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Research Article

Stability for a Class of Differential Equations with Nonconstant Delay

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Stability is investigated for the following differential equations with nonconstant delay $x'(t) = q(t)F(x(t)) - p(t)f(x(t-\tau(t)))$, where $p : [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to R$, $\tau : [0,+\infty) \to [0,r]$, and F and $f : R \to R$ with $xf(x) > 0$ for $x \neq 0$ and $|x| \leq a$ (a is a positive constant) are continuous functions. A criterion is given for the zero solution of this delay equation being uniformly stable and asymptotically stable.

1. Introduction

Delays are inherent in many physical and technological systems. In particular, pure delays are often used to ideally represent the effects of transmission, transportation, and inertia phenomena. Delay differential equations constitute basic mathematical models of real phenomena, for instance in biology, mechanics, and economics (cf., e.g., [1–17] and references therein). Stability analysis of delay differential equations is particularly relevant in control theory, where one cause of delay is the finite speed of communication. There have been a lot of results on the study of stability of delay differential equations. For example, we can see many earlier results on this issue from Burton's book [2]. Recently, in 2004, Butcher et al. [4] studied the stability properties of delay differential equations with time-periodic parameters. By employing a shifted Chebyshev polynomial approximation in each time interval with length equal to the delay and parametric excitation period, the system is reduced to a set of linear difference equations for the Chebyshev expansion coefficients of the state vector in the previous and current intervals. In 2005, Wahi and Chatterjee [16] used Galerkin-projection to reduce the infinite dimensional dynamics of a delay differential equation to one occurring on a finite number of modes. In 2009, Kalmár-Nagy [7] demonstrated that the method of steps for linear delay differential equation together with the inverse Laplace transform can be used to find a converging sequence of polynomial approximants to the transcendental function determining stability of the delay equation. Most recently, Berezansky and Braverman [3] gave some explicit conditions of asymptotic and exponential stability for the scalar nonautonomous linear delay differential equation with several delays and an arbitrary number of positive and negative coefficients.

This paper is concerned with the following differential equations with nonconstant delay:

$$x'(t) = q(t) F(x(t)) - p(t) f(x(t-\tau(t))), \qquad (1)$$

where $a : [0,+\infty) \to [0,+\infty)$, $q : [0,+\infty) \to R$, $\tau : [0,+\infty) \to [0,r]$, and F and $f : R \to R$ with

$$xf(x) > 0 \quad \text{for } x \neq 0,\ |x| \leq a \qquad (2)$$

(a is a positive constant) are continuous functions. We aim at giving general criterion for the zero solution of this delay equation being uniformly stable and asymptotically stable.

2. Main Result

Denote by $C[t_0 - r, t_0]$ the Banach space of continuous functions from $[t_0 - r, t_0]$ to R with the sup-norm

$$\|\varphi\|_{C[t_0-r,t_0]} = \max_{s\in[t_0-r,t_0]} \|\varphi(s)\|, \quad \text{for every } \varphi \in C[t_0 - r, t_0]. \tag{3}$$

We consider (1) for $t \geq t_0$ with the initial conditions (for any $t_0 \geq 0$)

$$x(t) = \varphi(t), \quad t_0 - r \leq t \leq t_0, \tag{4}$$

where $\varphi \in C[t_0 - r, t_0]$.

For an initial function $\varphi \in C[t_0 - r, t_0]$, we denote by $x(t; t_0, \varphi)$ the solution of (1) such that (4) holds.

Definition 1. The zero solution of (1) is said to be stable if for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta(t_0, \varepsilon) > 0$ such that if

$$\|\varphi\|_{C[t_0-r,t_0]} < \delta, \tag{5}$$

then

$$|x(t; t_0, \varphi)| < \varepsilon \quad \forall t \geq t_0. \tag{6}$$

The zero solution of (1) is uniformly stable if the above δ is independent of t_0.

