

## *Research Article*

# On the Gauss Map of Surfaces of Revolution with Lightlike Axis in Minkowski 3-Space

### Minghao Jin,<sup>1,2</sup> Donghe Pei,<sup>1</sup> and Shu Xu<sup>3</sup>

<sup>1</sup> School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China

<sup>2</sup> Department of Mathematics, Heilongjiang Institute of Technology, Harbin 150050, China

<sup>3</sup> Rear Services Office, Chongqing Police College, Chongqing 401331, China

Correspondence should be addressed to Donghe Pei; peidh340@nenu.edu.cn

Received 29 March 2013; Accepted 2 July 2013

Academic Editor: Ji Gao

Copyright © 2013 Minghao Jin et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

By studying the Gauss map *G* and Laplace operator  $\Delta^h$  of the second fundamental form *h*, we will classify surfaces of revolution with a lightlike axis in 3-dimensional Minkowski space and also obtain the surface of Enneper of the 2nd kind, the surface of Enneper of the 3rd kind, the de Sitter pseudosphere, and the hyperbolic pseudosphere that satisfy condition  $\Delta^h G = \Lambda G$ ,  $\Lambda$  being a 3 × 3 real matrix.

#### 1. Introduction

The Gauss map is a useful tool for studying surfaces in Euclidean space and pseudo-Euclidean space.

Suppose that M is a connected surface in  $\mathbb{R}^3$  and G is the Gauss map on M. According to a theorem proved by Ruh and Vilms [1], M has constant mean curvature if and only if

$$\Delta G = \|dG\|^2 G,\tag{1}$$

where  $\Delta$  is the Laplace operator on M that corresponds to the metric induced on M from  $\mathbb{R}^3$ . A special case of (1) is given by

$$\Delta G = \lambda G, \tag{2}$$

where the Gauss map *G* is an eigenfunction of the Laplacian  $\Delta$  on *M*. As a more general form of (1), Dillen et al. [2] proved that a surface of revolution *M* in  $\mathbb{R}^3$  satisfies the condition

$$\Delta G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(3, \mathbb{R}), \tag{3}$$

if and only if M is a plane, sphere, or cylinder. Baikoussis and Blair [3] proved that a ruled surface M in  $\mathbb{R}^3$  satisfies condition (3) if and only if M is a plane, helicoidal surface, or spiral surface in  $\mathbb{R}^3$ . Additionally, Choi and Alías et al. [4–6] completely classified the surfaces of revolution and ruled surfaces in 3-dimensional Minkowski space that satisfy condition (3). Kim and Yoon [7] studied ruled surfaces in  $\mathbb{R}_1^m$  such that

$$\Delta G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(N, \mathbb{R}), \quad N = \binom{m}{2}.$$
 (4)

Recently, an interesting question was raised: what surfaces of revolution without parabolic points in Euclidean or pseudo-Euclidean space satisfy the following condition?

$$\Delta^{h}G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(3, \mathbb{R}), \tag{5}$$

where  $\Delta^h$  is the Laplace operator with respect to the second fundamental form *h* of the surface. This operator is formally defined by

$$\Delta^{h} = -\frac{1}{\sqrt{|\mathcal{H}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^{i}} \left( \sqrt{|\mathcal{H}|} h^{ij} \frac{\partial}{\partial x^{j}} \right)$$
(6)

for the components  $h_{ij}$  (i, j = 1, 2) of the second fundamental form h on M, and we denote by  $(h^{ij})$  (resp.,  $\mathcal{H}$ ) the inverse matrix (resp., the determinant) of the matrix  $(h_{ii})$ .

In [8], the authors studied surfaces of revolution without parabolic points in Euclidean 3-space  $\mathbb{R}^3$  and presented

some classification theorems. In this paper, we will consider surfaces of revolution with lightlike axis in  $\mathbb{R}^3_1$  and present some classification results.

#### 2. Preliminaries

Let  $\mathbb{R}^3_1$  be a 3-dimensional Minkowski space with the scalar product and Lorentz cross-product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0^2 + x_1^2 + x_2^2,$$
  
 $\mathbf{x} \times \mathbf{y} = (x_2 y_1 - x_1 y_2, x_2 y_0 - x_0 y_2, x_0 y_1 - x_1 y_0)$ 
(7)

for every vector  $\mathbf{x} = (x_0, x_1, x_2)$  and  $\mathbf{y} = (y_0, y_1, y_2)$  in  $\mathbb{R}^3_1$ .

A vector **x** of  $\mathbb{R}_1^3$  is said to be spacelike if  $\langle \mathbf{x}, \mathbf{x} \rangle > 0$  or  $\mathbf{x} = \mathbf{0}$ , timelike if  $\langle \mathbf{x}, \mathbf{x} \rangle < 0$  and lightlike or null if  $\langle \mathbf{x}, \mathbf{x} \rangle = 0$  and  $\mathbf{x} \neq \mathbf{0}$ . A timelike or lightlike vector in  $\mathbb{R}_1^3$  is said to be causal. Let  $\gamma : I \to \mathbb{R}_1^3$  be a smooth curve in  $\mathbb{R}_1^3$ , where *I* is an interval in  $\mathbb{R}$ . We call  $\gamma$  spacelike, timelike, or lightlike curve if the tangent vector  $\gamma'$  at any point is spacelike, timelike, or lightlike, respectively.

