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Research Article

Pointwise Approximation of Functions from $L^p(w)_\beta$ by Linear Operators of Their Fourier Series

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We show the results, corresponding to theorem of Lal (2009), on the rate of pointwise approximation of functions from the pointwise integral Lipschitz classes by matrix summability means of their Fourier series as well as the theorems on norm approximations.

1. Introduction

Let L^p $(1 \le p < \infty)$ be the class of all 2π-periodic real-valued functions integrable in the Lebesgue sense with pth power over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \left(\int_Q |f(t)|^p dt\right)^{1/p}, \tag{1.1}$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_o(f)}{2} + \sum_{\nu=1}^{\infty} \left(a_\nu(f)\cos \nu x + b_\nu(f)\sin \nu x\right), \tag{1.2}$$

and conjugate one

$$\widetilde{S}f(x) := \sum_{\nu=1}^{\infty} \left(b_\nu(f)\cos \nu x - a_\nu(f)\sin \nu x\right) \tag{1.3}$$

with the partial sums $S_k f$ and $\widetilde{S}_k f$, respectively. We know that if $f \in L$, then

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^{\pi} \varphi_x(t)\frac{1}{2}\cot\frac{t}{2}dt = \lim_{\epsilon\to 0}\widetilde{f}(x,\epsilon), \tag{1.4}$$

where

$$\widetilde{f}(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi} \varphi_x(t)\frac{1}{2}\cot\frac{t}{2}dt \tag{1.5}$$

with

$$\varphi_x(t) := f(x+t) - f(x-t) \tag{1.6}$$

exists for almost all x [1, Th. (3.1)IV].

Let $A := (a_{n,k})$ be an infinite lower triangular matrix of real numbers such that

$$\begin{gathered} a_{n,k} \geq 0 \quad \text{when } k = 0,1,2,\ldots,n, \qquad a_{n,k} = 0 \quad \text{when } k > n, \\ \sum_{k=0}^{n} a_{n,k} = 1, \quad \text{where } n = 0,1,2,\ldots, \end{gathered} \tag{1.7}$$

and let the A-transformations of $(S_k f)$ and $(\widetilde{S}_k f)$ be given by

$$\begin{aligned} T_{n,A}f(x) &:= \sum_{k=0}^{n} a_{n,k} S_k f(x) \quad (n = 0,1,2,\ldots), \\ \widetilde{T}_{n,A}f(x) &:= \sum_{k=0}^{n} a_{n,k} \widetilde{S}_k f(x) \quad (n = 0,1,2,\ldots), \end{aligned} \tag{1.8}$$

respectively. Denote, for $m = 0,1,2,\ldots,n$,

$$A_{n,m} = \sum_{k=0}^{m} a_{n,k}, \qquad \overline{A}_{n,m} = \sum_{k=m}^{n} a_{n,k}. \tag{1.9}$$

We define two classes of sequences (see [2]).

A sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Rest Bounded Variation Sequence, or briefly $c \in$ RBVS, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c)c_m \tag{1.10}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

A sequence $c := (c_n)$ of nonnegative numbers will be called the Head Bounded Variation Sequence, or briefly $c \in$ HBVS, if it has the property

$$\sum_{k=0}^{m-1} |c_k - c_{k+1}| \leq K(c) c_m \tag{1.11}$$

for all natural numbers m, or only for all $m \leq N$ if the sequence c has only finite nonzero terms and the last nonzero terms is c_N.

Now, we define another class of sequences.

Followed by Leindler [3], a sequence $c := (c_n)$ of nonnegative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $c \in$ MRBVS, if it has the property

$$\sum_{k=m}^{\infty} |c_k - c_{k+1}| \leq K(c) \frac{1}{m+1} \sum_{k \geq m/2}^{m} c_k \tag{1.12}$$

for all natural numbers m, where $K(c)$ is a constant depending only on c.

Analogously, a sequence $c := (c_k)$ of nonnegative numbers will be called the Mean Head Bounded Variation Sequence, or briefly $c \in$ MHBVS, if it has the property

$$\sum_{k=0}^{n-m-1} |c_k - c_{k+1}| \leq K(c) \frac{1}{m+1} \sum_{k=n-m}^{n} c_k, \tag{1.13}$$

for all positive integer $m < n$, where the sequence c has only finite nonzero terms and the last nonzero term is c_n and where $K(c)$ is a constant depending only on c.

It is clear that (see [4])

$$\text{RBVS} \subsetneq \text{MRBVS}, \qquad \text{HBVS} \subsetneq \text{MHBVS}. \tag{1.14}$$

Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0 \leq K(\alpha_n) \leq K \tag{1.15}$$

holds for all n, where $K(\alpha_n)$ denotes the sequence of constants appearing in the inequalities (1.12) or (1.13) for the sequences $\alpha_n = (a_{n,k})_{k=0}^{n}$.