Definition 2. The zero solution of (1) is said to be asymptotically stable if it is stable and if for any $t_0 \geq 0$, there exists $\delta(t_0) > 0$ such that if

$$\|\varphi\|_{C[-r,0]} < \delta, \tag{7}$$

then

$$|x(t; t_0, \varphi)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{8}$$

Theorem 3. *Assume that*

(1) *the zero solution to* (1) *is unique;*

(2) *if q is nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, then*

$$\lim_{t\to+\infty} q(t) = 0, \qquad \lim_{t\to+\infty} \int_{t-\tau(t)}^{t} |q(s)|\, ds = 0, \qquad p(t) \geq \mu > 0, \quad t \geq 0, \tag{9}$$

for a constant μ;

(3) $\lim_{t\to+\infty} \int_{t-\tau(t)}^{t} p(s) ds = A$;

(4) *if $A \neq 0$, then*

$$|f(x)| \leq \frac{\lambda |x|}{2A}, \quad \text{for } x \in R, \tag{10}$$

where $0 < \lambda < 1$.

Then the zero solution of (1) *is uniformly stable.*

Proof. For each $\varepsilon > 0$, we set

$$S(f, \varepsilon) := \sup\{|f(x)|; |x| \leq \varepsilon\}, \tag{11}$$

and when q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, we set

$$S(F, \varepsilon) := \sup\{|F(x)|; |x| \leq \varepsilon\}, \tag{12}$$

$$I(\varepsilon) := \inf\left\{xf(y); xy > 0, \frac{1-\lambda}{2}\varepsilon \leq |x| \leq \varepsilon, \frac{1-\lambda}{2}\varepsilon \leq |y| \leq \varepsilon\right\}. \tag{13}$$

From (3) and (2), it follows that for every $\varepsilon > 0$, there exists $t(\varepsilon) > 0$ such that

$$\int_{t-\tau(t)}^{t} p(s)\, ds < \begin{cases} \dfrac{1-\lambda}{4(S(f,\varepsilon)+1)}\varepsilon, & \text{if } A = 0, \\ \dfrac{1-\lambda}{4(S(f,\varepsilon)+1)} \min\{A, 1\}\varepsilon + A, & \text{if } A \neq 0, \end{cases} \quad \forall t > t(\varepsilon), \tag{14}$$

and when q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, such that

$$\int_{t-\tau(t)}^{t} |q(s)|\, ds < \frac{1-\lambda}{4(S(F,\varepsilon)+1)}\varepsilon, \quad \forall t > t(\varepsilon), \tag{15}$$

$$|q(t)| \leq \mu \frac{I(\varepsilon)}{2(S(F,\varepsilon)+1)(\varepsilon+1)}, \quad \forall t > t(\varepsilon). \tag{16}$$

We claim that for any $\varepsilon > 0$ and $t_0 \geq t(\varepsilon)$, if

$$\|\varphi\|_{C[t_0-r,t_0]} < \frac{1-\lambda}{2}\varepsilon, \tag{17}$$

then

$$|x(t; t_0, \varphi)| < \varepsilon \quad \forall t \geq t_0, \tag{18}$$

which means that the zero solution of (1) is eventually uniformly stable. Actually, if this is not true, then there exist

$$\varepsilon_0 \leq \min\{a, 1\} \tag{19}$$

and a solution

$$x(t) := x(t; t_0, \varphi) \tag{20}$$

to (1) with $\|\varphi\|_{C[t_0-r,t_0]} < ((1-\lambda)/2)\varepsilon$ and

$$t_0 > t(\varepsilon_0) \tag{21}$$

such that there is a $\bar{t} > t_0$,

$$|x(\bar{t})| \geq \varepsilon_0. \tag{22}$$

Define

$$t_2 := \inf \{t \geq t_0; |x(t)| = \varepsilon_0\}, \tag{23}$$

$$t_1 := \sup \left\{t_0 \leq t < t_2; |x(t)| = \frac{1-\lambda}{2}\varepsilon_0\right\}, \qquad V(x) = x^2, \quad x \in R. \tag{24}$$