Let *I* be an open interval and  $\gamma : I \to \Pi$  a plane curve lying in a plane  $\Pi$  of  $\mathbb{R}_1^3$  and *l* a straight line in  $\Pi$  which does not intersect with the curve  $\gamma$ . A surface of revolution *M* with axis l in  $\mathbb{R}_1^3$  is defined to be invariant under the group of motions in  $\mathbb{R}_1^3$ , which fixes each point of the line *l* [9]. Because the present paper discusses the case of lightlike axis, without loss of generality, we may assume that the axis is the line spanned by vector (1, 1, 0) in the plane  $Ox_0x_1$ .

So, we choose the line spanned by the vector (1, 1, 0) as axis and express the suppose curve  $\gamma$  as follows:

$$\gamma(u) = \left(f(u), g(u), 0\right),\tag{8}$$

where f(u) is a smooth positive function and g(u) is a smooth function such that  $h(u) = f(u) - g(u) \neq 0$ . Then, the surface of revolution *M* with such axis may be given by

$$x(u,v) = \left(f(u) + \frac{v^2}{2}h(u), g(u) + \frac{v^2}{2}h(u), h(u)v\right).$$
(9)

Now, let us consider the Gauss map *G* on a surface *M* in  $\mathbb{R}^3_1$ . The map  $G: M \to Q^2(\varepsilon) \subset \mathbb{R}^3_1$ , which sends each point of *M* to the unit normal vector to *M* at that point, is called the Gauss map of surface *M*. Here,  $\varepsilon(=\pm 1)$  denotes the sign of the vector field *G* and  $Q^2(\varepsilon)$  is a 2-dimensional space form as follows:

$$Q^{2}(\varepsilon) = \begin{cases} S_{1}^{2}(1) & \text{in } \mathbb{R}_{1}^{3} \text{ if } \varepsilon = 1, \\ H^{2}(-1) & \text{in } \mathbb{R}_{1}^{3} \text{ if } \varepsilon = -1. \end{cases}$$
(10)

A surface  $M \in \mathbb{R}^3_1$  is called minimal if and only if its mean curvature *H* is zero. As de Woestijne ([10]) proved, we have the following theorems.

**Theorem 1** (see [10]). Every minimal, spacelike surface of revolution  $M \in \mathbb{R}^3_1$  is congruent to a part of one of the following surfaces:

- (1) a spacelike plane;
- (2) the catenoid of the 1st kind;
- (3) the catenoid of the 2nd kind;
- (4) the surface of Enneper of the 2nd kind.

**Theorem 2** (see [10]). Every minimal, timelike surface of revolution  $M \in \mathbb{R}^3_1$  is congruent to a part of one of the following surfaces:

- (1) a Lorentzian plane;
- (2) the catenoid of the 3rd kind;
- (3) *the catenoid of the 4th kind;*
- (4) the catenoid of the 5th kind;
- (5) the surface of Enneper of the 3rd kind.

Now, we consider some examples of surfaces of revolution which are mentioned in our theorems.

*Example 1* (The surface of Enneper of the 2nd kind is shown in Figure 1). The surface of Enneper of the 2nd kind is parameterized by

$$x(u,v) = \left(u^{3} - u - v^{2}u, u^{3} + u - v^{2}u, -2uv\right)$$
(11)

for u < 0. Then, the components of the first and the second fundamental forms are given by

$$g_{11} = 12u^{2}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = 4u^{2},$$
$$h_{11} = \frac{-24u^{2}}{|x_{u} \times x_{v}|}, \qquad h_{12} = h_{21} = 0, \qquad h_{22} = \frac{8u^{2}}{|x_{u} \times x_{v}|}.$$
(12)

So, the mean curvature *H* on the surface is

$$H = \frac{\left(-24u^2\right)\left(4u^2\right) + \left(8u^2\right)\left(12u^2\right)}{2\left(12u^2\right)\left(4u^2\right)\left|x_u \times x_v\right|} = 0.$$
 (13)

Therefore, the surface of Enneper of the 2nd kind is minimal.

*Example 2* (The surface of Enneper of the 3rd kind is shown in Figure 2). The surface of Enneper of the 3rd kind is parameterized by

$$x(u,v) = \left(-u^3 - u - v^2 u, -u^3 + u - v^2 u, -2uv\right)$$
(14)

for u < 0. Then, the components of the first and the second fundamental forms are given by

$$g_{11} = -12u^{2}, \qquad g_{12} = g_{21} = 0, \qquad g_{22} = 4u^{2},$$

$$h_{11} = \frac{24u^{2}}{|x_{u} \times x_{v}|}, \qquad h_{12} = h_{21} = 0, \qquad h_{22} = \frac{8u^{2}}{|x_{u} \times x_{v}|}.$$
(15)

So, the mean curvature H on the surface is

$$H = \frac{\left(24u^2\right)\left(4u^2\right) + \left(8u^2\right)\left(-12u^2\right)}{2\left(-12u^2\right)\left(4u^2\right)\left|x_u \times x_v\right|} = 0.$$
 (16)

Therefore, the surface of Enneper of the 3rd kind is minimal.



