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Research Article

Embedding Operators in Vector-Valued Weighted Besov Spaces and Applications

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The embedding theorems in weighted Besov-Lions type spaces $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$ in which E_0, E are two Banach spaces and $E_0 \subset E$ are studied. The most regular class of interpolation space E_α between E_0 and E is found such that the mixed differential operator D^α is bounded from $B^{l,s}_{p,q,\gamma}(\Omega; E_0, E)$ to $B^{s}_{p,q,\gamma}(\Omega; E_\alpha)$ and Ehrling-Nirenberg-Gagliardo type sharp estimates are established. By using these results, the uniform separability of degenerate abstract differential equations with parameters and the maximal B-regularity of Cauchy problem for abstract parabolic equations are obtained. The infinite systems of the degenerate partial differential equations and Cauchy problem for system of parabolic equations are further studied in applications.

1. Introduction

Embedding theorems in function spaces have been elaborated in [1–3]. A comprehensive introduction to the theory of embedding of function spaces and historical references may also be found in [4, 5]. Embedding theorems in abstract function spaces have been studied in [2, 6–18]. The anisotropic Sobolev spaces $W^l_p(\Omega; H_0, H)$, $\Omega \subset R^n$, and corresponding weighted spaces have been investigated in [11, 13–16, 18], respectively. Embedding theorems in Banach-valued Besov spaces have been studied in [6–8, 17, 19]. Moreover, boundary value problems (BVPs) for differential-operator equations (DOEs) have been studied in [4, 5, 20, 21]. The solvability and the spectrum of BVPs for elliptic DOEs have also been refined in [7, 13–18, 22–26]. A comprehensive introduction to the differential-operator equations and historical references may be found in [4, 5]. In these works, Hilbert-valued function spaces essentially have been considered.

Let $l = (l_1, l_2, \ldots, l_n)$ and $s = (s_1, s_2, \ldots, s_n)$. Let E_0 and E be Banach spaces such that E_0 is continuously and densely embedded in E. In the present paper, the weighted

Banach-valued Besov space $B^{l,s}_{p,\theta,\gamma}(\Omega; E_0, E)$ is to be introduced. The smoothest interpolation class E_α between E_0, E (i.e., to find the possible small σ_α for $E_\alpha = (E_0, E)_{\sigma_\alpha,p}$) is found such that the appropriate mixed differential operators D^α are bounded from $B^{l,s}_{p,\theta,\gamma}(\Omega; E_0, E)$ to $B^{s}_{p,\theta,\gamma}(\Omega; E_\alpha)$. By applying these results, the maximal B-regularity of certain classes of anisotropic partial DOE with parameters is derived. The paper is organized as follows. Section 2 collects notations and definitions. Section 3 presents embedding theorems in Besov-Lions space $B^{l,s}_{p,\theta,\gamma}(\Omega; E_0, E)$. Section 4 contains applications of corresponding embedding theorems to vector-valued function spaces, and Section 5 is devoted to applications of these embedding theorems to anisotropic DOE with parameters for which the uniformly maximal B-regularity is obtained. Then, in Section 6, by using these results, the maximal B-regularity of parabolic Cauchy problem is shown. In Section 7, this DOE is applied to BVP and Cauchy problem for infinite systems of quasielliptic and parabolic PDE, respectively.

2. Notations and Definitions

Let E be a Banach space and $\gamma = \gamma(x)$ a nonnegative measurable weighted function defined on a domain $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable E-valued functions that are defined on Ω with the norm

$$\begin{gathered} \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)dx\right)^{1/p}, \quad 1 \le p < \infty, \\ \|f\|_{L_{\infty,\gamma}(\Omega;E)} = \operatorname{ess\,sup}_{x\in\Omega}\left[\|f(x)\|_E \gamma(x)\right]. \end{gathered} \tag{2.1}$$

Let $h \in R$, $m \in N$, and e_i, $i = 1, 2, \ldots, n$ be the standard unit vectors in R^n. Let (see [1, Section 16])

$$\begin{aligned} \Delta_i(h)f(x) &= f(x + he_i) - f(x), \ldots, \Delta_i^m(h)f(x) \\ &= \Delta_i(h)\left[\Delta_i^{m-1}(h)f(x)\right] = \sum_{k=0}^{m}(-1)^{m+k}C_m^k f(x + khe_i). \end{aligned} \tag{2.2}$$

Let

$$\Delta_i^m(\Omega, h) = \left\{ \begin{array}{ll} \Delta_i^m(h), & \text{for } [x, x + mye_i] \subset \Omega, \\ 0, & \text{for } [x, x + mye_i] \subset \dfrac{R^n}{\Omega} \end{array} \right\}. \tag{2.3}$$

Let $L^*_\theta(E)$ be a E-valued function space such that

$$\|u\|_{L^*_\theta(E)} = \left(\int_0^\infty \|u(t)\|_E^\theta \frac{dt}{t}\right)^{1/\theta} < \infty. \tag{2.4}$$

Let m_i be positive integers, k_i nonnegative integers, s_i positive numbers, and $m_i > s_i - k_i > 0$, $i = 1, 2, \ldots, n$, $s = (s_1, s_2, \ldots, s_n)$, $1 \le p \le \infty$, $1 \le \theta \le \infty$, $0 < y_0 < \infty$. Let F denote the Fourier transform. The Banach-valued Besov space $B^s_{p,\theta,\gamma}(\Omega; E)$ is defined as

$$B^s_{p,\theta,\gamma}(\Omega; E) = \left\{ f : f \in L_p(\Omega; E), \|f\|_{B^s_{p,\theta}(\Omega;E)} = \|f\|_{L_{p,\gamma}(\Omega;E)} \right.$$

$$\left. + \sum_{i=1}^{n} \left(\int_0^{h_0} h^{-[(s_i-k_i)q+1]} \left\| \Delta_i^{m_i}(h,\Omega) D_i^{k_i} f \right\|^{\theta}_{L_{p,\gamma}(\Omega;E)} dy \right)^{1/\theta} < \infty \right\}, \quad 1 \le \theta < \infty,$$

$$\|f\|_{B^s_{p,\theta,\gamma}(\Omega;E)} = \sum_{i=1}^{n} \sup_{0<h<h_0} \frac{\left\| \Delta_i^{m_i}(h,\Omega) D_i^{k_i} f \right\|_{L_{p,\gamma}(\Omega;E)}}{h^{s_i-k_i}} \quad \text{for } \theta = \infty. \tag{2.5}$$

For $E = R$ and $\gamma(x) \equiv 1$, we obtain a scalar-valued anisotropic Besov space $B^s_{p,\theta,\gamma}(\Omega)$ [1, Section 18].

