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Research Article

A Note on Noneffective Weights in Variable Lebesgue Spaces

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We study noneffective weights in the framework of variable exponent Lebesgue spaces, and we show that $L^{p(\cdot)}(\Omega) = L^{p(\cdot)}_{\omega}(\Omega)$ if and only if $\omega(x)^{1/p(x)} \sim$ constant in the set where $p(\cdot) < \infty$, and $\omega(x) \sim$ constant in the set where $p(\cdot) = \infty$.

1. Introduction

The variable Lebesgue spaces generalize the classical Lebesgue spaces L_p, where the constant exponent p is replaced by a function $p(\cdot)$. They fall within the scope of Musielak-Orlicz spaces (see [1]) hence the general theory implies their basic properties. Nevertheless, they have appeared as an individual subject of study, also in other areas in recent years. One sees their formal similarity with the standard L_p spaces hence natural questions arise about their various common properties; further, interesting applications of them have been found in mathematical modelling of some physical processes. All this has triggered study of the variable exponent Lebesgue spaces oriented and related to the theory of integrable and weakly differentiable functions. Note in passing that at the very beginning one hits on a quite unpleasant fact that standard L_p techniques cannot be applied here in a straightforward manner; for example, the shift operator is never bounded, with the only exception when the exponent is constant. For more information we refer the reader to the surveys by Diening et al. [2], and Samko [3].

In this note we study the noneffective weights in the framework of variable exponent Lebesgue spaces. Our point of departure is the paper by Hudzik and Krbec [4], where the noneffective weights have been introduced and explored in the framework of Orlicz spaces $L_\Phi(w)$. It has been proved that (we omit here notation and details) if a Young function Φ satisfies the Δ_2 condition at infinity, a weight w is noneffective (i.e., $L_\Phi(w) = L_\Phi$) if and only if $w \sim$ constant; moreover, the fact that noneffective weights are exactly the trivial ones (i.e., the weights equivalent to a constant) may happen also if the Δ_2 condition at infinity is not satisfied, and a necessary and sufficient condition on Φ for the nonexistence of essentially unbounded noneffective weights is given.

To state our results, we first give some basic definitions. For information on the basic properties of the variable exponent Lebesgue spaces, see the recent book by Diening et al. [5], or the pioneering papers [6, 7].

Given a set $\Omega \subset \mathbb{R}^n$, an exponent function is a measurable function $p(\cdot) : \Omega \to [1, \infty]$. We denote the set of all such functions by $\mathcal{P}(\Omega)$. Given $p(\cdot) \in \mathcal{P}(\Omega)$, let $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$, and for any $E \subset \Omega$, let

$$p_-(E) = \operatorname*{ess\,inf}_{x\in E} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x\in E} p(x). \tag{1.1}$$

For brevity, we denote $p_- = p_-(\Omega)$ and $p_+ = p_+(\Omega)$.

Given a weight w in Ω (i.e., $w : \Omega \to]0, \infty[$, $w \in L^1_{\mathrm{loc}}(\Omega)$), we define the modular functional

$$\rho_w(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)} w(x)dx + \|f(w)\|_{L^\infty(\Omega_\infty)}. \tag{1.2}$$

If $|\Omega_\infty| = 0$, then we set the last term equal to 0; if $|\Omega \setminus \Omega_\infty| = 0$, then $\rho_w(f) = \|fw\|_{L^\infty(\Omega_\infty)}$. We define the space $L^{p(\cdot)}_w(\Omega)$ to be the set of measurable functions such that, for some $\lambda > 0$, $\rho_w(f/\lambda) < \infty$. This is a Banach function space when equipped with the norm

$$\|f\|_{p(\cdot),w} = \inf\left\{\lambda > 0 : \rho_w\left(\frac{f}{\lambda}\right) \le 1\right\}. \tag{1.3}$$

When $p(\cdot) = p$, a constant, then $L^{p(\cdot)}_w(\Omega) = L^p_w(\Omega)$ with equality of norms. When $w \equiv 1$, we will denote by $\rho(\cdot)$ the modular functional, $\|f\|_{p(\cdot)}$ the norm, and $L^{p(\cdot)}(\Omega)$ the space.