FIGURE 1: The surface of Enneper of the 2nd kind.



FIGURE 2: The surface of Enneper of the 3rd kind.

*Example 3* (The de Sitter pseudosphere is shown in Figure 3). The de Sitter pseudosphere with radius 1 can be expressed as

$$x(u, v) = (\sinh u, \cosh u \cos v, \cosh u \sin v).$$
(17)

Then, its Gauss map G and Laplacian are given by

$$G = (-\sinh u, -\cosh u \cos v, -\cosh u \sin v),$$

$$\Delta^{h} = \frac{\partial^{2}}{\partial u^{2}} - \frac{1}{\cosh^{2} u} \frac{\partial^{2}}{\partial v^{2}} + \frac{\sinh u}{\cosh u} \frac{\partial}{\partial u}.$$
 (18)

By a straight computation, we get

$$\Delta^h G = (-2\sinh u, -2\cosh u\cos v, -2\cosh u\sin v), \quad (19)$$

which means

$$\Delta^{h}G = \begin{pmatrix} 2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2 \end{pmatrix} G;$$
 (20)

that is, the de Sitter pseudosphere satisfies condition (1).

*Example 4* (The hyperbolic pseudosphere is shown in Figure 4). The hyperbolic pseudosphere with radius 1 is parameterized by

$$x(u, v) = (\cosh u, \sinh u \cos v, \sinh u \sin v).$$
(21)



FIGURE 3: The de Sitter pseudosphere.



FIGURE 4: The future hyperbolic pseudosphere.

Then, its Gauss map G and Laplacian are given by

$$G = (-\cosh u, -\sinh u \cos v, -\sinh u \sin v),$$

$$\Delta^{h} = -\frac{\partial^{2}}{\partial u^{2}} - \frac{1}{\sinh^{2} u} \frac{\partial^{2}}{\partial v^{2}} - \frac{\cosh u}{\sinh u} \frac{\partial}{\partial u}.$$
 (22)

By a straight computation, we get

$$\Delta^h G = (2\cosh u, 2\sinh u\cos v, 2\sinh u\sin v).$$
(23)

So, we have

$$\Delta^{h}G = \begin{pmatrix} -2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -2 \end{pmatrix} G;$$
(24)

that is, the hyperbolic pseudosphere satisfies condition (1).

# 3. The Surface of Revolution with Lightlike Axis

In this section, we will classify the surfaces of revolution with lightlike axis in  $\mathbb{R}^3_1$  that satisfy condition (5).

**Theorem 3.** The only surfaces of revolution with lightlike axis in  $\mathbb{R}^3_1$ , whose Gauss map G satisfies

$$\Delta^{h}G = \Lambda G, \quad \Lambda \in Mat(3, \mathbb{R}), \quad (25)$$

*are locally the surface of Enneper of the 2nd kind, the surface of Enneper of the 3rd kind, the de Sitter pseudosphere, and the hyperbolic pseudosphere.* 

*Proof.* Let *M* be a surface of revolution with lightlike axis as (9); then we may assume that the profile curve  $\gamma$  is of unit speed; thus

$$\left\langle \gamma',\gamma'\right\rangle = -f^{\prime 2}\left(u\right) + g^{\prime 2}\left(u\right) = \varepsilon\left(\pm 1\right). \tag{26}$$

Without lost of generality, we assume that h = f(u)-g(u) > 0and give a detailed proof just for the case  $\varepsilon = 1$ .

Then, we may put

$$f'(u) = \sinh t, \qquad g'(u) = \cosh t \tag{27}$$

for the smooth function t = t(u). Using the natural frame  $\{x_u, x_v\}$  of *M* defined by

$$\begin{aligned} x_{u} &= \left( f' + \frac{v^{2}}{2}h', g' + \frac{v^{2}}{2}h', h'v \right), \qquad x_{v} = (vh, vh, h), \\ x_{uu} &= \left( f'' + \frac{v^{2}}{2}h'', g'' + \frac{v^{2}}{2}h'', h''v \right), \\ x_{uv} &= \left( vh', vh', 0 \right), \qquad x_{vv} = (h, h, 0), \end{aligned}$$
(28)

we obtain the components of the first and the second fundamental forms of the surface as follows:

$$g_{11} = \langle x_{u}, x_{u} \rangle = 1,$$

$$g_{12} = g_{21} = \langle x_{u}, x_{v} \rangle = 0,$$

$$g_{22} = \langle x_{v}, x_{v} \rangle = h^{2},$$

$$h_{11} = \langle x_{uu}, G \rangle = f''g' - f'g'' = t',$$

$$h_{12} = h_{21} = \langle x_{uv}, G \rangle = 0,$$

$$h_{22} = \langle x_{vv}, G \rangle = -hh',$$
(29)

where Gauss map *G* is defined by  $(x_u \times x_v)/|x_u \times x_v| = (-g' + (v^2/2)h', -f' + (v^2/2)h', vh')$ .