Now, we can give the conditions to be used later on. We assume that, for all n and $0 \leq m < n$,

$$\begin{aligned} \sum_{k=m}^{n-1} |a_{n,k} - a_{n,k+1}| &\leq K \frac{1}{m+1} \sum_{k \geq m/2}^{m} a_{n,k}, \\ \sum_{k=0}^{n-m-1} |a_{n,k} - a_{n,k+1}| &\leq K \frac{1}{m+1} \sum_{k=n-m}^{n} a_{n,k}, \end{aligned} \tag{1.16}$$

where K is the same as above, hold if $\alpha_n = (a_{n,k})_{k=0}^{n}$ belong to MRBVS or MHBVS, for $n = 0, 1, 2, \ldots$, respectively.

As a measure of approximation of functions by the above means, we use the generalized pointwise moduli of continuity of f in the space L^p defined for $\beta \geq 0$ by the formulas

$$\begin{aligned} \tilde{w}_{x,\beta} f(\delta)_{L^p} &:= \left\{ \frac{1}{\delta^{1+\beta p}} \int_0^{\delta} \left(|\varphi_x(t)| \left| \sin \frac{t}{2} \right|^{\beta} \right)^p dt \right\}^{1/p}, \\ w_{x,\beta} f(\delta)_{L^p} &:= \left\{ \frac{1}{\delta^{1+\beta p}} \int_0^{\delta} \left(|\varphi_x(t)| \left| \sin \frac{t}{2} \right|^{\beta} \right)^p dt \right\}^{1/p}, \end{aligned} \tag{1.17}$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x). \tag{1.18}$$

It is clear that, for $\beta > \alpha \geq 0$,

$$w_{x,\beta} f(\delta)_{L^p} \leq w_{x,\alpha} f(\delta)_{L^p}, \qquad \tilde{w}_{x,\beta} f(\delta)_{L^p} \leq \tilde{w}_{x,\alpha} f(\delta)_{L^p}. \tag{1.19}$$

It is easily seen that $w_{x,0} f(\cdot)_{L^p} = w_x f(\cdot)_{L^p}$ and $\tilde{w}_{x,0} f(\cdot)_{L^p} = \tilde{w}_x f(\cdot)_{L^p}$ are the classical pointwise moduli of continuity.

The deviation $T_{n,A} f - f$ with special form of matrix A was estimated in the norm of L^p by Lal [5, Theorem 2, page 347] as follows.

Theorem A. *If*

$$f \in L_{\beta}^{p}(\omega) = \left\{ f \in L^p : \omega f(\delta)_{L_{\beta}^{p}} := \sup_{0 \leq |t| \leq \delta} \left\{ \int_0^{\pi} |\varphi_x(t)|^p \left| \sin \frac{x}{2} \right|^{\beta p} dx \right\}^{1/p} \leq \omega(\delta) \right\}, \tag{1.20}$$

$$\frac{\omega(t)}{t} \text{ is a decreasing function of } t, \tag{1.21}$$

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{t|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} t \, dt \right\}^{1/p} = O\left((n+1)^{-1} \right), \tag{1.22}$$

$$\left\{ \int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\gamma}|\varphi_x(t)|}{\omega(t)} \right)^p \sin^{\beta p} t \, dt \right\}^{1/p} = O\left((n+1)^{\gamma} \right) \quad \left(0 < \gamma < \frac{1}{p} \right), \tag{1.23}$$

then

$$\left\| \frac{1}{n+1} \sum_{\nu=0}^{n} \frac{1}{P_\nu} \sum_{k=0}^{\nu} p_{\nu-k} S_k f - f \right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} \omega\left(\frac{1}{n+1} \right) \right), \tag{1.24}$$

where $P_n = \sum_{\nu=0}^{n} p_\nu, (p_\nu)$ *is a nonnegative and nonincreasing sequence, and the function* ω *of modulus of continuity type will be defined in the next section.*

In this note we show that the conditions (1.21), (1.22), and (1.23) are superfluous when we use the pointwise modulus of continuity.

In our theorems, we will consider the pointwise deviations $T_{n,A}f - f, \widetilde{T}_{n,A}f - \widetilde{f}, \widetilde{T}_{n,A}f - \widetilde{f}(\cdot,\ 2\pi/(n+2))$ with the matrix whose rows belong to the classes of sequences MRBVS and MHBVS. Consequently, we also give some results on norm approximation.

We will write $I_1 \ll I_2$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_1 \le K I_2$.

2. Statement of the Results

Let us define, for a fixed x, a function w_x (or $\widetilde{w}_x$) of modulus of continuity type on the interval $[0, 2\pi]$, that is, a nondecreasing continuous function having the following properties:

$$\begin{gathered} w_x(0) = 0, \qquad w_x(\delta_1 + \delta_2) \le w_x(\delta_1) + w_x(\delta_2), \\ (\text{or } \widetilde{w}_x(0) = 0,\ \widetilde{w}_x(\delta_1 + \delta_2) \le \widetilde{w}_x(\delta_1) + \widetilde{w}_x(\delta_2)), \end{gathered} \tag{2.1}$$

for any $0 \le \delta_1 \le \delta_2 \le \delta_1 + \delta_2 \le 2\pi$. It is easy to conclude that the function $\delta^{-1} w_x(\delta)$ nondecreases in δ. Let