2. Noneffective Weights for Variable Exponent Lebesgue Spaces

We begin by stating the following general embedding theorem, which is a consequence of a classical result by Ishii [8].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$, $p_1(\cdot)$, $p_2(\cdot)$ be exponents in $\mathcal{P}(\Omega)$, $(p_1)_+ < \infty$, $(p_2)_+ < \infty$, w_1, w_2, weights. The continuous embedding $L^{p_1(\cdot)}_{w_1}(\Omega) \subset L^{p_2(\cdot)}_{w_2}(\Omega)$ holds if and only if there exist positive constants K_1, K_2, and $h \in L^1(\Omega)$, such that*

$$t^{p_2(x)} w_2(x) \le K_1 (K_2 t)^{p_1(\cdot)} w_1(x) + h(x) \quad \text{for a.a. } x \in \Omega \text{ and all } t > 0. \tag{2.1}$$

At first we consider the case $p_+ < \infty$, where it is possible to show that the noneffective weights must be the trivial ones.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$, and let w be weight. If $p_+ < \infty$, then $L^{p(\cdot)}(\Omega) = L_w^{p(\cdot)}(\Omega)$ if and only if $w \sim$ constant.*

Proof. If $w \sim$ constant, then obviously $L^{p(\cdot)}(\Omega) = L_w^{p(\cdot)}(\Omega)$, therefore we need to prove only the converse implication.

Since, in particular, $L_w^{p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$, then by Theorem 2.1 and the boundedness of $p(\cdot)$ there exist $c_1 > 0$ and $h_1 \in L^1(\Omega)$ such that

$$t^{p(x)} \leq c_1 t^{p(\cdot)} w(x) + h_1(x) \quad \text{for a.a. } x \in \Omega \text{ and all } t > 0, \tag{2.2}$$

that is,

$$1 \leq c_1 w(x) + \frac{h_1(x)}{t^{p(x)}} \quad \text{for a.a. } x \in \Omega \text{ and all } t > 0, \tag{2.3}$$

from which, letting $t \to \infty$,

$$\frac{1}{c_1} \leq w(x) \quad \text{for a.a. } x \in \Omega. \tag{2.4}$$

The upper bound follows, in the same way, from the opposite inclusion. □

We consider now the case of unbounded exponents.

Let us first remark that, if $p \equiv \infty$, as in the previous case, all noneffective weights must be trivial.

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^n$, and let w be weight. It is $L^\infty(\Omega) = L_w^\infty(\Omega)$ if and only if $w \sim$ constant.*

Proof. As above, we show only that, if $L_w^\infty(\Omega) = L^\infty(\Omega)$, then $w \sim$ constant. Arguing by contradiction, it is $w \notin L^\infty(\Omega)$ or $w^{-1} \notin L^\infty(\Omega)$. In the first case, set

$$E_n = |\{x \in \Omega : w(x) > n\}|, \quad n \in \mathbb{N}, \tag{2.5}$$

so that $|E_n| > 0$ for all n. Testing the inequality $\|fw\|_\infty \leq c_2 \|f\|_\infty$ with $f = \chi_{E_n}$, we get $n < \|w\|_{L^\infty(E_n)} \leq c_2$ for all n, which is absurd. In the second case we argue similarly, and the proposition is proved. □

Proposition 2.3 is a special case of the following more general result, which in fact includes also Theorem 2.2. We will see (Theorem 2.5) that the assumption $p_+(\Omega \setminus \Omega_\infty) < \infty$ is in fact optimal.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$, and let w be weight. If $p_+(\Omega \setminus \Omega_\infty) < \infty$, then $L^{p(\cdot)}(\Omega) = L_w^{p(\cdot)}(\Omega)$ if and only if $w \sim$ constant.*