Then, together with (21) and (22), we obtain

$$t(\varepsilon_0) < t_1 < t_2, \qquad V(x(t_1)) = \frac{(1-\lambda)^2}{4}\varepsilon_0^2, \qquad V(x(t_2)) > \varepsilon_0^2, \tag{25}$$

and, for $t \in (t_1, t_2)$,

$$\frac{(1-\lambda)^2}{4}\varepsilon_0^2 < V(x(t)) < \varepsilon_0^2, \tag{26}$$

and for arbitrary $\eta > 0$, there exists $\xi \in [t_2 - \eta, t_2]$ such that

$$V'(x(\xi)) > 0. \tag{27}$$

Therefore,

$$V'(x(t_2)) \geq 0. \tag{28}$$

This implies that

$$t_1 \geq t_2 - \tau(t_2). \tag{29}$$

In fact, if

$$t_1 < t_2 - \tau(t_2), \tag{30}$$

then by (23)–(25), we have

$$\frac{1-\lambda}{2}\varepsilon_0 \leq |x(t_2 - \tau(t_2))| \leq \varepsilon_0, \qquad t_2 - \tau(t_2) > t(\varepsilon_0). \tag{31}$$

It is not hard to see that we can choose t_1 and t_2 above to make $x(t)$ have constant sign in $[t_1, t_2]$.

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{32}$$

where b is a positive real number.

In this case, if $q(t) \equiv 0$, then

$$V'(x(t_2)) = -2p(t_2)x(t_2)f(x(t_2 - \tau(t_2))) < 0, \tag{33}$$

which contradicts with (28). Moreover, if

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{34}$$

for a positive real number b, then it is clear that we can require $\varepsilon_0 < b$. Hence,

$$V'(x(t_2)) = -2p(t_2)x(t_2)f(x(t_2 - \tau(t_2))) < 0, \tag{35}$$

which contradicts with (28) too.

Consequently, in this case we have the following observation.

Case I-1. If $A = 0$, then we deduce by (23), (24), (1), and (11) that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\
&\leq |x(t_2) - x(t_1)| \\
&\leq \int_{t_1}^{t_2} p(s)|f(x(s - \tau(s)))|\,ds \\
&\quad + \int_{t_1}^{t_2} |q(s)||F(x(s))|\,ds \\
&\leq S(f, \varepsilon_0)\int_{t_1}^{t_2} p(s)\,ds \\
&\leq S(f, \varepsilon_0)\int_{t_2 - \tau(t_2)}^{t_2} p(s)\,ds \\
&< \frac{\varepsilon_0}{2}.
\end{aligned} \tag{36}$$

This is clearly impossible.

Case I-2. If $A \neq 0$, then we deduce by (23), (24), (1), (11), and (14) that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\
&\leq |x(t_2) - x(t_1)| \\
&\leq \int_{t_1}^{t_2} p(s)|f(x(s - \tau(s)))|\,ds \\
&\quad + \int_{t_1}^{t_2} |q(s)||F(x(s))|\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\int_{t_1}^{t_2} p(s)\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\int_{t_2 - \tau(t_2)}^{t_2} p(s)\,ds \\
&\leq \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f, \varepsilon_0) + 1)}\min\{A, 1\}\varepsilon_0 + A\right) \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{37}$$

This is clearly impossible too.

Therefore, in this case, the zero solution of (1) is eventually uniformly stable. This, together with assumption (1), implies that the zero solution of (1) is uniformly stable.

Case II. q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$.

In this case, by virtue of (1), and assumption (2), (12), (13), and (16), we get

$$\begin{aligned} V'(x(t_2)) &= -2p(t_2)x(t_2)f(x(t_2-\tau(t_2))) \\ &\quad + 2x(t_2)q(t_2)F(x(t_2)) \\ &\le -2\mu I(\varepsilon_0) + 2\varepsilon_0\mu\frac{I(\varepsilon_0)}{2(S(F,\varepsilon_0)+1)(\varepsilon_0+1)}S(F,\varepsilon) \\ &\le -\mu I(\varepsilon) \\ &< 0, \end{aligned} \tag{38}$$

which contradicts with (28).