$$\begin{aligned} L^p(w_x)_\beta &= \left\{ f \in L^p : w_{x,\beta} f(\delta)_{L^p} \le w_x(\delta) \right\}, \\ L^p(\widetilde{w}_x)_\beta &= \left\{ f \in L^p : \widetilde{w}_{x,\beta} f(\delta)_{L^p} \le \widetilde{w}_x(\delta) \right\}, \\ L^p(w)_\beta &= \left\{ f \in L^p : \left\| w_{\cdot,\beta} f(\delta)_{L^p} \right\|_{L^p} \le w(\delta) \right\}, \\ L^p(\widetilde{w})_\beta &= \left\{ f \in L^p : \left\| \widetilde{w}_{\cdot,\beta} f(\delta)_{L^p} \right\|_{L^p} \le \widetilde{w}(\delta) \right\}, \end{aligned} \tag{2.2}$$

where $w_x, \widetilde{w}_x \widetilde{w}$, and w are also the functions of modulus of continuity type. It is clear that, for $\beta > \alpha \ge 0$,

$$\begin{aligned} L^p(w_x)_\alpha &\subset L^p(w_x)_\beta, \qquad L^p(w)_\alpha \subset L^p(w)_\beta, \\ L^p(\widetilde{w}_x)_\alpha &\subset L^p(\widetilde{w}_x)_\beta, \qquad L^p(\widetilde{w})_\alpha \subset L^p(\widetilde{w})_\beta. \end{aligned} \tag{2.3}$$

Now, we can formulate our main results on the degrees of pointwise summability.

Theorem 2.1. *Let* $f \in L^p(w_x)_\beta$ *with* $\beta < 1 - (1/p)$. *If* $(a_{n,k})_{k=0}^{n} \in$ MRBVS *is such that* $A_{n,\tau} = O(\tau/(n+1))$, *where* $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, *then*

$$\left| T_{n,A} f(x) - f(x) \right| = O\left((n+1)^{\beta + (1/p)} w_x\left(\frac{\pi}{n+1} \right) \right), \tag{2.4}$$

for considered x.

Theorem 2.2. *Let $f \in L^p(w_x)_\beta$ with $\beta < 1 - (1/p)$. If $(a_{n,k})_{k=0}^n \in$ MHBVS is such that $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left|T_{n,A}f(x) - f(x)\right| = O\left((n+1)^{\beta+(1/p)} w_x\left(\frac{\pi}{n+1}\right)\right), \tag{2.5}$$

for considered x.

Theorem 2.3. *Let $f \in L^p(\widetilde{w}_x)_\beta$ with $\beta < 2 - (1/p)$. If $(a_{n,k})_{k=0}^n \in$ MRBVS is such that $A_{n,\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{2\pi}{n+2}\right)\right| = O\left((n+1)^{\beta+(1/p)} \widetilde{w}_x\left(\frac{\pi}{n+1}\right)\right), \tag{2.6}$$

for considered x.

Theorem 2.4. *Let $f \in L^p(\widetilde{w}_x)_\beta$ with $\beta < 2 - (1/p)$. If $(a_{n,k})_{k=0}^n \in$ MHBVS is such that $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{2\pi}{n+2}\right)\right| = O\left((n+1)^{\beta+(1/p)} \widetilde{w}_x\left(\frac{\pi}{n+1}\right)\right), \tag{2.7}$$

for considered x.

Theorem 2.5. *Let $f \in L^p(\widetilde{w}_x)_\beta$, and*

$$\left\{\int_0^{2\pi/(n+2)} \frac{1}{t}\left(\frac{\widetilde{w}_x(t)}{\sin^\beta(t/2)}\right)^q dt\right\}^{1/q} = O\left((n+1)^\beta \widetilde{w}_x\left(\frac{\pi}{n+1}\right)\right) \tag{2.8}$$

holds with $q = p(p-1)^{-1}$ when $\beta > 0$ or with $q = 1$ when $\beta = 0$. If $(a_{n,k})_{k=0}^n \in$ MRBVS is such that $A_{n,\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O\left((n+1)^{\beta+(1/p)} \widetilde{w}_x\left(\frac{\pi}{n+1}\right)\right), \tag{2.9}$$

for considered x such that $\widetilde{f}(x)$ exists.

Theorem 2.6. *Let $f \in L^p(\widetilde{w}_x)_\beta$, and (2.8) holds with $q = p(p-1)^{-1}$ when $\beta > 0$ or with $q = 1$ when $\beta = 0$. If $(a_{n,k})_{k=0}^n \in$ MHBVS and $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}(x)\right| = O\left((n+1)^{\beta+(1/p)} \widetilde{w}_x\left(\frac{\pi}{n+1}\right)\right), \tag{2.10}$$

for considered x such that $\widetilde{f}(x)$ exists.

Consequently, we formulate the results on norm approximation.