So, the matrix  $(h_{ij})$  is composed by second fundamental form h as follows:

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} t' & 0 \\ 0 & -hh' \end{pmatrix}.$$
 (30)

Since  $\mathcal{H} = h_{11}h_{22} - h_{12}^2 = 0$  makes Laplacian  $\Delta^h$  degenerate, so we can assume that  $\mathcal{H} \neq 0$  for every *t*. Then, the mean curvature *H* on *M* is given by

$$H = \frac{h^2 t' - hh'}{2h^2} = \frac{1}{2} \left( t' - \frac{h'}{h} \right).$$
(31)

By a straightforward computation, the Laplacian  $\Delta^h$  of the second fundamental form *h* on *M* with the help of (2), (27), and (29) turns out to be

$$\Delta^{h} = -\frac{1}{t'}\frac{\partial^{2}}{\partial u^{2}} + \frac{1}{hh'}\frac{\partial^{2}}{\partial v^{2}} + \left(\frac{t''}{2t'^{2}} - \frac{h'}{2ht'} - \frac{h''}{2h't'}\right)\frac{\partial}{\partial u}.$$
(32)

Accordingly, we get

$$\Delta^{h}G = \begin{pmatrix} \left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^{2}} - \frac{h'h''}{4t'h} - \frac{h''^{2}}{4t'h'}\right)v^{2} + \frac{g'''}{t'} - \frac{t''g''}{2t'^{2}} + \frac{h'g''}{2t'h} + \frac{h''g''}{2t'h'} + \frac{1}{h} \\ \left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^{2}} - \frac{h'h''}{4t'h} - \frac{h''^{2}}{4t'h'}\right)v^{2} + \frac{f'''}{t'} - \frac{t''f''}{2t'^{2}} + \frac{h'f''}{2t'h} + \frac{h''f''}{2t'h'} + \frac{1}{h} \\ \left(-\frac{h'''}{t'} + \frac{t''h''}{2t'^{2}} - \frac{h'h''}{2t'h} - \frac{h'h''}{2t'h} - \frac{h''^{2}}{2t'h'}\right)v$$
(33)

By the assumption (25) and the above equation, we get the following system of differential equations:

$$\left( -\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{11} + a_{12}}{2}h' \right) v^2 - a_{13}h'v + \frac{g'''}{t'} - \frac{t''g''}{2t'^2} + \frac{h'g''}{2t'h} + \frac{h''g''}{2t'h'}$$

$$+ \frac{1}{h} + a_{11}g' + a_{12}f' = 0,$$

$$\left( -\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{21} + a_{22}}{2}h' \right)v^2$$

$$- a_{23}h'v + \frac{f'''}{t'} - \frac{t''f''}{2t'^2} + \frac{h'f''}{2t'h} + \frac{h''f''}{2t'h'}$$

$$+ \frac{1}{h} + a_{21}g' + a_{22}f' = 0,$$

$$- \frac{a_{31} + a_{32}}{2}h'v^{2} + \left(-\frac{h'''}{t'} + \frac{t''h''}{2t'^{2}} - \frac{h'h''}{2t'h} - \frac{h''^{2}}{2t'h'} - a_{33}h'\right)v$$

$$+ a_{31}g' + a_{32}f' = 0,$$

$$(34)$$

where  $a_{ij}$  (*i*, *j* = 1, 2, 3) denote the components of the matrix  $\Lambda$  given by (25).

In order to prove the theorem, we have to solve the above system of ordinary differential equations. So, we get three systems of ODE, equivalently:

$$\begin{aligned} -\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{11} + a_{12}}{2}h' &= 0, \\ -a_{13}h' &= 0, \\ \\ \frac{g'''}{t'} - \frac{t''g''}{2t'^2} + \frac{h'g''}{2t'h} + \frac{h''g''}{2t'h'} + \frac{1}{h} + a_{11}g' + a_{12}f' &= 0, \\ -\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{21} + a_{22}}{2}h' &= 0 \\ -a_{23}h' &= 0, \end{aligned}$$

$$\frac{f'''}{t'} - \frac{t''f''}{2t'^2} + \frac{h'f''}{2t'h} + \frac{h''f''}{2t'h'} + \frac{1}{h} + a_{21}g' + a_{22}f' = 0,$$
  
$$-\frac{a_{31} + a_{32}}{2}h' = 0,$$
  
$$-\frac{h'''}{t'} + \frac{t''h''}{2t'^2} - \frac{h'h''}{2t'h} - \frac{h''^2}{2t'h'} - a_{33}h' = 0,$$
  
$$a_{31}g' + a_{32}f' = 0.$$
(35)