Consequently, in this case we have the following observation,

Case II-1. If $A = 0$, then we deduce by (23), (24), (1), (11), (12), (14), and (15) that

$$\begin{aligned} \frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_2)| - |x(t_1)| \\ &\le |x(t_2) - x(t_1)| \\ &\le \int_{t_1}^{t_2} p(s)\left|f(x(s-\tau(s)))\right| ds \\ &\quad + \int_{t_1}^{t_2} |q(s)||F(x(s))| ds \\ &\le S(f,\varepsilon)\int_{t_1}^{t_2} p(s)\,ds \\ &\quad + S(F,\varepsilon)\int_{t_1}^{t_2}|q(s)|\,ds \\ &\le S(f,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds \\ &\quad + S(F,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2}|q(s)|\,ds \\ &< \frac{1-\lambda}{4}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\ &< \frac{\varepsilon}{2}. \end{aligned} \tag{39}$$

This is a contradiction.

Case II-2. If $A \neq 0$, then we deduce by (23), (24), (1), (11), (12), (14), and (15) that

$$\begin{aligned} \frac{\varepsilon}{2} + \frac{\lambda}{2}\varepsilon &= |x(t_2)| - |x(t_1)| \\ &\le |x(t_2) - x(t_1)| \\ &\le \int_{t_1}^{t_2} p(s)\left|f(x(s-\tau(s)))\right| ds \\ &\quad + \int_{t_1}^{t_2} |q(s)||F(x(s))| ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\int_{t_1}^{t_2} p(s)\,ds + S(F,\varepsilon)\int_{t_1}^{t_2}|q(s)|\,ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\int_{t_2-\tau(t_2)}^{t_2} p(s)\,ds + S(F,\varepsilon)\int_{t_2-\tau(t_2)}^{t_2}|q(s)|\,ds \\ &\le \frac{\lambda\varepsilon_0}{2A}\left(\frac{1-\lambda}{4(S(f,\varepsilon_0)+1)}\min\{A,1\}\varepsilon_0 + A\right) \\ &\quad + \frac{1-\lambda}{4}\varepsilon \\ &< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\ &= \frac{\varepsilon}{2}. \end{aligned} \tag{40}$$

This is a contradiction too.

Therefore, in this case, the zero solution of (1) is eventually uniformly stable. This, together with assumption (1), implies that the zero solution of (1) is uniformly stable. □

Theorem 4. *Assume that*

(1) *the zero solution to* (1) *is unique;*

(2) *if* $q(t) \equiv 0$ *or*

$$F(x) \equiv 0 \quad \text{for } |x| \le b, \tag{41}$$

for a positive real number b, then

$$\int_0^{+\infty} p(s)\,ds = +\infty; \tag{42}$$

(3) *if q is nontrivial function and* $F(\cdot)$ *is nontrivial in any interval* $[-b,b]$ $(b>0)$, *then*

$$\begin{gathered} \lim_{t\to+\infty} q(t) = 0, \qquad \lim_{t\to+\infty}\int_{t-\tau(t)}^{t}|q(s)|\,ds = 0, \\ \int_0^{+\infty} q(s)\,ds < +\infty, \\ p(t) \ge \mu > 0, \quad t \ge 0, \end{gathered} \tag{43}$$

for a constant μ;

(4) $\lim_{t\to+\infty}\int_{t-\tau(t)}^{t} p(s)ds = A$;

(5) *if* $A \neq 0$, *then*

$$\left|f(x)\right| \le \frac{\lambda|x|}{2A}, \quad \text{for } x \in R, \tag{44}$$

where $0 < \lambda < 1$. *Then the zero solution of* (1) *is asymptotically stable.*

Proof. It follows from Theorem 3 that the zero solution of (1) is uniformly stable; that is, for arbitrarily given $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$\|\varphi\|_{C[t_0-r,t_0]} < \delta, \tag{45}$$

then

$$|x(t; t_0, \varphi)| < \varepsilon \quad \forall t \geq t_0. \tag{46}$$

Next, we will prove that

$$|x(t; t_0, \varphi)| \longrightarrow 0, \quad \text{as } t \longrightarrow +\infty. \tag{47}$$