Theorem 2.7. *Let $f \in L^p(w)_\beta$ with $\beta < 1-(1/p)$, $(a_{n,k})_{k=0}^n \in$ MRBVS is such that $A_{n,\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left\|T_{n,A}f(\cdot) - f(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} w\left(\frac{\pi}{n+1}\right)\right). \tag{2.11}$$

Theorem 2.8. *Let $f \in L^p(w)_\beta$ with $\beta < 1-(1/p)$, $(a_{n,k})_{k=0}^n \in$ MHBVS is such that $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left\|T_{n,A}f(\cdot) - f(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} w\left(\frac{\pi}{n+1}\right)\right). \tag{2.12}$$

Theorem 2.9. *Let $f \in L^p(\tilde{w})_\beta$ with $\beta < 1-(1/p)$, $(a_{n,k})_{k=0}^n \in$ MRBVS is such that $A_{n,\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left\|\tilde{T}_{n,A}f(\cdot) - \tilde{f}\left(\cdot, \frac{2\pi}{n+2}\right)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} \tilde{w}\left(\frac{\pi}{n+1}\right)\right). \tag{2.13}$$

Theorem 2.10. *Let $f \in L^p(\tilde{w})_\beta$ with $\beta < 1-(1/p)$, $(a_{n,k})_{k=0}^n \in$ MHBVS is such that $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left\|\tilde{T}_{n,A}f(\cdot) - \tilde{f}\left(\cdot, \frac{2\pi}{n+2}\right)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} \tilde{w}\left(\frac{\pi}{n+1}\right)\right). \tag{2.14}$$

Theorem 2.11. *Let $f \in L^p(\tilde{w})_\beta$, and*

$$\left\{\int_0^{2\pi/(n+2)} \frac{1}{t}\left(\frac{\tilde{w}(t)}{\sin^\beta(t/2)}\right)^q dt\right\}^{1/q} = O\left((n+1)^\beta \tilde{w}\left(\frac{\pi}{n+1}\right)\right) \tag{2.15}$$

holds with $q = p(p-1)^{-1}$. If $(a_{n,k})_{k=0}^n \in$ MRBVS and $A_{n,\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then

$$\left\|\tilde{T}_{n,A}f(\cdot) - \tilde{f}(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} \tilde{w}\left(\frac{\pi}{n+1}\right)\right). \tag{2.16}$$

Theorem 2.12. *Let $f \in L^p(\tilde{w}_x)_\beta$, and (2.15) holds with $q = p(p-1)^{-1}$. If $(a_{n,k})_{k=0}^n \in$ MHBVS is such that $\overline{A}_{n,n-2\tau} = O(\tau/(n+1))$, where $\tau = [\pi/t](2\pi/(n+2) \le t \le \pi)$, then*

$$\left\|\tilde{T}_{n,A}f(\cdot) - \tilde{f}(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta+(1/p)} \tilde{w}\left(\frac{\pi}{n+1}\right)\right). \tag{2.17}$$

Remark 2.13. In the case $p \ge 1$ (specially if $p = 1$), we can suppose that the expression $t^{-\beta} w_x(t)$ nondecreases in t instead of the assumption $\beta < 1-(1/p)$.

Remark 2.14. Under additional assumptions $a_{nn} = O(1/n)$, $\beta = 0$, and $w_x(t) = O(t^\alpha)(0 < \alpha \leq 1)$, the degree of approximation in Theorem 2.1 is following $O(n^{(1/p)-\alpha})$. The same degree of approximation we obtain in Theorem 2.2 under the assumption $a_{n0} = O(1/n)$. In case of the remaining theorems, we can use the same remarks.

Remark 2.15. If we consider the classical modulus of continuity $\omega_0 f(\delta)_{L^p}$ of the function f instead of the modulus w, then the condition $\|w_{\cdot,\beta} f(\delta)_{L^p}\|_{L^p} \leq \omega_0 f(\delta)_{L^p}$ holds for every function $f \in L^p$ and thus $L^p(w)_\beta = L^p$. The same remark for conjugate functions holds too.

Remark 2.16. Our theorems will be also true if we consider function f from $L^p_\beta(\omega)$ with the following norm:

$$\|f\|_{L^p_\beta} := \|f(\cdot)\|_{L^p_\beta} = \left(\int_Q |f(t)|^p \left|\sin\frac{t}{2}\right|^{\beta p} dt\right)^{1/p}. \tag{2.18}$$

Remark 2.17. We can observe that, taking $a_{nk} = 1/(n+1)\sum_{\nu=k}^n p_{\nu-k}/P_\nu$, we obtain the mean considered by Lal [5], and if (p_ν) is monotonic with respect to ν, then $(a_{n,k})_{k=0}^n \in$ MRBVS. Therefore, if (p_ν) is a nonincreasing sequence such that $P_\tau \sum_{\nu=\tau}^n P_\nu^{-1} = O(\tau)$, then, from Theorem 2.1, we obtain the corrected form of the result of Lal [5] (i.e., with condition $\{\int_0^{\pi/(n+1)} (|\varphi_x(t)|/\omega(t))^p \sin^{\beta p} t\, dt\}^{1/p} = O_x((n+1)^{-1/p})$ instead of (1.22)).

3. Auxiliary Results

We begin this section by some notations following Zygmund [1].