From (35), we easily deduce that  $a_{13} = a_{23} = a_{31} = a_{32} = 0$ and  $a_{33} = (a_{11} + a_{22})/2 = a_{11} + a_{12} = a_{21} + a_{22}$ . We put  $a_{11} = \lambda$ and  $a_{22} = \mu$ . Therefore, the matrix  $\Lambda$  satisfies

$$\Lambda = \begin{pmatrix} \lambda & \frac{1}{2}(\mu - \lambda) & 0\\ \frac{1}{2}(\lambda - \mu) & \mu & 0\\ 0 & 0 & \frac{1}{2}(\lambda + \mu) \end{pmatrix}.$$
 (36)

Then, three systems (35) now reduce to the following equations:

$$\frac{g'''}{t'} - \frac{t''g''}{2t'^2} + \frac{h'g''}{2t'h'} + \frac{h''g''}{2t'h'} + \frac{1}{h} = -\lambda g' - \frac{\mu - \lambda}{2} f', \quad (37)$$

$$\frac{f'''}{t'} - \frac{t''f''}{2t'^2} + \frac{h'f''}{2t'h'} + \frac{h''f''}{2t'h} + \frac{h''f''}{2t'h} + \frac{1}{h} = -\mu f' - \frac{\lambda - \mu}{2}g', \quad (38)$$

$$-\frac{h'''}{t'} + \frac{t''h''}{2t'^2} - \frac{h'h''}{2t'h} - \frac{h''^2}{2t'h'} = \frac{\mu + \lambda}{2}h'.$$
 (39)

By the computation  $(37) \times \cosh t - (38) \times \sinh t$  and using  $f' = \sinh t$ ,  $f'' = t' \cosh t$ ,  $f''' = t'^2 \sinh t + t'' \cosh t$ ,  $g' = \cosh t$ ,  $g'' = t' \sinh t$ , and  $g''' = t'^2 \cosh t + t'' \sinh t$ , we easily get

$$t' - \frac{h'}{h} = -\lambda \cosh^2 t + \mu \sinh^2 t + (\lambda - \mu) \sinh t \cosh t.$$
(40)

On the other hand, substituting h'' = -h't' and  $h''' = h'(t'^2 - t'')$  into (39) equivalently, we get the following equation:

$$t'' - 3t'^{2} + \frac{h'}{h}t' = (\lambda + \mu)t'.$$
(41)

Now, we discuss five cases according to the constants  $\lambda$  and  $\mu$ .

*Case 1* ( $\lambda = \mu = 0$ ). In this case, we easily get t' - (h'/h) = 0, which implies that the mean curvature *H* vanishes identically because of (31). Therefore, the surface is minimal; from Theorem 1 it is the surface of Enneper of the 2nd kind. Furthermore, a surface of Enneper of the 2nd kind satisfies the condition (25).

*Case2* ( $\lambda = \mu \neq 0$ ). By (40), we get

$$t' = \frac{h'}{h} - \lambda. \tag{42}$$

Differentiating (42) with respect to u, we have

$$t'' = -\frac{h'}{h}t' - \left(\frac{h'}{h}\right)^2.$$
 (43)

Substituting (42) and (43) into (41), we get

$$4\left(\frac{h'}{h}\right)^2 + 4\lambda\frac{h'}{h} + \lambda^2 = 0 \tag{44}$$

from which

9

$$\frac{h'}{h} = \frac{\lambda}{2}.$$
(45)

Furthermore, (45) together with (42) becomes  $t' = -(\lambda/2)$ ; that is,

$$t(u) = -\frac{\lambda}{2}u + k, \quad k \in \mathbb{R}.$$
 (46)

On the other hand, by (27), (45), and (46), we have

$$f(u) = -\frac{2}{\lambda} \cosh\left(-\frac{\lambda}{2}u + k\right) + c,$$

$$(u) = -\frac{2}{\lambda} \sinh\left(-\frac{\lambda}{2}u + k\right) + c, \quad c \in \mathbb{R}.$$

$$(47)$$

Then, the surface *M* has the following expression:

$$x(u,v) = \left(-\frac{2}{\lambda}\cosh\left(-\frac{\lambda}{2}u+k\right) + \frac{v^2}{2}h+c, -\frac{2}{\lambda}\sinh\left(-\frac{\lambda}{2}+k\right) + \frac{v^2}{2}h+c, hv\right),$$
(48)

where  $h = f - g = -(2/\lambda)e^{(\lambda/2)-k}$ ,  $c, k \in \mathbb{R}$ . From this, we easily get

$$\langle x(u,v) - \mathbf{C}, x(u,v) - \mathbf{C} \rangle = -\left(\frac{2}{\lambda}\right)^2, \quad \mathbf{C} = (c,c,0).$$
 (49)

This equation means that the surface M is contained in the hyperbolic pseudosphere  $H^2(-(2/|\lambda|))$  centered at **C** with radius  $2/|\lambda|$ . Also, the hyperbolic pseudosphere satisfies condition (25).