Let $\mathbf{C}$ be the set of complex numbers and

$$S_\varphi = \{\lambda;\ \lambda \in \mathbf{C}, |\arg \lambda| \le \varphi\} \cup \{0\}, \quad 0 \le \varphi < \pi. \tag{2.6}$$

A linear operator A is said to be a φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and $\|(A + \lambda I)^{-1}\|_{L(E)} \le M(1 + |\lambda|)^{-1}$ with $\lambda \in S_\varphi$, $\varphi \in [0\ \pi)$, where I is the identity operator in E and $L(E)$ is the space of bounded linear operators in E.

It is known [3, Section 1.15.1] that there exist the fractional powers A^θ of the positive operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with a graph norm defined as

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\| A^\theta u \right\|^p \right)^{1/p}, \quad 1 \le p < \infty,\ -\infty < \theta < \infty. \tag{2.7}$$

The operator $A(t)$ is said to be φ-positive in E uniformly with respect to t with bound $M > 0$ if $D(A(t))$ is independent of t, $D(A(t))$ is dense in E, and $\|(A(t) + \lambda I)^{-1}\| \le M(1 + |\lambda|)^{-1}$ for all $\lambda \in S_\varphi$, $0 \le \varphi < \pi$, where M does not depend on t and λ.

Let $l = (l_1, l_2, \ldots, l_n)$, $s = (s_1, s_2, \ldots, s_n)$, where l_k are positive integers. Let $B^{l,s}_{p,\theta,\gamma}(\Omega; E)$ denote a E-valued weighted Sobolev-Besov space of functions $u \in B^s_{p,\theta,\gamma}(\Omega; E)$ that have generalized derivatives $D_k^{l_k} u = (\partial^{l_k} / \partial x_k^{l_k})\ u \in B^s_{p,\theta,\gamma}(\Omega; E)$, $k = 1, 2, \ldots, n$, with the norm

$$\|u\|_{B^{l,s}_{p,\theta,\gamma}(\Omega;E)} = \|u\|_{B^s_{p,\theta,\gamma}(\Omega;E)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{B^s_{p,\theta,\gamma}(\Omega;E)} < \infty. \tag{2.8}$$

Suppose E_0 is continuously and densely embedded into E. Let $B^{l,s}_{p,\theta,\gamma}(\Omega; E_0, E)$ denote the space with the norm

$$\|u\|_{B^{l,s}_{p,\theta,\gamma}} = \|u\|_{B^{l,s}_{p,\theta,\gamma}(\Omega;E_0,E)} = \|u\|_{B^{s}_{p,\theta,\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \left\| D_k^{l_k} u \right\|_{B^{s}_{p,\theta,\gamma}(\Omega;E)} < \infty. \tag{2.9}$$

Let $t = (t_1, t_2, \ldots, t_n)$, where t_k are parameters. We define the following parameterized norm in $B^{l,s}_{p,\theta}(\Omega; E_0, E)$:

$$\|u\|_{B^{l,s}_{p,\theta,\gamma,t}(\Omega;E_0,E)} = \|u\|_{B^{s}_{p,\theta,\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \left\| t_k D_k^{l_k} u \right\|_{B^{s}_{p,\theta,\gamma}(\Omega;E)} < \infty. \tag{2.10}$$

Let m be a positive integer. $C^{(m)}(\Omega; E)$ denotes the spaces of E-valued bounded and m-times continuously differentiable functions on Ω. For two sequences $\{a_j\}_1^\infty$ and $\{b_j\}_1^\infty$ of positive numbers, the expression $a_j \sim b_j$ means that there exist positive numbers C_1 and C_2 such that

$$C_1 a_j \le b_j \le C_2 a_j. \tag{2.11}$$

Let E_1, and E_2 be two Banach spaces. Let F denote the Fourier transformation and let h be some parameter. We say that the function Ψ_h dependent of h is a uniform collection of multipliers if there exists a positive constant M independent of h such that $\|F^{-1}\Psi_h F u\|_{B^{s}_{q,\theta,\gamma}(R^n;E_2)} \le M\|u\|_{B^{s}_{p,\theta,\gamma}(R^n;E_1)}$ for all $u \in B^{s}_{p,\theta,\gamma}(R^n; E_1)$. The set of all multipliers from $B^{s}_{p,\theta,\gamma}(R^n; E_1)$ to $B^{s}_{q,\theta,\gamma}(R^n; E_2)$ will be denoted by $M^{q,\theta,\gamma}_{p,\theta,\gamma}(E_1, E_2)$. For $E_1 = E_2 = E$, it will be denoted by $M^{q,\theta,\gamma}_{p,\theta,\gamma}(E)$. The exposition of the theory of Fourier multipliers and some related references can be found in [3, Sections 2.2.1–2.2.4]. In weighted L_p spaces, Fourier multipliers have been investigated in several studies like [27, 28]. Operator-valued Fourier multipliers in Banach-valued L_p spaces studied, for example, in [4, 6, 25, 27–33].