First, we show that

$$\liminf_{t \to +\infty} |x(t; t_0, \varphi)| = 0. \tag{48}$$

Suppose that this is not true. Then

$$\liminf_{t \to +\infty} |x(t; t_0, \varphi)| > 0. \tag{49}$$

Hence, for the arbitrarily given

$$0 < \varepsilon < \min\{a, b\}, \tag{50}$$

there exist $0 < \varepsilon_0 < \varepsilon$ and $T > t_0$ such that

$$x(t; t_0, \varphi) > \varepsilon_0 \quad \forall t \geq T, \tag{51}$$

or

$$x(t; t_0, \varphi) < -\varepsilon_0 \quad \forall t \geq T. \tag{52}$$

Let us now consider

$$x(t; t_0, \varphi) > \varepsilon_0 \quad \forall t \geq T. \tag{53}$$

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \leq b, \tag{54}$$

for a positive real number b, we obtain by assumption (2), (46), (50), and (53)

$$\begin{aligned} x(t) &= x(T+r) - \int_{T+r}^{t} p(s) f(x(s-\tau(s)))\, ds \\ &\quad + \int_{T+r}^{t} q(s) F(x(s))\, ds \\ &\leq x(T+r) - \inf\{f(x); x \in [\varepsilon_0, \varepsilon]\} \int_{T+r}^{t} p(s)\, ds. \end{aligned} \tag{55}$$

This implies that

$$x(t) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \tag{56}$$

which contradicts with (53).

Case II. When q is a nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$, we obtain by assumptions (3), (46), (50), and (53)

$$\begin{aligned} x(t) &= x(T+r) - \int_{T+r}^{t} p(s) f(x(s-\tau(s)))\, ds \\ &\quad + \int_{T+r}^{t} q(s) F(x(s))\, ds \\ &\leq x(T+r) - \mu \inf\{f(x); x \in [\varepsilon_0, \varepsilon]\}(t - T - r) \\ &\quad + \sup\{|F(x)|; x \in (\varepsilon_0, \varepsilon)\} \int_{T+r}^{t} |q(s)|\, ds. \end{aligned} \tag{57}$$

This, together with assumption (2), implies that

$$x(t) \longrightarrow -\infty \quad \text{as } t \longrightarrow +\infty, \tag{58}$$

which contradicts with (53).

Moreover, in a similar way, we can prove that

$$x(t; t_0, \varphi) < -\varepsilon \quad \forall t \geq T \tag{59}$$

is impossible.

Therefore, (48) is true.

Based on (48), we will show that

$$\limsup_{t \to +\infty} |x(t; t_0, \varphi)| = 0. \tag{60}$$

Actually, if this is not true, that is,

$$\limsup_{t \to +\infty} |x(t; t_0, \varphi)| > 0, \tag{61}$$

then by (48) we see that there are ε_0 with

$$0 < \varepsilon_0 < \min\{a, b, 1\}, \tag{62}$$

and two sequences $\{\theta_n\}$ and $\{t_n\}$ such that

$$\begin{gathered} \theta_n < t_n, \quad n = 1, 2, \ldots, \\ \theta_n \longrightarrow +\infty \qquad t_n \longrightarrow +\infty \quad \text{as } n \longrightarrow +\infty, \\ V(x(\theta_n)) = \frac{(1-\lambda)^2}{4}\varepsilon_0^2, \qquad V(x(t_n)) > \varepsilon_0^2, \\ V'(x(t_n)) > 0, \end{gathered} \tag{63}$$

and for $t \in (\theta_n, t_n)$,

$$\frac{(1-\lambda)^2}{4}\varepsilon_0^2 < V(x(t)) < \varepsilon_0^2. \tag{64}$$

By the same reason as that in the proof of Theorem 3, we know that

$$t_n - \tau(t_n) \leq \theta_n \leq t_n. \tag{65}$$

Define $S(f, \varepsilon)$, $S(F, \varepsilon)$, $I(\varepsilon)$, and $t(\varepsilon)$ as those in the proof of Theorem 3. Then when n is large enough, we have

$$t_n > t(\varepsilon). \tag{66}$$

Case I. When $q(t) \equiv 0$ or

$$F(x) \equiv 0 \quad \text{for } |x| \le b, \tag{67}$$

where b is a positive real number.

