(1-r)\sum_{n=0}^{\infty}\widehat{f}(n)e^{\mathrm{int}}r^n}{1-r} = \frac{\widetilde{D}_{\{\widehat{f}(n)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)}{1-r}, \end{aligned} \tag{2.5}$$

thus

$$f(z) = \frac{\widetilde{D}_{\{\widehat{f}(n)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)}{1-r} \tag{2.6}$$

for every $z = re^{it} \in \mathbb{D}$, where, as usual, $r = |z|$ and $t = \arg(z)$. Now, assertion (a) is immediate from the definition of considering space and formula (2.6).

Let us prove (b). Indeed, it follows from the condition $\widehat{f}(n) = O(n^{-1})$ that the diagonal operator $D_{\{n\widehat{f}(n)\}}$ is bounded in H^2 (and hence $D_{\{(n+1)\widehat{f}(n+1)e^{\mathrm{int}}\}}$ is bounded for every fixed $t \in [0, 2\pi)$). Then, we have

$$\begin{aligned} f'(z) &= \left(\sum_{n=0}^{\infty}\widehat{f}(n)z^n\right)' = \sum_{n=1}^{\infty}n\widehat{f}(n)z^{n-1} \\ &= \sum_{n=0}^{\infty}(n+1)\widehat{f}(n+1)z^n = \sum_{n=0}^{\infty}(n+1)\widehat{f}(n+1)e^{\mathrm{int}}r^n \\ &= \frac{(1-r)\sum_{n=0}^{\infty}(n+1)\widehat{f}(n+1)e^{\mathrm{int}}r^n}{1-r} \\ &= \frac{\widetilde{D}_{\{(n+1)\widehat{f}(n+1)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)}{1-r}, \end{aligned} \tag{2.7}$$

thus

$$f'(z) = \frac{\widetilde{D}_{\{(n+1)\widehat{f}(n+1)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)}{1-r}. \tag{2.8}$$

Therefore, by using formula (2.8), we have that

$$\int_{\mathbb{D}} \left(1-|z|^2\right)^{p(t)} \left|f'(z)\right|^{p(t)} \frac{dA(z)}{\left(1-|z|^2\right)^2} < +\infty \tag{2.9}$$

if and only if

$$\int_0^{2\pi}\int_0^1 \left|\widetilde{D}_{\{(n+1)\hat{f}(n+1)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)\right|^{p(t)} \frac{\left(1-r^2\right)^{p(t)}}{(1-r)^{p(t)}} \frac{r}{\left(1-r^2\right)^2} dr\, dt < +\infty, \tag{2.10}$$

that is,

$$\int_0^{2\pi}\int_0^1 \left|\widetilde{D}_{\{(n+1)\hat{f}(n+1)e^{\mathrm{int}}\}}\left(\sqrt{r}\right)\right|^{p(t)} \frac{(1+r)^{p(t)-2}}{(1-r)^2} r\, dr\, dt < +\infty, \tag{2.11}$$

as desired. The theorem is proved. □

We remark that, in case of classical Hardy space, Theorem 2.1 shed some light on the following old problem for the Hardy space functions (see Privalov [12] and Duren [3]): how an H^p function can be recognized by the behavior of its Taylor coefficients?

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