Let $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be multi-indexes and

$$\xi^\beta = \xi_1^{\beta_1} \xi_2^{\beta_2}, \ldots, \xi_n^{\beta_n}, \qquad U_n = \{\beta : |\beta| \le n\}, \qquad \eta = \frac{1}{p} - \frac{1}{q}. \tag{2.12}$$

Definition 2.1. A Banach space E satisfies a B-multiplier condition with respect to p, q, θ, s (or with respect to p, θ, s for $p = q$), and the weight γ, when $\Psi \in C^n(R^n; B(E))$, $1 \le p \le q \le \infty$, $\beta \in U_n$ and $\xi \in V_n$, if the estimate $(1 + |\xi|)^{|\beta|+\eta} \|D^\beta \Psi(\xi)\|_{L(E)} \le C$, $k = 0, 1, \ldots, |\beta|$ implies $\Psi \in M^{q,\theta,\gamma}_{p,\theta,\gamma}(E)$.

It is well known (e.g., see [32]) that any Hilbert space satisfies the B-multiplier condition. There are, however, Banach spaces which are not Hilbert spaces but satisfy the

B-multiplier condition (see [7, 30]). However, additional conditions are needed for operator-valued multipliers in L_p spaces, for example, UMD spaces (e.g., see [25, 33]). Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be nonnegative and $l_1, l_2, \ldots, l_n$ positive integers:

$$|\alpha : l| = \sum_{k=1}^{n} \frac{\alpha_k}{l_k}, \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ l = (l_1, l_2, \ldots, l_n),$$
$$D^{\alpha} = D_1^{\alpha_1} D_2^{\alpha_2} \cdots \mathrm{D}_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}, \quad |\alpha| = \sum_{k=!}^{n} \alpha_k. \tag{2.13}$$

Consider the following differential-operator equation:

$$Lu = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + A_\lambda u + \sum_{|\alpha:2l|<1} \alpha(t) A_\alpha(x) D^\alpha u = f, \tag{2.14}$$

where $A(x)$, $A_\alpha(x)$ are linear operators in a Banach space E, a_k are complex-valued functions and t_k are some parameters $\alpha(t) = \prod_{k=1}^{n} t_k^{\alpha_k/2l_k}$. For $l_1 = l_2 = \cdots = l_n = m$, we obtain the elliptic class of DOE.

The function belonging to $B_{p,\theta,\gamma}^{l,s}(R^n; E(A), E)$ and satisfying (2.14) a.e. on R^n is said to be a solution of (2.14) on R^n.

Definition 2.2. The problem (2.14) is said to be uniform weighted B-separable (or weighted $B_{p,\theta,\gamma}^{s}(\Omega; E)$-separable) if, for all $f \in B_{p,\theta,\gamma}^{s}(\Omega; E)$, the problem (2.14) has a unique solution $u \in B_{p,\theta,\gamma}^{l,s}(\Omega; E(A), E)$ and the following estimate holds:

$$\|Au\|_{B_{p,\theta,\gamma}^{s}(\Omega;E)} + \sum_{k=1}^{n} t_k \left\| D_k^{2l_k} u \right\|_{B_{p,\theta,\gamma}^{s}(\Omega;E)} \le C \|f\|_{B_{p,\theta,\gamma}^{s}(\Omega;E)}. \tag{2.15}$$

Consider the following degenerate DOE:

$$Lu = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{[2l_k]} u + A_\lambda u + \sum_{|\alpha:2l|<1} \alpha(t) A_\alpha(x) D^{[\alpha]} u = f, \tag{2.16}$$

where $A(x)$, $A_\alpha(x)$ are linear operators in a Banach space E, a_k are complex-valued functions, t_k are some parameters and

$$D_{x_k}^{[i]} = \left(\gamma(x_k) \frac{\partial}{\partial x_k} \right)^i, \quad k = 1, 2, \ldots, n. \tag{2.17}$$

Remark 2.3. Under the substitution

$$\tau_k = \int_0^{x_k} \gamma^{-1}(y)\, dy, \tag{2.18}$$

$B^s_{p,\theta,\gamma}(R^n;E)$, $B^{[l],s}_{p,\theta,\gamma}(R^n;E(A),E)$ are mapped isomorphically onto the spaces $B^s_{p,\theta,\tilde{\gamma}}(R^n;E)$, $B^{l,s}_{p,\theta,\tilde{\gamma}}(R^n;E(A),E)$, respectively, where

$$\gamma = \prod_{k=1}^{n}\gamma(x_k), \qquad \tilde{\gamma} = \tilde{\gamma}(\tau) = \prod_{k=1}^{n}\gamma(x_k(\tau_k)). \tag{2.19}$$

Moreover, under the substitution (2.18), the degenerate problem (2.16) is mapped to the undegenerate problem (2.14).

3. Embedding Theorems

Let

$$\varkappa = \sum_{k=1}^{n}\frac{\alpha_k + 1/p - 1/q}{l_k}, \qquad \nu(l) = \max_{k,j\in\{1,2,\dots,n\}}\left[\frac{1}{l_k} - \frac{1}{l_j}\right], \qquad \sigma(t) = \prod_{k=1}^{n} t_k^{(\alpha_k+1/p-1/q)/l_k}. \tag{3.1}$$

Theorem 3.1. *Suppose the following conditions hold:*

(1) *E is a Banach space satisfying the B-multiplier condition with respect to p, q, s;*

(2) *$t = (t_1, t_2, \dots, t_n)$, $0 < t_k \le T < \infty$, $k = 1, 2, \dots, n$, $1 < p \le q < \infty$, $\theta \in [1, \infty]$;*

(3) *l_k are positive, α_k nonnegative integers such that $0 < \varkappa + \nu(l) \le 1$ and $0 \le \mu \le 1 - \varkappa - \nu(l)$;*

(4) *A is a φ-positive operator in E.*

Then, the embedding $D^\alpha B^{l,s}_{p,\theta,\gamma}(R^n;E(A),E) \subset B^s_{q,\theta,\gamma}(R^n;E(A^{1-\varkappa-\mu}))$ is continuous, and there exists a constant $C_\mu > 0$, depending only on μ such that

$$\sigma(t)\|D^\alpha u\|_{B^s_{q,\theta,\gamma}(R^n;E(A^{1-\varkappa-\mu}))} \le C_\mu\left[h^\mu\|u\|_{B^{l,s}_{p,\theta,\gamma,t}} + h^{-(1-\mu)}\|u\|_{B^s_{p,\theta,\gamma}(R^n;E)}\right] \tag{3.2}$$

for all $u \in B^{l,s}_{p,\theta,\gamma}(R^n;E(A),E)$ and $0 < h \le h_0 < \infty$.