It is clear that

$$\begin{aligned}
\widetilde{S}_k f(x) &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\widetilde{D}_k(t)dt,\\
S_k f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_k(t)dt,\\
\widetilde{T}_{n,A} f(x) &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^n a_{n,k}\widetilde{D_k}(t)dt,\\
T_{n,A} f(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^n a_{n,k}D_k(t)dt,
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
\widetilde{D_k}(t) &= \sum_{\nu=0}^k \sin\nu t = \frac{\cos(t/2) - \cos((2k+1)t/2)}{2\sin(t/2)},\\
D_k(t) &= \frac{1}{2} + \sum_{\nu=1}^k \cos\nu t = \frac{\sin((2k+1)t/2)}{2\sin(t/2)}.
\end{aligned} \tag{3.2}$$

Hence,

$$\begin{aligned}\widetilde{T}_{n,A}f(x)-\widetilde{f}\left(x,\frac{2\pi}{n+2}\right)&=-\frac{1}{\pi}\int_0^{2\pi/(n+2)}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k}(t)dt\\&\quad+\frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^\circ}(t)dt,\\\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)&=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D_k^\circ}(t)dt,\end{aligned}\tag{3.3}$$

where

$$\begin{aligned}\widetilde{D_k^\circ}(t)&=\frac{\cos((2k+1)t/2)}{2\sin(t/2)},\\T_{n,A}f(x)-f(x)&=\frac{1}{\pi}\int_0^{\pi}\varphi_x(t)\sum_{k=0}^{n}a_{n,k}D_k(t)dt.\end{aligned}\tag{3.4}$$

Now, we formulate some estimates for the conjugate Dirichlet kernel.

Lemma 3.1 (see [1]). *If* $0<|t|\le\pi/2$, *then*

$$\left|\widetilde{D_k^\circ}(t)\right|\le\frac{\pi}{2|t|},\qquad\left|\widetilde{D_k}(t)\right|\le\frac{\pi}{|t|},\tag{3.5}$$

and, for any real t, *we have*

$$\left|\widetilde{D_k}(t)\right|\le\frac{1}{2}k(k+1)|t|,\qquad\left|\widetilde{D_k}(t)\right|\le k+1.\tag{3.6}$$

Lemma 3.2 (cf. [2, 6]). *If* $(a_{n,k})_{k=0}^{n}\in$ MHBVS, *then*

$$\left|\sum_{k=0}^{n}a_{n,k}D_k(t)\right|=O\left(t^{-1}\overline{A}_{n,n-2\tau}\right),\tag{3.7}$$

and if $(a_{n,k})_{k=0}^{n}\in$ MRBVS, *then*

$$\left|\sum_{k=0}^{n}a_{n,k}D_k(t)\right|=O\left(t^{-1}A_{n,\tau}\right),\tag{3.8}$$

for $2\pi/n\le t\le\pi$ $(n=2,3,\ldots)$, *where* $\tau=[\pi/t]$.

Lemma 3.3 (see [7]). *If* $(a_{n,k})_{k=0}^{n} \in$ MHBVS, *then*

$$\left|\sum_{k=0}^{n} a_{n,k}\widetilde{D}_k^{\circ}(t)\right| = O\left(t^{-1}\overline{A}_{n,n-2\tau}\right), \tag{3.9}$$

and if $(a_{n,k})_{k=0}^{n} \in$ MRBVS, *then*

$$\left|\sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_k^{\circ}(t)\right| = O\left(t^{-1}A_{n,\tau}\right), \tag{3.10}$$

for $2\pi/n \le t \le \pi$ $(n = 2, 3, \ldots)$, *where* $\tau = [\pi/t]$.

4. Proofs of the Results

4.1. Proof of Theorem 2.1

As usual,

$$\begin{aligned} T_{n,A}f(x) - f(x) &= \frac{1}{\pi}\int_0^{2\pi/(n+2)} \varphi_x(t)\sum_{k=0}^{\infty} a_{n,k}D_k(t)dt \\ &\quad + \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi} \varphi_x(t)\sum_{k=0}^{\infty} a_{n,k}D_k(t)dt = I_1 + I_2, \\ \left|T_{n,A}f(x) - f(x)\right| &\le |I_1| + |I_2|. \end{aligned} \tag{4.1}$$

By the Hölder inequality $((1/p) + (1/q) = 1)$ and Lemma 3.1, for $\beta < 1 - (1/p)$,

$$\begin{aligned} |I_1| &\le \frac{(n+1)}{\pi}\int_0^{2\pi/(n+2)} \left|\varphi_x(t)\right| dt \\ &\le \frac{(n+1)}{\pi}\left\{\int_0^{2\pi/(n+2)} \left[\left|\varphi_x(t)\right|\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{1/p}\left\{\int_0^{2\pi/(n+2)} \left[\frac{1}{\sin^{\beta}(t/2)}\right]^q dt\right\}^{1/q} \\ &\ll (n+1)^{1-\beta-(1/p)} w_x\left(\frac{2\pi}{n+2}\right)\left\{\int_0^{2\pi/(n+2)} \left[\frac{1}{t^{\beta}}\right]^q dt\right\}^{1/q} \\ &\ll w_x\left(\frac{2\pi}{n+2}\right) \le 2w_x\left(\frac{\pi}{n+1}\right). \end{aligned} \tag{4.2}$$