*Case 3* ( $\lambda \neq 0, \mu = 0$ ). In this case, (40) becomes  $t' - (h'/h) = -\lambda \cosh^2 t + \lambda \sinh t \cosh t$ ; that is,

$$t' = \frac{h'}{h} - \lambda \cosh^2 t + \lambda \sinh t \cosh t$$
(50)

and thus

$$t'' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 - 2\lambda t' \sinh t \cosh t$$
  
+  $\lambda t' \sinh^2 t + \lambda t' \cosh^2 t.$  (51)

Substituting (50) and (51) into (41), we get

$$\Phi_1 h^2 + \Phi_2 h + \Phi_3 = 0, \tag{52}$$

where we put

$$\Phi_{1} = \lambda^{2} \left(-3 \cosh^{4} t + 8 \sinh t \cosh^{3} t -7 \sinh^{2} t \cosh^{2} t + 2 \sinh^{3} t \cosh t\right),$$

$$\Phi_{2} = \lambda \left(-6 \cosh^{3} t + 14 \sinh t \cosh^{2} t -10 \sinh^{2} t \cosh t + 2 \sinh^{3} t\right),$$

$$\Phi_{3} = -4 \sinh^{2} t + 8 \sinh t \cosh t - 4 \cosh^{2} t.$$
(53)

Differentiating (52) and using (50), we find

$$\Psi_1 f^2 + \Psi_2 f + \Psi_3 = 0, \tag{54}$$

where

$$\Psi_{1} = \lambda^{2} \left(-4 \sinh^{8} t \cosh t + 40 \sinh^{7} t \cosh^{2} t - 182 \sinh^{6} t \cosh^{3} t + 474 \sinh^{5} t \cosh^{4} t - 760 \sinh^{4} t \cosh^{5} t + 764 \sinh^{3} t \cosh^{6} t - 470 \sinh^{2} t \cosh^{7} t + 162 \sinh t \cosh^{8} t - 24 \cosh^{9} t\right),$$

$$\begin{split} \Psi_{2} &= \lambda \left( -4 \sinh^{8} t + 44 \sinh^{7} t \cosh t \right. \\ &\quad - 220 \sinh^{6} t \cosh^{2} t + 612 \sinh^{5} t \cosh^{3} t \\ &\quad - 1020 \sinh^{4} t \cosh^{2} t + 1044 \sinh^{3} t \cosh^{5} t \\ &\quad - 644 \sinh^{2} t \cosh^{4} t + 1044 \sinh^{3} t \cosh^{5} t \\ &\quad - 644 \sinh^{2} t \cosh^{6} t + 220 \sinh t \cosh^{7} t \\ &\quad - 32 \cosh^{8} t \right), \\ \Psi_{3} &= 8 \sinh^{7} t - 64 \sinh^{6} t \cosh t \\ &\quad + 208 \sinh^{5} t \cosh^{2} t - 360 \sinh^{4} t \cosh^{3} t \\ &\quad + 360 \sinh^{3} t \cosh^{4} t - 208 \sinh^{2} t \cosh^{5} t \\ &\quad + 64 \sinh t \cosh^{6} t - 8 \cosh^{7} t. \end{split}$$

Combining (52) and (54), we show that

$$\chi_1 f + \chi_2 = 0, (56)$$

(55)

where  $\chi_1 = \Phi_2 \Psi_1 - \Phi_1 \Psi_2$ ,  $\chi_2 = \Phi_3 \Psi_1 - \Phi_1 \Psi_3$ .

Differentiating once again this equation and using the same algebraic techniques above, we find the following trigonometric polynomial in  $\sinh t$  and  $\cosh t$  satisfying

$$\mu^2 \left( \sum_{i=1}^{31} c_i \sinh^{31-i} t \cosh^{5+i} t \right) = 0, \tag{57}$$

where  $c_1 = 1024$ ,  $c_2 = -24064$ ,..., and  $c_{31} = 170496$  are nonzero coefficients of the function  $\sinh^{31-i}t\cosh^{5+i}t$ . Since this polynomial is equal to zero for every *t*, all its coefficients must be zero. Thus, we have  $\mu = 0$ , which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis in this case.

*Case 4* ( $\lambda = 0, \mu \neq 0$ ). In this case, (40) becomes  $t' - (h'/h) = \mu \sinh^2 t - \mu \sinh t \cosh t$ ; that is,

$$t' = \frac{h'}{h} + \mu \sinh^2 t - \mu \sinh t \cosh t \tag{58}$$

and thus

$$t'' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 + 2\mu t' \sinh t \cosh t$$
  
$$-\mu t' \sinh^2 t - \mu t' \cosh^2 t.$$
 (59)

Substituting (58) and (59) into (41), we get

$$\iota_1 h^2 + \iota_2 h + \iota_3 = 0, (60)$$

where we put

$$\begin{aligned}
\iota_{1} &= \mu^{2} \left( -3 \sinh^{4} t + 8 \sinh^{3} t \cosh t \\
&-7 \sinh^{2} t \cosh^{2} t + 2 \sinh t \cosh^{3} t \right), \\
\iota_{2} &= \mu \left( -6 \sinh^{3} t + 14 \sinh^{2} t \cosh t \\
&- 10 \sinh t \cosh^{2} t + 2 \cosh^{3} t \right), \\
\iota_{3} &= -4 \sinh^{2} t + 8 \sinh t \cosh t - 4 \cosh^{2} t.
\end{aligned}$$
(61)
Differentiating (60) and using (58), we find

$$\kappa_1 f^2 + \kappa_2 f + \kappa_3 = 0, \tag{62}$$

where

$$\kappa_{1} = \mu^{2} \left( 108 \sinh^{9} t - 246 \sinh^{8} t \cosh t \right.$$
  