Proof. Denoting Fu by $\hat{u}$, it is clear that

$$\|D^\alpha u\|_{B^s_{q,\theta,\gamma}(R^n;E(A^{1-\varkappa-\mu}))} \sim \left\|F^{-1}(i\xi)^\alpha A^{1-\varkappa-\mu}\hat{u}\right\|_{B^s_{q,\theta,\gamma}(R^n;E)}. \tag{3.3}$$

Similarly, from the definition of $B^{l,s}_{p,\theta,\gamma}(R^n;E(A),E)$, we have

$$\begin{aligned}\|u\|_{B^{l,s}_{p,\theta,\gamma,t}(R^n;E(A),E)} &= \|u\|_{B^s_{p,\theta,\gamma}(R^n;E(A))} + \sum_{k=1}^{n}\left\|t_k D_k^{l_k} u\right\|_{B^s_{p,\theta,\gamma}(R^n;E)} \\ &\sim \left\|F^{-1}A\hat{u}\right\|_{B^s_{p,\theta,\gamma}(R^n;E)} + \sum_{k=1}^{n}\left\|t_k F^{-1}\left[(i\xi_k)^{l_k}\hat{u}\right]\right\|_{B^s_{p,\theta,\gamma}(R^n;E)}.\end{aligned} \tag{3.4}$$

Thus, proving the inequality (3.2) is equivalent to proving

$$\sigma(t)\left\|F^{-1}\left[(i\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right]\right\|_{X_s} \le h^{\mu}\left\|F^{-1}A\widehat{u}\right\|_{X_s} + h^{\mu}\sum_{k=1}^{n}\left\|t_kF^{-1}\left[(i\xi_k)^{l_k}\widehat{u}\right]\right\|_{X_s} + h^{-(1-\mu)}\|u\|_{X_s}. \quad (3.5)$$

So, the inequality (3.2) will be followed if we prove the following inequality:

$$\sigma(t)\left\|F^{-1}\left[(i\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right]\right\|_{X_s} \le C_{\mu}\left\|F^{-1}\left[h^{\mu}\left(A+\varphi(t,\xi)\right)\right]\widehat{u}\right\|_{X_s} \quad (3.6)$$

for a suitable $C_{\mu} > 0$ and for all $u \in B^{l,s}_{p,\theta,\gamma}(R^n; E(A), E)$, where

$$\varphi = \varphi(t,\xi) = \sum_{k=1}^{n} t_k|\xi_k|^{l_k} + h^{-1}, \qquad X_s = B^{s}_{p,\theta,\gamma}(R^n; E). \quad (3.7)$$

Let us express the left-hand side of (3.6) as

$$\begin{aligned}\sigma(t)&\left\|F^{-1}\left[(i\xi)^{\alpha}A^{1-\varkappa-\mu}\widehat{u}\right]\right\|_{B^{s}_{q,\theta,\gamma}(R^n;E)}\\ &= \sigma(t)\left\|F^{-1}(i\xi)^{\alpha}A^{1-\varkappa-\mu}\left[h^{\mu}\left(A+\varphi\right)\right]^{-1}\left[h^{\mu}\left(A+\varphi\right)\right]\right\|_{B^{s}_{q,\theta,\gamma}(R^n;E)}.\end{aligned} \quad (3.8)$$

(Since A is a positive operator in E and $-\varphi(t,\xi) \in S(\varphi)$, it is possible.) By virtue of Definition 2.1, it is clear that the inequality (3.6) will be followed immediately from (3.8) if we can prove that the operator-function $\Psi_t = \Psi_{t,h,\mu} = \sigma(t)(i\xi)^{\alpha}A^{1-\varkappa-\mu}[h^{\mu}(A+\varphi)]^{-1}$ is a multiplier in $M^{q,\theta,\gamma}_{p,\theta,\gamma}(E)$, which is uniform with respect to h and t. Since E satisfies the multiplier condition with respect to p and q, it suffices to show the following estimate:

$$|\xi|^{k+\eta}\left\|D^{\beta}\Psi_t(\xi)\right\|_{L(E)} \le C, \quad k = 0,1,\ldots,|\beta| \quad (3.9)$$

for all $\beta \in U_n$, $\xi \in R^n/\{\xi_k = 0\}$ and $\eta = 1/p - 1/q$. In a way similar to [18, Lemma 3.1], we obtain that $|\xi|^{-\eta}\|\Psi_t(\xi)\|_{L(E)} \le M_{\mu}$ for all $\xi \in R^n$. This shows that the inequality (3.9) is satisfied for $\beta = (0,\ldots,0)$. We next consider (3.9) for $\beta = (\beta_1,\ldots,\beta_n)$, where $\beta_k = 1$ and $\beta = 0$ for $j \ne k$. By using the condition $\varkappa + \nu(l) \le 1$ and well-known inequality $y_1^{\alpha_1}y_2^{\alpha_2}\cdots y_n^{\alpha_n} \le C(1+\sum_{k=1}^{n} y_k^{l_k})$, $y_k \ge 0$ and by reasoning according to [18, Theorem 3.1], we have

$$|\xi|^{1+\eta}|\xi_k|\|D_k\Psi_t(\xi)\|_{L(E)} \le M_{\mu}, \quad k = 1,2,\ldots,n. \quad (3.10)$$

Repeating the above process, we obtain the estimate (3.9). Thus, the operator-function $\Psi_{t,h,\mu}(\xi)$ is a uniform collection of multiplier, that is, $\Psi_{t,h,\mu} \in \Phi_h \subset M^{q,\theta,\gamma}_{p,\theta,\gamma}(E)$. This completes the proof of the Theorem 3.1. □

It is possible to state Theorem 3.1 in a more general setting. For this, we use the extension operator in $B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E)$.