Using Lemma 3.2 and the Hölder inequality $((1/p)+(1/q)=1)$,

$$
\begin{aligned}
|I_2| &\ll \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\frac{|\varphi_x(t)|}{t}A_{n,\tau}\,dt \ll \frac{1}{\pi(n+1)}\int_{2\pi/(n+2)}^{\pi}\frac{|\varphi_x(t)|}{t^2}dt \\
&\le \frac{1}{\pi(n+1)}\int_{\pi/(n+2)}^{\pi}\frac{|\varphi_x(t)|\sin^{\beta}(t/2)}{t^2\sin^{\beta}(t/2)}dt \le \frac{\pi^{\beta-1}}{(n+1)}\int_{\pi/(n+1)}^{\pi}\frac{|\varphi_x(t)|\sin^{\beta}(t/2)}{t^{2+\beta}}dt \\
&= \frac{\pi^{\beta-1}}{n+1}\left\{\left[\frac{1}{t^{2+\beta}}\int_0^t|\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]_{t=\pi/(n+1)}^{\pi}+(2+\beta)\int_{\pi/(n+1)}^{\pi}\frac{\int_0^t|\varphi_x(u)|\sin^{\beta}(u/2)du}{t^{3+\beta}}dt\right\} \\
&\le \frac{1}{\pi^3(n+1)}\int_0^{\pi}|\varphi_x(u)|\sin^{\beta}\frac{u}{2}du \\
&\quad+\frac{\pi^{\beta-1}(2+\beta)}{n+1}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{t^{-1}\int_0^t|\varphi_x(u)|\sin^{\beta}(u/2)du}{w_x(t)}\right]^p dt\right\}^{1/p} \\
&\quad\cdot\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{w_x(t)}{t^{2+\beta}}\right]^q dt\right\}^{1/q} \\
&\le \frac{1}{\pi^2(n+1)}\left\{\frac{1}{\pi}\int_0^{\pi}\left(|\varphi_x(u)|\sin^{\beta}\frac{u}{2}\right)^p du\right\}^{1/p} \\
&\quad+\frac{\pi^{\beta-1}(2+\beta)}{n+1}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{\left\{t^{-1}\int_0^t\left(|\varphi_x(u)|\sin^{\beta}(u/2)\right)^p du\right\}^{1/p}}{w_x(t)}\right]^p dt\right\}^{1/p} \\
&\quad\cdot\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{w_x(t)}{t^{2+\beta}}\right]^q dt\right\}^{1/q} \\
&\ll \frac{1}{n+1}w_x(\pi)+\frac{1}{n+1}\left\{\int_{\pi/(n+1)}^{\pi}t^{\beta p}dt\right\}^{1/p}\frac{w_x(\pi/(n+1))}{\pi/(n+1)}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{1}{t^{1+\beta}}\right]^q dt\right\}^{1/q}.
\end{aligned}
\tag{4.3}
$$

Since $\beta+(1/p)>0$, we have

$$
\begin{aligned}
|I_2| &\ll \frac{1}{n+1}w_x(\pi)+\left\{\left[\frac{t^{(-1-\beta)q+1}}{(-1-\beta)q+1}\right]_{t=\pi/(n+1)}^{\pi}\right\}^{1/q}w_x\left(\frac{\pi}{n+1}\right) \\
&\ll \frac{1}{n+1}w_x(\pi)+(n+1)^{\beta+(1/p)}w_x\left(\frac{\pi}{n+1}\right) \ll (n+1)^{\beta+(1/p)}w_x\left(\frac{\pi}{n+1}\right).
\end{aligned}
\tag{4.4}
$$

Collecting these estimates, we obtain the desired result.

4.2. Proof of Theorem 2.2

The proof is the same as the proof of Theorem 2.1, the only difference is that, in place of $A_{n,\tau}$, we have to write the quantity $\overline{A}_{n,n-2\tau}$ for which we suppose the same order.

4.3. Proof of Theorem 2.3

We start with the obvious relations

$$\begin{aligned}
\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{2\pi}{n+1}\right) &= -\frac{1}{\pi}\int_0^{2\pi/(n+2)} \varphi_x(t)\sum_{k=0}^{n} a_{nk}\widetilde{D_k}(t)dt \\
&\quad + \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi} \varphi_x(t)\sum_{k=0}^{n} a_{nk}\widetilde{D_k^\circ}(t) = \widetilde{I}_1 + \widetilde{I}_2^\circ, \\
\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}\left(x, \frac{2\pi}{n+2}\right)\right| &\le \left|\widetilde{I}_1\right| + \left|\widetilde{I}_2^\circ\right|.
\end{aligned} \tag{4.5}$$

By the Hölder inequality $((1/p) + (1/q) = 1)$ and Lemma 3.1, we have

$$\begin{aligned}
\left|\widetilde{I}_1\right| &\le (n+1)^2 \int_0^{2\pi/(n+2)} t\left|\varphi_x(t)\right|dt \\
&\le (n+1)^2 \left\{\int_0^{2\pi/(n+2)} \left[\left|\varphi_x(t)\right|\sin^\beta\frac{t}{2}\right]^p dt\right\}^{1/p} \left\{\int_0^{2\pi/(n+2)} \left[\frac{t}{\sin^\beta(t/2)}\right]^q dt\right\}^{1/q} \\
&\ll (n+1)^{2-\beta-(1/p)}\widetilde{w}_x\left(\frac{2\pi}{n+2}\right)\left\{\int_0^{2\pi/(n+2)} \left[t^{1-\beta}\right]^q dt\right\}^{1/q} \ll \widetilde{w}_x\left(\frac{2\pi}{n+2}\right) \ll \widetilde{w}_x\left(\frac{\pi}{n+1}\right)
\end{aligned} \tag{4.6}$$

for $\beta < 2 - (1/p)$.