+ 106 \sinh^{7} t \cosh^{2} t - 64 \sinh^{6} t \cosh^{3} t   
+ 424 \sinh^{5} t \cosh^{4} t - 530 \sinh^{4} t \cosh^{5} t   
+ 238 \sinh^{3} t \cosh^{6} t - 40 \sinh^{2} t \cosh^{7} t   
+ 4 \sinh t \cosh^{8} t \right),

$$\kappa_2 = \mu \left( 116 \sinh^8 t - 220 \sinh^7 t \cosh t \right)$$

$$- 28 \sinh^{6} t \cosh^{2} t + 76 \sinh^{5} t \cosh^{3} t$$

$$+ 432 \sinh^{4} t \cosh^{4} t - 612 \sinh^{3} t \cosh^{5} t$$

$$+ 276 \sinh^{2} t \cosh^{6} t$$
(63)

$$-44 \sinh t \cosh^7 t + 4 \cosh^8 t$$

 $\kappa_3 = 8 \sinh^7 t - 64 \sinh^6 t \cosh t$ 

$$+ 208 \sinh^5 t \cosh^2 t - 360 \sinh^4 t \cosh^3 t$$

+ 360 sinh<sup>3</sup> t cosh<sup>4</sup> t - 208 sinh<sup>2</sup> t cosh<sup>5</sup> t

+ 64 
$$\sinh t \cosh^6 t - 8 \cosh^7 t$$
.

Combining (60) and (62), we show that

$$\omega_1 f + \omega_2 = 0, \tag{64}$$

where  $\omega_1 = \iota_2 \kappa_1 - \iota_1 \kappa_2$ ,  $\omega_2 = \iota_3 \kappa_1 - \iota_1 \kappa_3$ .

Differentiating once again this equation and using the same method above, we find the following trigonometric polynomial in  $\sinh t$  and  $\cosh t$  satisfying

$$\mu^2 \left( \sum_{i=1}^{31} c_i \sinh^{37-i} t \cosh^{i-1} t \right) = 0, \tag{65}$$

where  $c_1 = 86420736$ ,  $c_2 = -4471635456$ , ..., and  $c_{31} = -8192$  are nonzero coefficients of the function  $\sinh^{37-i}t\cosh^{i-1}t$ . Since this polynomial is equal to zero for every *t*, all its

coefficients must be zero. Thus, we have  $\mu = 0$ , which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis.

*Case 5*  $(\lambda \neq 0, \mu \neq 0, \lambda \neq \mu)$ . In this case, (40) is unchanged; that is,

$$t' = \frac{h'}{h} - \lambda \cosh^2 t + \mu \sinh^2 t + (\lambda - \mu) \sinh t \cosh t \quad (66)$$

and thus

$$t'' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 - 2\left(\lambda - \mu\right)t'\sinh t\cosh t + \left(\lambda - \mu\right)t'\sinh^2 t + \left(\lambda - \mu\right)t'\cosh^2 t.$$
(67)

Substituting (66) and (67) into (41), we get

$$P_1h^2 + P_2h + P_3 = 0, (68)$$

where we put

$$P_{1} = (2\lambda\mu - 5\mu^{2})\sinh^{4}t + (2\lambda^{2} - 12\lambda\mu + 10\mu^{2})\sinh^{3}t \cosh t + (-7\lambda^{2} + 18\lambda\mu - 5\mu^{2})\sinh^{2}t \cosh^{2}t + (8\lambda^{2} - 8\lambda\mu)\sinh t \cosh^{3}t - 3\lambda^{2}\cosh^{4}t,$$
(69)  
$$P_{2} = (-4\lambda - 2\mu)\sinh^{3}t + (-4\lambda + 10\mu)\sinh^{2}t \cosh t + (14\lambda - 8\mu)\sinh t \cosh^{2}t - 6\lambda\cosh^{3}t P_{3} = -4\sinh^{2}t + 8\sinh t \cosh t - 4\cosh^{2}t.$$

Differentiating (68) and using (66), we find

$$Q_1 f^2 + Q_2 f + Q_3 = 0, (70)$$

where

$$Q_{1} = \left(8\lambda^{3}\mu - 44\lambda^{2}\mu^{2} + 16\lambda\mu^{3} + 20\mu^{4}\right)\sinh^{9}t \\ + \left(8\lambda^{4} - 100\lambda^{3}\mu + 144\lambda^{2}\mu^{2} + 118\lambda\mu^{3} - 170\mu^{4}\right)\sinh^{8}t\cosh t \\ + \dots + \left(-48\lambda^{4} + 48\lambda^{3}\mu + 42\lambda^{2}\mu^{2} - 24\lambda\mu^{3} - 18\mu^{4}\right)\cosh^{9}t,$$