Condition 1. Let A be a φ-positive operator in Banach spaces E satisfying the B-multiplier condition. Let a region $\Omega \subset R^n$ be such that there exists a bounded linear extension operator from $B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E)$ to $B^{l,s}_{p,\theta,\gamma}(R^n; E(A), E)$, for $1 \le p,\ \theta \le \infty$.

Remark 3.2. If $\Omega \subset R^n$ is a region satisfying a strong l-horn condition (see [23, Section 18]) $E = R,\ A = I$, then there exists a bounded linear extension operator from $B^s_{p,\theta,\gamma}(\Omega) = B^s_{p,\theta,\gamma}(\Omega; R, R)$ to $B^s_{p,\theta,\gamma}(R^n) = B^s_{p,\theta,\gamma}(R^n; R, R)$.

Theorem 3.3. *Suppose all conditions of Theorem 3.1 and Condition 1 are satisfied. Then, the embedding $D^\alpha B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E) \subset B^s_{q,\theta,\gamma}(\Omega; E(A^{1-\varkappa-\mu}))$ is continuous and there exists a constant C_μ depending only on μ such that*

$$\sigma(t)\|D^\alpha u\|_{B^s_{q,\theta,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} \le C_\mu \left[h^\mu \|u\|_{B^{l,s}_{p,\theta,\gamma,t}} + h^{-(1-\mu)}\|u\|_{B^s_{p,\theta,\gamma}(\Omega;E)}\right] \tag{3.11}$$

for all $u \in B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E)$ and $0 < h \le h_0 < \infty$.

Proof. It suffices to prove the estimate (3.11). Let P be a bounded linear extension operator from $B^s_{q,\theta,\gamma}(\Omega; E)$ to $B^s_{q,\theta,\gamma}(R^n; E)$ and also from $B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E)$ to $B^{l,s}_{p,\theta,\gamma}(R^n; E(A), E)$. Let P_Ω be a restriction operator from R^n to Ω. Then, for any $u \in B^{l,s}_{p,\theta}(\Omega; E(A), E)$, we have

$$\begin{aligned}\|D^\alpha u\|_{B^s_{q,\theta,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} &= \|D^\alpha P_\Omega P u\|_{B^s_{q,\theta,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} \\ &\le C\mu\left[h^\mu \|u\|_{B^{l,s}_{p,\theta,\gamma}(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{B^s_{p,\theta,\gamma}(\Omega;E)}\right].\end{aligned} \tag{3.12}$$

□

Result 1. Let all conditions of Theorem 3.3 hold. Then, for all $u \in B^{l,s}_{p,\theta,\gamma}(\Omega; E(A), E)$ we get

$$\|D^\alpha u\|_{B^s_{q,\theta,\gamma}(\Omega;E(A^{1-\varkappa-\mu}))} \le C_\mu \|u\|^{1-\mu}_{B^{l,s}_{p,\theta,\gamma}(\Omega;E(A),E)} \|u\|^{\mu}_{B^s_{p,\theta,\gamma}(\Omega;E)}. \tag{3.13}$$

Indeed, setting $h = \|u\|_{B^s_{p,\theta,\gamma}(\Omega;E)} \cdot \|u\|^{-1}_{B^{l,s}_{p,\theta,\gamma}(\Omega;E(A),E)}$ in (3.13), we obtain (3.11).

Result 2. If $l_1 = l_2 = \cdots = l_n = m$ and $s_1 = s_2 = \cdots = s_n = \sigma$, then we obtain that embedding $D^\alpha B^{m,\sigma}_{p,\theta,\gamma}(\Omega; E(A), E) \subset B^\sigma_{q,\theta,\gamma}(\Omega; E(A^{1-\varkappa}))$ for $\varkappa = |\alpha|/m$ and the corresponding estimate (3.11). For $E = \mathbf{C}$, $A = I$, we obtain the embedding of weighted Besov spaces $D^\alpha B^{l,s}_{p,\theta,\gamma}(\Omega) \subset B^s_{q,\theta,\gamma}(\Omega)$.

4. Application to Vector-Valued Functions

Let $s > 0$, and consider the space [3, Section 1.18.2]

$$l_q^\sigma = \{u; u = \{u_i\}_1^\infty, u_i \in C\}, \qquad \|u\|_{l_q^\sigma} = \left(\sum_{i=1}^{\infty} 2^{iq\sigma}|u_i|^q\right)^{1/q} < \infty. \tag{4.1}$$

Note that $l_q^0 = l_q$. Let A be an infinite matrix defined in l_q such that $D(A) = l_q^\sigma$, $A = [\delta_{ij}2^{\sigma i}]$, where $\delta_{ij} = 0$, when $i \neq j, \delta_{ij} = 1$, when $i = j$, $i, j = 1, 2, \ldots, \infty$. It is clear to see that A is positive in l_q. Then, by Theorem 3.3, we obtain the embedding

$$D^\alpha B_{p_1,\theta,\gamma}^{l,s}\left(\Omega; l_q^\sigma, l_q\right) \subset B_{p_2,\theta,\gamma}^{s}\left(\Omega; l_q^{\sigma(1-\varkappa-\mu)}\right), \qquad \varkappa = \sum_{k=1}^{n} \frac{\alpha_k + 1/p_1 - 1/p_2}{l_k}, \tag{4.2}$$

and the corresponding estimate (3.11), where $0 \le \mu + \nu(l) \le 1 - \varkappa$.

It should be noted that the above embedding has not been obtained with classical methods up to this time.