Using Lemma 3.3 and the Hölder inequality $((1/p) + (1/q) = 1)$, we get

$$\begin{aligned}
\left|\widetilde{I}_2^\circ\right| &\ll \frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi} \frac{\left|\varphi_x(t)\right|}{t} A_{n,\tau}\, dt \ll \frac{1}{\pi(n+1)}\int_{2\pi/(n+2)}^{\pi} \frac{\left|\varphi_x(t)\right|}{t^2}dt \\
&\le \frac{1}{\pi(n+1)}\int_{\pi/(n+1)}^{\pi} \frac{\left|\varphi_x(t)\right|}{t^2}dt \\
&\le \frac{\pi^{\beta-1}}{n+1}\int_{\pi/(n+1)}^{\pi} \frac{\left|\varphi_x(t)\right|\sin^\beta(t/2)}{t^{2+\beta}}dt \le \frac{1}{\pi^3(n+1)}\int_0^{\pi} \left|\varphi_x(u)\right|\sin^\beta\frac{t}{2}du
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\pi^{\beta-1}(2+\beta)}{n+1}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{t^{-1}\int_0^t|\psi_x(u)|\sin^{\beta}(u/2)du}{\widetilde{w}_x(t)}\right]^p dt\right\}^{1/p}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{\widetilde{w}_x(t)}{t^{2+\beta}}\right]^q dt\right\}^{1/q}\\
&\ll \frac{1}{\pi^2(n+1)}\left\{\frac{1}{\pi}\int_0^{\pi}\left(|\psi_x(u)|\sin^{\beta}\frac{u}{2}\right)^p du\right\}^{1/p}\\
&+\frac{\pi^{\beta-1}(2+\beta)}{n+1}\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{\left\{t^{-1}\int_0^t\left(|\psi_x(u)|\sin^{\beta}(u/2)\right)^p du\right\}^{1/p}}{\widetilde{w}_x(t)}\right]^p dt\right\}^{1/p}\\
&\cdot\left\{\int_{\pi/(n+1)}^{\pi}\left[\frac{\widetilde{w}_x(t)}{t^{2+\beta}}\right]^q dt\right\}^{1/q}.
\end{aligned}
\tag{4.7}
$$

Since $\beta+(1/p)>0$, we have

$$
\begin{aligned}
\left|\widetilde{I}_2^{\circ}\right| &\ll \frac{1}{n+1}\widetilde{w}_x(\pi)+\widetilde{w}_x\left(\frac{\pi}{n+1}\right)\left\{\left[\frac{t^{(-1-\beta)q+1}}{(-1-\beta)q+1}\right]_{t=\pi/(n+1)}^{\pi}\right\}^{1/q}\\
&\ll (n+1)^{\beta+(1/p)}\widetilde{w}_x\left(\frac{\pi}{n+1}\right),
\end{aligned}
\tag{4.8}
$$

and our proof is complete.

4.4. Proof of Theorem 2.4

The proof is the same as the proof of Theorem 2.3, the only difference is that, in place of $A_{n,\tau}$, we have to write the quantity $\overline{A}_{n,n-2\tau}$ for which we suppose the same order.

4.5. Proof of Theorem 2.5

We start with the obvious relations

$$
\begin{aligned}
\widetilde{T}_{n,A}f(x)-\widetilde{f}(x) &= \frac{1}{\pi}\int_0^{2\pi/(n+2)}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D}_k^{\circ}(t)dt\\
&\quad+\frac{1}{\pi}\int_{2\pi/(n+2)}^{\pi}\psi_x(t)\sum_{k=0}^{n}a_{n,k}\widetilde{D}_k^{\circ}(t)dt=\widetilde{I}_1^{\circ}+\widetilde{I}_2^{\circ},\\
\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}(x)\right| &\le \left|\widetilde{I}_1^{\circ}\right|+\left|\widetilde{I}_2^{\circ}\right|.
\end{aligned}
\tag{4.9}
$$

Lemma 3.1 gives

$$\begin{aligned}
\left|\tilde{I}_1^{\circ}\right| &\ll \frac{1}{\pi}\int_0^{2\pi/(n+2)} \frac{|\varphi_x(t)|}{t}dt \le \pi^{\beta-1}\int_0^{2\pi/(n+2)} \frac{|\varphi_x(t)|\sin^{\beta}(t/2)}{t^{1+\beta}}dt \\
&= \pi^{\beta-1}\left[\frac{1}{t^{1+\beta}}\int_0^t |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]_{t=0}^{2\pi/(n+2)} \\
&\quad + \pi^{\beta-1}(1+\beta)\int_0^{2\pi/(n+2)} \frac{t^{-1}\int_0^t |\varphi_x(u)|\sin^{\beta}(u/2)du}{t^{1+\beta}}dt.
\end{aligned} \tag{4.10}$$