Case I-1. If $A = 0$, then we deduce that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s) \left| f(x(s-\tau(s))) \right| ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)| \, |F(x(s))| \, ds \\
&\le S(f, \varepsilon_0) \int_{\theta_n}^{t_n} p(s) \, ds \\
&\le S(f, \varepsilon_0) \int_{t_n-\tau(t_n)}^{t_n} p(s) \, ds \\
&< \frac{\varepsilon_0}{2}.
\end{aligned} \tag{68}$$

This is impossible.

Case I-2. If $A \neq 0$, then we obtain

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s) \left| f(x(s-\tau(s))) \right| ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)| \, |F(x(s))| \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \int_{\theta_n}^{t_n} p(s) \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \int_{t_n-\tau(t_n)}^{t_n} p(s) \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \left(\frac{1-\lambda}{4(S(f, \varepsilon_0) + 1)} \min\{A, 1\} \varepsilon_0 + A \right) \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{\lambda}{2}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{69}$$

This is clearly impossible too.

Consequently, (60) is true in this case.

Case II. When q is nontrivial function and $F(\cdot)$ is nontrivial in any interval $[-b, b]$ $(b > 0)$.

Case II-1. If $A = 0$, then we deduce that

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s) \left| f(x(s-\tau(s))) \right| ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)| \, |F(x(s))| \, ds \\
&\le S(f, \varepsilon) \int_{\theta_n}^{t_n} p(s) \, ds \\
&\quad + S(F, \varepsilon) \int_{\theta_n}^{t_n} |q(s)| \, ds \\
&\le S(f, \varepsilon) \int_{t_n-\tau(t_n)}^{t_n} p(s) \, ds \\
&\quad + S(F, \varepsilon) \int_{t_n-\tau(t_n)}^{t_n} |q(s)| \, ds \\
&< \frac{1-\lambda}{4}\varepsilon + \frac{1-\lambda}{4}\varepsilon \\
&< \frac{\varepsilon}{2}.
\end{aligned} \tag{70}$$

This is a contradiction.

Case II-2. If $A \neq 0$, then we obtain

$$\begin{aligned}
\frac{\varepsilon_0}{2} + \frac{\lambda}{2}\varepsilon_0 &= |x(t_n)| - |x(\theta_n)| \\
&\le |x(t_n) - x(\theta_n)| \\
&\le \int_{\theta_n}^{t_n} p(s) \left| f(x(s-\tau(s))) \right| ds \\
&\quad + \int_{\theta_n}^{t_n} |q(s)| \, |F(x(s))| \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \int_{\theta_n}^{t_n} p(s) \, ds + S(F, \varepsilon) \int_{\theta_n}^{t_n} |q(s)| \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \int_{t_n-\tau(t_n)}^{t_n} p(s) \, ds \\
&\quad + S(F, \varepsilon) \int_{t_n-\tau(t_n)}^{t_n} |q(s)| \, ds \\
&\le \frac{\lambda\varepsilon_0}{2A} \left(\frac{1-\lambda}{4(S(f, \varepsilon_0) + 1)} \min\{A, 1\} \varepsilon_0 + A \right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1-\lambda}{4}\varepsilon \\
&<\frac{1-\lambda}{4}\varepsilon+\frac{\lambda}{2}\varepsilon+\frac{1-\lambda}{4}\varepsilon \\
&=\frac{\varepsilon}{2}.
\end{aligned}
\tag{71}
$$

This is a contradiction too.

Therefore, (60) is true in this case. So, (60) holds truly. This means that the zero solution of (4) is asymptotically stable. □

Remark 5. Our results are new comparing with the results in [2, 3] since $\tau(t)$ could go to 0 or a big number as $t \to +\infty$ and in this case $p(t)$ also could be very large in our theorems. Moreover, for the case of $A = 0$, the condition on f in our results is very weak.

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