5. B-Separable DOE in R^n with Parameters

Let us consider the differential-operator equation (2.14). Let

$$X_s = B_{p,\theta,\gamma}^{s}(R^n; E), \qquad Y_s = B_{p,\theta}^{s}(R^n; E). \tag{5.1}$$

Theorem 5.1. *Suppose the following conditions hold:*

(1) $s_k > 0$, $1 < p < \infty$, $1 \le \theta < \infty$, $t_k > 0$, $k = 1, 2, \ldots, n$;

(2) *E is a Banach space satisfying the B-multiplier condition;*

(3) *A is a φ-positive operator in E and*

$$A_\alpha(x)A^{-(1-|\alpha:2l|-\mu)} \in L_\infty(R^n; L(E)), \quad 0 < \mu < 1 - |\alpha : 2l|. \tag{5.2}$$

Then, for all $f \in B_{p,\theta,\gamma}^{s}(R^n; E)$ and for sufficiently large $|\lambda| > 0$, $\lambda \in S(\varphi)$, the equation (2.18) has a unique solution $u(x)$ that belongs to space $B_{p,\theta,\gamma}^{2l,s}(R^n; E(A), E)$ and the following uniform coercive estimate holds:

$$\sum_{k=1}^{n} t_k \left\| D_k^{2l_k} u \right\|_{B_{p,\theta,\gamma}^{s}(R^n;E)} + \|Au\|_{B_{p,\theta,\gamma}^{s}(R^n;E)} \le C\|f\|_{B_{p,\theta,\gamma}^{s}(R^n;E)}. \tag{5.3}$$

Proof. At first, we will consider the principal part of (2.14), that is, the differential-operator equation

$$(L_0 + \lambda)\, u = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + (A + \lambda) u = f. \tag{5.4}$$

Then, by applying the Fourier transform to (5.4), we obtain

$$\sum_{k=1}^{n} t_k \xi_k^{2l_k} \widehat{u}(\xi) + (A+\lambda)\widehat{u}(\xi) = f^{\frown}(\xi). \tag{5.5}$$

Since $\sum_{k=1}^{n} t_k \xi_k^{2l_k} \geq 0$ for all $\xi = (\xi_1, \ldots, \xi_n) \in R^n$, we can say that $\omega = \omega(t, \lambda, \xi) = \lambda + \sum_{k=1}^{n} t_k \xi_k^{2l_k} \in S(\varphi)$ for all $\xi \in R^n$, that is, operator $A + \omega$ is invertible in E. Hence, (5.5) implies that the solution of (5.4) can be represented in the form $u(x) = F^{-1}(A+\omega)^{-1} f^{\frown}$. It is clear to see that the operator-function $\varphi_{\lambda,t}(\xi) = [A+\omega]^{-1}$ is a multiplier in $B^s_{p,\theta,\gamma}(R^n; E)$ uniformly with respect to $\lambda \in S(\varphi)$. Actually, by definition of the positive operator, for all $\xi \in R^n$ and $\lambda \geq 0$, we get

$$\left\|\varphi_\lambda(\xi)\right\|_{L(E)} = \left\|(A+\omega)^{-1}\right\| \leq M(1+|\omega|)^{-1} \leq M_0. \tag{5.6}$$

Moreover, since $D_k \varphi_{\lambda,t}(\xi) = 2l_k t_k (A+\omega)^{-2} \xi_k^{2l_k - 1}$, then $\|\xi_k D_k \varphi_{\lambda,t}\|_{L(E)} \leq M$. By using this estimate for $\beta \in U_n$, we get

$$|\xi|^\beta \left\|D_\xi^\beta \varphi_{\lambda,t}(\xi)\right\|_{L(E)} \leq C. \tag{5.7}$$

In a similar way to Theorem 3.1, we prove that $\varphi_{k,\lambda,t}(\xi) = \xi_k^{2l_k} \varphi_{\lambda,t}$, $k = 1, 2, \ldots, n$, and $\varphi_{0,\lambda,t} = A\varphi_{\lambda,t}$ satisfy the estimates

$$(1+|\xi|)^{|\beta|} \left\|D_\xi^\beta \varphi_{k,\lambda,t}(\xi)\right\|_{L(E)} \leq C, \qquad (1+|\xi|)^{|\beta|} \left\|D_\xi^\beta \varphi_{0,\lambda,t}(\xi)\right\|_{L(E)} \leq C. \tag{5.8}$$

Since the space E satisfies the multiplier condition with respect to p, then, in view of estimates (5.7) and (5.8), we obtain that the operator-functions $\varphi_{\lambda,t}$, $\varphi_{k,\lambda,t}$, $\varphi_{0,\lambda,t}$ are multipliers in $B^s_{p,\theta,\gamma}(R^n; E)$. Then, we obtain that there exists a unique solution of (5.4) for $f \in B^s_{p,\theta,\gamma}(R^n; E)$ and the following estimate holds:

$$\sum_{k=1}^{n} t_k \left\|D_k^{2l_k} u\right\|_{B^s_{p,\theta,\gamma}} + \|Au\|_{B^s_{p,\theta,\gamma}} \leq C\|f\|_{B^s_{p,\theta,\gamma}}. \tag{5.9}$$

Consider now the differential operator G_{0t} generated by problem (5.4), that is,

$$D(G_{0t}) = B^{2l,s}_{p,\theta,\gamma}(R^n; E(A), E), \qquad G_{0t} u = \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u + Au. \tag{5.10}$$

The estimate (5.9) implies that the operator $G_{0t} + \varkappa$ has a bounded inverse from $B^s_{p,\theta,\gamma}(R^n; E)$ into $B^{2l,s}_{p,\theta,\gamma}(R^n; E(A), E)$ for all $\varkappa \geq 0$. Let G_t denote the differential operator in $B^s_{p,\theta,\gamma}(R^n; E)$

generated by problem (2.14). In view of (2.18) condition, by virtue of Theorem 3.1, for all $u \in B^{2l,s}_{p,\theta,\gamma}(R^n; E(A), E)$, we have

$$\begin{aligned}\|L_1 u\|_{B^s_{p,\theta,\gamma}} &\le \sum_{|\alpha:2l|<1} \alpha(t) \left\| A^{1-|\alpha:2l|-\mu} D^\alpha u \right\|_{B^s_{p,\theta,\gamma}} \\ &\le C\left[h^\mu \left(\sum_{k=1}^{n} t_k \left\| D_k^{2l_k} u \right\|_{B^s_{p,\theta,\gamma}} + \|Au\|_{B^s_{p,\theta,\gamma}} \right) + h^{-(1-\mu)} \|u\|_{B^s_{p,\theta,\gamma}} \right].\end{aligned} \tag{5.11}$$