If $\beta > 0$ and $q > 1$, then, by the Hölder inequality $((1/p) + (1/q) = 1)$,

$$\begin{aligned}
\left|\tilde{I}_1^{\circ}\right| &\le \pi^{\beta-1}\left(\frac{n+2}{2\pi}\right)^{1+\beta}\int_0^{2\pi/(n+2)} |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du \\
&\quad + \pi^{\beta-1}(1+\beta)\left\{\int_0^{2\pi/(n+2)}\left[\frac{t^{-1}\int_0^t |\varphi_x(u)|\sin^{\beta}(u/2)du}{t^{1/p}\tilde{w}_x(t)}\right]^p dt\right\}^{1/p} \\
&\quad \cdot \left\{\int_0^{2\pi/(n+2)}\left[\frac{\tilde{w}_x(t)}{t^{(1/q)+\beta}}\right]^q dt\right\}^{1/q} \\
&\le \pi^{\beta-1}\left(\frac{n+2}{2\pi}\right)^{\beta}\left\{\frac{n+2}{2\pi}\int_0^{2\pi/(n+2)}\left(|\varphi_x(u)|\sin^{\beta}\frac{u}{2}\right)^p du\right\}^{1/p} \\
&\quad + \pi^{\beta-1}(1+\beta)\left\{\int_0^{2\pi/(n+2)}\left[\frac{\left\{t^{-1}\int_0^t\left(|\varphi_x(u)|\sin^{\beta}(u/2)\right)^p du\right\}^{1/p}}{t^{1/p}\tilde{w}_x(t)}\right]^p dt\right\}^{1/p} \\
&\quad \cdot \left\{\int_0^{2\pi/(n+2)}\left[\frac{\tilde{w}_x(t)}{t^{(1/q)+\beta}}\right]^q dt\right\}^{1/q} \\
&\ll \tilde{w}_x\left(\frac{2\pi}{n+2}\right) + \left\{\int_0^{2\pi/(n+2)} t^{\beta p-1}dt\right\}^{1/p}\left\{\int_0^{2\pi/(n+2)}\frac{1}{t}\left[\frac{\tilde{w}_x(t)}{\sin^{\beta}(t/2)}\right]^q dt\right\}^{1/q} \\
&\ll \tilde{w}_x\left(\frac{2\pi}{n+2}\right) + (n+1)^{-\beta}\left\{\int_0^{2\pi/(n+2)}\frac{1}{t}\left[\frac{\tilde{w}_x(t)}{\sin^{\beta}(t/2)}\right]^q dt\right\}^{1/q},
\end{aligned} \tag{4.11}$$

or if $\beta = 0$, then

$$\left|\tilde{I}_1^{\circ}\right| \ll \tilde{w}_x\left(\frac{2\pi}{n+2}\right) + \int_0^{2\pi/(n+2)} \frac{\tilde{w}_x(t)}{t}dt. \tag{4.12}$$

Therefore, using (2.8), we have

$$\left|\widetilde{I}_1^{\circ}\right| \ll \widetilde{w}_x\left(\frac{\pi}{n+1}\right). \tag{4.13}$$

The same estimation as in the proof of Theorem 2.3 gives

$$\left|\widetilde{I}_2^{\circ}\right| \ll (n+1)^{\beta+(1/p)}\widetilde{w}_x\left(\frac{\pi}{n+1}\right). \tag{4.14}$$

Collecting these estimates, we obtain the desired result.

4.6. Proof of Theorem 2.6

For the proof, we use the analogical remark as these in the proofs of Theorems 2.2 and 2.4.

4.7. Proofs of Theorems from 2.7 to 2.12

The proofs are similar to these above. We have only to use the generalized Minkowski inequality.

For example, in case of Theorem 2.7, we get

$$\begin{aligned}
\left\|T_{n,A}f(\cdot)-f(\cdot)\right\|_{L^p} &\le \|I_1\|_{L^p}+\|I_2\|_{L^p} \le \left\|w_{\cdot,\beta}f\left(\frac{2\pi}{n+2}\right)_{L^p}\right\|_{L^p}+\|I_2\|_{L^p} \\
&\ll \left\|w_{\cdot,\beta}f\left(\frac{\pi}{n+1}\right)_{L^p}\right\|_{L^p}+\frac{1}{n+1}\left\|w_{\cdot,\beta}f(\pi)_{L^p}\right\|_{L^p} \\
&\quad +(n+1)^{\beta+(1/p)}\left\|w_{\cdot,\beta}f\left(\frac{\pi}{n+1}\right)_{L^p}\right\|_{L^p} \\
&\ll \frac{1}{n+1}w(\pi)+(n+1)^{\beta+(1/p)}w\left(\frac{\pi}{n+1}\right) \\
&= O\left((n+1)^{\beta+(1/p)}w\left(\frac{\pi}{n+1}\right)\right).
\end{aligned} \tag{4.15}$$

This completes the proof of Theorem 2.7.

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