Theorem 2.4 is a consequence of the following result, where in $\Omega \setminus \Omega_\infty$ (possibly empty) an unbounded exponent is allowed.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$, $p(\cdot) \in \mathcal{P}(\Omega)$, and let w be weight. It is $L^{p(\cdot)}(\Omega) = L_w^{p(\cdot)}(\Omega)$ if and only if*

$$w(x)^{1/p(x)} \sim \text{constant} \quad \text{for a.a. } x \in \Omega \setminus \Omega_\infty, \qquad w \sim \text{constant} \quad \text{for a.a. } x \in \Omega_\infty. \tag{2.6}$$

Proof. If $|\Omega \setminus \Omega_\infty| = 0$, then the theorem follows from Proposition 2.3, therefore we may assume that $|\Omega \setminus \Omega_\infty| > 0$. We now argue as in the proof of Theorem 2.2. Since $L_w^{p(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$, it is also $L_w^{p(\cdot)}(\Omega \setminus \Omega_\infty) \subset L^{p(\cdot)}(\Omega \setminus \Omega_\infty)$, and therefore, applying Theorem 2.1, there exist positive constants K_1, K_2, and $h_1 \in L^1(\Omega \setminus \Omega_\infty)$, such that

$$t^{p(x)} \leq K_1 (K_2 t)^{p(\cdot)} w(x) + h_1(x) \quad \text{for a.a. } x \in \Omega \setminus \Omega_\infty \text{ and all } t > 0, \tag{2.7}$$

and arguing similarly as before we get $K_1^{-1} K_2^{-p(x)} \leq w(x)$, from which we get the existence of a constant K_0 such that $K_0^{-p(x)-1} \leq w(x)$. Starting from the opposite inclusion, we get the existence of a constant K_3 such that $w(x) \leq K_3^{p(x)+1}$. In conclusion,

$$w(x)^{1/(p(x)+1)} \sim \text{constant} \quad \text{for a.a. } x \in \Omega \setminus \Omega_\infty, \tag{2.8}$$

but this is equivalent to say that

$$w(x)^{1/p(x)} \sim \text{constant} \quad \text{for a.a. } x \in \Omega \setminus \Omega_\infty \tag{2.9}$$

because

$$\frac{p(x)}{p(x)+1} \sim \text{constant} \quad \text{for a.a. } x \in \Omega \setminus \Omega_\infty. \tag{2.10}$$

If $|\Omega_\infty| > 0$, the second part of the statement, about Ω_∞, follows analogously, from the fact that $L_w^{p(\cdot)}(\Omega_\infty) = L^{p(\cdot)}(\Omega_\infty)$ and then using Proposition 2.3.Vice versa, if (2.6) is true, the functional

$$\rho_w(f) = \int_{\Omega \setminus \Omega_\infty} \left| f(x) w(x)^{1/p(x)} \right|^{p(x)} dx + \|fw\|_{L^\infty(\Omega_\infty)} \tag{2.11}$$

can be majorized and minorized with $\rho(\lambda f)$, where $\lambda > 0$ comes from assumption (2.6), and this implies the equivalence of the norms of $L^{p(\cdot)}(\Omega)$ and $L_w^{p(\cdot)}(\Omega)$, and therefore the theorem is proved. □

Example 2.6. Let $\Omega = (0,1) \subset \mathbb{R}$, $p(x) = 1/x$, $w(x) = 2^{1/x}$. Then w is unbounded, and, since $w(x)^{1/p(x)} = 2$, it is also noneffective:

$$L_w^{p(\cdot)}(0,1) = \mathrm{L}^{p(\cdot)}(0,1). \tag{2.12}$$

This can be checked also directly, in fact

$$\rho_w(f) = \int_0^1 |f(x)|^{p(x)}\, w(x)dx = \int_0^1 |f(x)|^{1/x} 2^{1/x} dx = \rho(2f), \tag{2.13}$$

and therefore $\|f\|_{p(\cdot),w} = 2\|f\|_{p(\cdot)}$.

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