Then, from (5.11), we have

$$\|L_1 u\|_{B^s_{p,\theta,\gamma}} \le C\left[h^\mu \|(G_{0t}+\lambda)u\|_{B^s_{p,\theta,\gamma}} + h^{-(1-\mu)} \|u\|_{B^s_{p,\theta,\gamma}} \right]. \tag{5.12}$$

Since $\|u\|_{B^s_{p,\theta,\gamma}} = (1/\lambda)\|(G_{0t}+\lambda)u - G_{0t}u\|_{B^s_{p,\theta,\gamma}}$ for all $u \in B^{2l,s}_{p,\theta,\gamma}(R^n; E(A), E)$, we get

$$\|u\|_{B^s_{p,\theta,\gamma}} \le \frac{1}{|\lambda|} \left[\|(G_{0t}+\lambda)u\|_{B^s_{p,\theta,\gamma}} + \|G_{0t}u\|_{B^s_{p,\theta,\gamma}} \right]. \tag{5.13}$$

From estimates (5.11)–(5.13), we obtain

$$\|L_1 u\|_{X_s} \le Ch^\mu \|(G_{0t}+\lambda)u\|_{X_s} + C_1 |\lambda|^{-1} h^{-(1-\mu)} \|(G_{0t}+\lambda)u\|_{X_s}. \tag{5.14}$$

Then, by choosing h and λ, such that $Ch^\mu < 1$, $C_1|\lambda|^{-1}h^{-(1-\mu)} < 1$ from (5.14), we get the following uniform estimate:

$$\left\| L_1 (G_{0t}+\lambda)^{-1} \right\|_{L(E)} < 1. \tag{5.15}$$

Then, using the estimates of (5.9), (5.15) and the perturbation theory of linear operators, we obtain that the operator $G_t + \lambda$ is invertible from $B^s_{p,\theta,\gamma}(R^n, E)$ into $B^{2l,s}_{p,\theta,\gamma}(R^n; E(A), E)$. This implies the estimate (5.3). □

Result 1. Let all conditions of Theorem 5.1 hold. Then,

(1) for $f \in B^s_{p,\theta}(R^n, E)$, $\lambda \in S(\varphi)$, (2.16) has a unique solution $u \in B^{[2l],s}_{p,\theta,\gamma}(R^n; E(A), E)$ and

$$\sum_{k=1}^{n} t_k \left\| D_k^{[2l_k]} u \right\|_{B^s_{p,\theta}(R^n;E)} + \|Au\|_{B^s_{p,\theta}(R^n;E)} \le C\|f\|_{B^s_{p,\theta}(R^n;E)}, \tag{5.16}$$

(2) the operator Q_t has a resolvent $(Q_t+\lambda)^{-1}$ for $|\arg\lambda| \le \varphi$, and the following uniform estimate holds:

$$\sum_{|\alpha:2l|\le 1} \alpha(t)|\lambda|^{1-|\alpha:2l|} \left\| D^{[\alpha]}(Q_t+\lambda)^{-1} \right\|_{L(Y_s)} + \left\| A(Q_t+\lambda)^{-1} \right\|_{L(Y_s)} \le C. \tag{5.17}$$

Remark 5.2. Result 1 implies that operator Q_t is uniformly positive in $B^s_{p,\theta}(R^n;E)$. Then, by virtue of [3, Section 1.14.5], the operator Q_t is a generator of an analytic semigroup in $B^s_{p,\theta}(R^n;E)$.

6. Cauchy Problem for Degenerate Parabolic DOE with Parameters

Consider the Cauchy problem for the degenerate parabolic DOE

$$\frac{\partial u}{\partial y} + \sum_{k=1}^{n}(-1)^{l_k} t_k D_k^{[2l_k]} u + Au + \sum_{|\alpha:2l|<1} \alpha(t) A_\alpha(x) D^{[\alpha]} u = f(y,x), \qquad u(0,x) = 0, \tag{6.1}$$

where A and $A_\alpha(x)$ are linear operators in a Banach space in E. Let $F = B^s_{p,\theta}(R^n;E)$.

Theorem 6.1. *Assume all conditions of Theorem 5.1 hold for $\varphi \in (\pi/2,\pi)$ and $s > 0$. Then, for $f \in B^s_{p,\theta}(R_+;F)$, (6.1) has a unique solution $u \in B^1_{p,q}(R_+;D(Q_t),F)$ satisfying*

$$\left\| \frac{\partial u}{\partial y} \right\|_{B^s_{p,\theta}(R_+;F)} + \sum_{k=1}^{n} \left\| t_k D_k^{[2l_k]} u \right\|_{B^s_{p,\theta}(R_+;F)} + \|Au\|_{B^s_{p,\theta}(R_+;F)} \le C\|f\|_{B^s_{p,\theta}(R_+;F)}. \tag{6.2}$$

Proof. The problem (6.1) can be expressed as

$$\frac{du}{dy} + Q_t u(y) = f(y), \quad u(0) = 0,\ y \in (0,\infty). \tag{6.3}$$

Result 1 implies the uniform positivity of G_t. So, by [6, Application D], we obtain that, for $f \in B^s_{p,\theta}(R_+;F)$, the Cauchy problem (6.3) has a unique solution $u \in B^{1+s}_{p,\theta}(R_+;D(Q_t),F)$ satisfying

$$\|D_t u\|_{B^s_{p,\theta}(R_+;F)} + \|Q_t u\|_{B^s_{p,\theta}(R_+;F)} \le C\|f\|_{B^s_{p,q}(R_+;F)}. \tag{6.4}$$