$$Q_{2} = \left(8\lambda^{3} - 52\lambda^{2}\mu + 4\lambda\mu^{2} - 20\mu^{3}\right)\sinh^{8}t \\ + \left(-64\lambda^{3} + 180\lambda^{2}\mu - 28\lambda\mu^{2} - 40\mu^{3}\right) \\ \times \sinh^{7}t \cosh t + \cdots \\ + \left(-80\lambda^{3} + 80\lambda^{2}\mu + 48\lambda\mu^{2} - 24\mu^{3}\right) \\ \times \cosh^{8}t, \\Q_{3} = \left(8\lambda^{2} - 32\lambda\mu\right)\sinh^{7}t \\ + \left(-64\lambda^{2} + 176\lambda\mu - 40\mu^{2}\right)\sinh^{6}t \cosh t \\ + \cdots + \left(-32\lambda^{2} + 32\lambda\mu + 24\mu^{2}\right)\cosh^{7}t.$$
(71)

Combining (68) and (70), we show that

$$R_1 f + R_2 = 0, (72)$$

where  $R_1 = P_2Q_1 - P_1Q_2$ ,  $R_2 = P_3Q_1 - P_1Q_3$ .

Differentiating once again this equation and using the same algebraic techniques above, we find the following trigonometric polynomial in  $\sinh t$  and  $\cosh t$  satisfying

$$\sum_{i=1}^{37} c_i \left(\lambda, \mu\right) \sinh^{37-i} t \cosh^{i-1} t = 0, \tag{73}$$

where

$$c_{1}(\lambda,\mu) = -331776\lambda^{11}\mu^{3} + 6819840\lambda^{10}\mu^{4} + \dots - 4352000\mu^{14},$$
  

$$\vdots \qquad (74)$$

$$c_{37}(\lambda,\mu) = -18874368\lambda^{14} + 54263808\lambda^{13}\mu^{1}$$

$$+\cdots+2985984\mu^{14},$$

where  $c_i(\lambda, \mu)$  (i = 1, ..., 37) are the known polynomials in  $\lambda$  and  $\mu$ . Since this polynomial is equal to zero for every t, all its coefficients must be zero. Therefore,  $\lambda = \mu = 0$ , which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis in this case.

When  $\varepsilon = -1$ , we can assume that  $f'(u) = \cosh t$  and  $g'(u) = \sinh t$ . Using the same algebraic techniques as for  $\varepsilon = 1$ , we easily prove from theorem (9) that the surfaces of Enneper of the 3rd kind and the de Sitter pseudosphere satisfy condition (25). This completes the proof.

#### Acknowledgments

The authors were supported by NSF of China (no. 11271063) and NSF of Heilongjiang Institute of Technology (no. 2012QJ1 9).

#### References

- E. A. Ruh and J. Vilms, "The tension field of the Gauss map," *Transactions of the American Mathematical Society*, vol. 149, pp. 569–573, 1970.
- [2] F. Dillen, J. Pas, and L. Verstraelen, "On the Gauss map of surfaces of revolution," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 18, no. 3, pp. 239–246, 1990.
- [3] C. Baikoussis and D. E. Blair, "On the Gauss map of ruled surfaces," *Glasgow Mathematical Journal*, vol. 34, no. 3, pp. 355– 359, 1992.
- [4] L. J. Alías, A. Ferrández, P. Lucas, and M. A. Meroño, "On the Gauss map of *B*-scrolls," *Tsukuba Journal of Mathematics*, vol. 22, no. 2, pp. 371–377, 1998.
- [5] S. M. Choi, "On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space," *Tsukuba Journal of Mathematics*, vol. 19, no. 2, pp. 351–367, 1995.
- [6] S. M. Choi, "On the Gauss map of ruled surfaces in a 3dimensional Minkowski space," *Tsukuba Journal of Mathematics*, vol. 19, no. 2, pp. 285–304, 1995.
- [7] Y. H. Kim and D. W. Yoon, "Classification of ruled surfaces in Minkowski 3-spaces," *Journal of Geometry and Physics*, vol. 49, no. 1, pp. 89–100, 2004.
- [8] Y. H. Kim, C. W. Lee, and D. W. Yoon, "On the Gauss map of surfaces of revolution without parabolic points," *Bulletin of the Korean Mathematical Society*, vol. 46, no. 6, pp. 1141–1149, 2009.
- [9] C. C. Beneki, G. Kaimakamis, and B. J. Papantoniou, "Helicoidal surfaces in three-dimensional Minkowski space," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 586– 614, 2002.
- [10] I. V. de Woestijne, Minimal Surfaces in the 3-Dimensional Minkowski Space, Geometry and Topology of Submanifolds: II, World Scientific, Singapore, 1990.



The Scientific World Journal





**Decision Sciences** 







Journal of Probability and Statistics



Hindawi Submit your manuscripts at





International Journal of Differential Equations





International Journal of Combinatorics





Mathematical Problems in Engineering



Abstract and Applied Analysis



Discrete Dynamics in Nature and Society







Journal of Function Spaces



International Journal of Stochastic Analysis



Journal of Optimization