In view of Result 1, the operator Q_t is uniform separable in F; therefore, the estimate (6.4) implies (6.2). □

7. Infinite Systems of the Quasielliptic Equation

Consider the following infinity systems:

$$\begin{aligned}(L+\lambda)u_m &= \sum_{k=1}^{n}(-1)^{l_k} t_k D_k^{2l_k} u_m(x) + (d_m+\lambda)u_m(x) + \sum_{|\alpha:l|<1} \alpha(t) \sum_{k=1}^{\infty} d_{\alpha k m}(x) D^\alpha u_m \\ &= f_m(x), \quad x \in R^n,\ m = 1,2,\ldots,\infty.\end{aligned} \tag{7.1}$$

Let

$$\begin{gathered} d(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \quad du = \{d_m u_m\}, \\ Q(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \quad Qu = \{d_m u_m\}, \\ l_q(Q) == \left\{ u : u \in l_q, \|u\|_{l_q(Q)} = \|Qu\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{1/q} < \infty \right\}, \end{gathered} \tag{7.2}$$

and let t_k be positive parameters. Let O_t denote the differential operator in $B = L(B^s_{p,\theta,\gamma}(R^n; l_q))$ generated by problem (7.1).

Theorem 7.1. *Let $a_\alpha \in C_b(R^n)$, $d_m \in C_b(R^n)$, $d_{\alpha k m} \in L_\infty(R^n)$ such that $\max_\alpha \sup_m \sum_{k=1}^{\infty} d_{\alpha k m}(x) d_k^{-(1-|\alpha:l|-\mu)} < M$ for all $x \in R^n$, $p, q \in (1, \infty)$, $\theta \in [1, \infty]$ and $0 < \mu < 1 - |\alpha : l|$.*

Then,

(a) *for all $f(x) = \{f_m(x)\}_1^\infty \in B^s_{p,\theta,\gamma}(R^n; l_q)$, for $|\arg \lambda| \le \varphi$ and for sufficiently large $|\lambda|$, the problem (7.1) has a unique solution $u = \{u_m(x)\}_1^\infty$ that belongs to space $B^{s+2l}_{p,\theta,\gamma}(R^n, l_q(d), l_q)$ and the uniform coercive estimate holds*

$$\sum_{|\alpha:2l| \le 1} \|D^\alpha u\|_{B^s_{p,\theta,\gamma}(R^n;l_q)} + \|Qu\|_{B^s_{p,\theta,\gamma}(R^n;l_q)} \le C \|f\|_{B^s_{p,\theta,\gamma}(R^n;l_q)}, \tag{7.3}$$

(b) *for $|\arg \lambda| \le \varphi$ and for sufficiently large $|\lambda|$, there exists a resolvent $(O_t + \lambda)^{-1}$ of operator O_t and*

$$\sum_{|\alpha:2l| \le 1} \alpha(t) |\lambda|^{1-|\alpha:l|} \left\| D^\alpha (O_t + \lambda)^{-1} \right\|_B + \left\| Q (O_t + \lambda)^{-1} \right\|_B \le M. \tag{7.4}$$

Proof. Really, let $E = l_q$, $A(x)$, and let $A_\alpha(x)$ be infinite matrices such that

$$A = [d_m(x) \delta_{km}], \qquad A_\alpha(x) = [d_{\alpha k m}(x)], \quad k, m = 1, 2, \ldots, \infty. \tag{7.5}$$

It is clear that the operator A is positive in l_q. Therefore, from Theorem 6.1, we obtain that the problem (7.1) has a unique solution $u \in B^{s+2l}_{p,\theta,\gamma}(R^n; l_q(Q), l_q)$ for all $f \in B^s_{p,\theta,\gamma}(R^n; l_q)$, $|\arg \lambda| \le \varphi$, sufficiently large $|\lambda|$ and estimate (7.3) holds. From estimate (7.3), we obtain (7.4). □

8. Cauchy Problem for Infinite Systems of Parabolic Equations

Consider the following infinity systems of parabolic Cauchy problem

$$\begin{gathered} \frac{\partial u_m}{\partial y} + \sum_{k=1}^{n} (-1)^{l_k} t_k D_k^{2l_k} u_m + d_m(x) u_m + \sum_{|\alpha:l|<1} \alpha(t) d_{\alpha k m}(x) D^\alpha u_m \\ = f_m(y, x), \quad y \in R_+, x \in R^n, \ u_m(0, x) = 0, \ m = 1, 2, \ldots, \infty. \end{gathered} \tag{8.1}$$

Theorem 8.1. *Let all conditions of Theorem 7.1 hold. Then, the parabolic systems* (8.1) *for sufficiently large* $\varkappa > 0$ *have a unique solution* $u \in B^{1,s+2l}_{p,\theta,\gamma}(R^n; l_q(Q), l_q)$, *and the following estimate holds:*

$$\left\| \frac{\partial u}{\partial y} \right\|_{B^s_{p,\theta}(R^{n+1}_+;l_q)} + \sum_{k=1}^{n} \left\| t_k D_k^{[2l_k]} u \right\|_{B^s_{p,\theta}(R^{n+1}_+;l_q)} + \|Qu\|_{B^s_{p,\theta}(R^{n+1}_+;l_q)} \leq C \|f\|_{B^s_{p,\theta}(R^{n+1}_+;l_q)}. \tag{8.2}$$

Proof. Really, let $E = l_q$, and let A and $A_\alpha(x)$ be infinite matrices, such that

$$A = [d_m(x)\delta_{km}], \qquad A_\alpha(x) = [d_{\alpha km}(x)], \quad k, m = 1, 2, \ldots \infty. \tag{8.3}$$

Then, the problem (8.1) can be expressed in the form (6.3), where

$$A = [d_m(x)\delta_{km}], \qquad A_\alpha(x) = [d_{\alpha km}(x)], \quad k, m = 1, 2, \ldots \infty. \tag{8.4}$$

Then, by virtue of Theorems 6.1 and 7.1, we obtain the assertion. □

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