Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 198405, 9 pages
http://dx.doi.org/10.1155/2013/198405

Research Article

Generalizations of Hölder's and Some Related Integral Inequalities on Fractal Space

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Received 5 May 2013; Accepted 8 July 2013

Academic Editor: Miguel Martin

Based on the local fractional calculus, we establish some new generalizations of Hölder's inequality. By using it, some related results on the generalized integral inequality in fractal space are investigated in detail.

1. Introduction

Let $p > 1$, $1/p + 1/q = 1$, $f(x)$, and $g(x)$ be continuous real-valued functions on $[a, b]$. Then, the famous Hölders inequality reads as

$$\int_a^b |f(x)\, g(x)|\, dx \le \left(\int_a^b |f(x)|^p dx\right)^{1/p} \left(\int_a^b |g(x)|^q dx\right)^{1/q}. \tag{1}$$

The renowned inequality of Hölder [1] is well celebrated for its beauty and its wide range of important applications to real and complex analysis and functional analysis, as well as many disciplines in applied mathematics. A large number of new proofs, various generalizations, refinements, variations, and applications of Hölder inequality have been investigated in the literature in [2–11]. Recently, it comes to our attention that an interesting local fractional integral Hölder's inequality, which was established by Yang [12], is as follows.

Let $f(x), g(x) \in C_\alpha(a, b)$, $p > 1$, $1/p + 1/q = 1$. Then,

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|\, (dx)^\alpha \\ &\quad \le \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p} \\ &\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^\alpha\right)^{1/q}. \end{aligned} \tag{2}$$

Recently, the local fractional calculus has attracted a lot of interest for scientists and engineers. Local fractional derivative had been introduced in [12–36]; that is the local fractional derivative was structured in [12–18, 26, 30–36], Jumarie gave the modified Riemann-Liouville derivative in [19, 20], the fractal derivative was considered in [21–25, 27–29], and the generalized fractal derivative was proposed by Chen et al. [25]. As a consequence, the theory of local fractional calculus becomes important for modelling problems for fractal mathematics and engineering on Cantor sets and it plays important role in many applications in several fields such as the theoretical physics [14, 18], the elasticity and fracture mechanics [14], the heat conduction theory [14, 16, 27], signal analysis [12, 13], the fluid mechanics [14], tensor analysis [14], Fourier and wavelet analysis [12, 13], optimization method [14], and complex analysis [12, 13]. For example, the local fractional Fokker-Planck equation was proposed in [18]. The local fractional Stieltjes transform was established in [37]. The fractal heat conduction problems were presented in [14, 27]. Local fractional improper integral was obtained in [38]. The principles of virtual work, minimum potential, and complementary energy in the mechanics of fractal media were investigated in [14]. Mean value theorems for local fractional integrals were considered in [39]. The diffusion problems in fractal media were reported in [24].

The purpose of this work is to establish some generalizations of inequality (2) and give its corresponding reverse version. Moreover, the obtained results will be applied to

establish local fractional integral reverse Minkowski inequality, Dresher's inequality, and its corresponding reverse version. This paper is divided into four sections. In Section 2, we recall some basic facts about local fractional calculus; in Section 3, we give some generalizations of the local fractional integral Hölder inequality and establish its corresponding reverse version; in Section 4, we apply the obtained results to establish reverse Minkowski inequality, Dresher's inequality, and its reverse form involving local fractional integral; some extensions of Minkowski and Dreshers inequalities are considered also.

2. Preliminaries

In this section, we recall the basic notions of local fractional calculus (see [12–14]).

2.1. Local Fractional Continuity of Functions. In order to study the local fractional continuity of nondifferentiable functions on fractal sets, we first give the following results.

Lemma 1 (see [14]). *Assume that F is a subset of the real line and is a fractal. Let $f : (F, d) \to (\Omega', d')$ be a bi-Lipschitz mapping. Then, there exist two positive constants ρ, τ, and $F \subset R$,*

$$\rho^s H^s(F) \le H^s(f(F)) \le \tau^s H^s(F), \tag{3}$$

such that for all $x_1, x_2 \in F$,

$$\rho^\alpha |x_1 - x_2|^\alpha \le |f(x_1) - f(x_2)| \le \tau^\alpha |x_1 - x_2|^\alpha. \tag{4}$$

From Lemma 1, we obtain easily

$$|f(x_1) - f(x_2)| \le \tau^\alpha |x_1 - x_2|^\alpha \tag{5}$$

such that

$$|f(x_1) - f(x_2)| \le \varepsilon^\alpha, \tag{6}$$

where α is fractal dimension of F. The result that is directly deduced from fractal geometry is related to fractal coarse-grained mass function $\gamma^\alpha[F, a, b]$ which reads

$$\gamma^\alpha[F, a, b] = \frac{H^\alpha(F \cap (a, b))}{\Gamma(1+\alpha)} \tag{7}$$

with

$$H^\alpha(F \cap (a, b)) = (b - a)^\alpha, \tag{8}$$

where H^α is α dimensional Hausdorff measure.

Notice that we consider the dimensions of any fractal spaces (e.g., Cantor spaces or like-Cantor spaces) as a positive number. It looks like Euclidean space because its dimension is also a positive number. The detailed results had been considered in [12–14].

Definition 2 (see [12, 14]). If there exists

$$|f(x) - f(x_0)| \le \varepsilon^\alpha, \tag{9}$$

with $|x - x_0|^\alpha \le \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then $f(x)$ is called local fractional continuous at $x = x_0$, denoted by $\lim_{x \to x_0} f(x) = f(x_0)$. $f(x)$ is local fractional continuous on the interval (a, b), denoted by

$$f(x) \in C_\alpha(a, b) \tag{10}$$

if (9) holds for $x \in (a, b)$.

Definition 3 (see [13, 14]). Assume that $f(x)$ is a nondifferentiable function of exponent α, $0 < \alpha \le 1$, which satisfies Hölder function of exponent α, then, for $x, y \in X$ such that

$$|f(x) - f(y)| \le C|x - y|^\alpha. \tag{11}$$

Definition 4 (see [13, 14]). A function $f(x)$ is continuous of order α, $0 < \alpha \le 1$, or shortly α continuous, if

$$|f(x) - f(x_0)| \le o\left((x - x_0)^\alpha\right). \tag{12}$$

Remark 5. Compared with (12), (9) is standard definition of local fractional continuity. Here, (11) is unified local fractional continuity [14].

2.2. Local Fractional Derivatives and Integrals

Definition 6 (see [12–14]). Let $f(x) \in C_\alpha(a, b)$. Local fractional derivative of $f(x)$ of order α at $x = x_0$ is given by

$$f^{(\alpha)}(x_0) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha(f(x) - f(x_0))}{(x - x_0)^\alpha}, \tag{13}$$

where $\Delta^\alpha(f(x) - f(x_0)) \cong \Gamma(1+\alpha)\Delta(f(x) - f(x_0))$.

For any $x \in (a, b)$, there exists

$$f^{(\alpha)}(x) = D_x^{(\alpha)} f(x) \tag{14}$$

denoted by

$$f(x) \in D_x^{(\alpha)}(a, b). \tag{15}$$

Local fractional derivative of high order is derived as

$$f^{(k\alpha)}(x) = \overbrace{D_x^{(\alpha)} \cdots D_x^{(\alpha)}}^{k \text{ times}} f(x), \tag{16}$$

and local fractional partial derivative of high order is derived as

$$\frac{\partial^{k\alpha} f(x)}{\partial x^{k\alpha}} = \overbrace{\frac{\partial^\alpha}{\partial x^\alpha} \cdots \frac{\partial^\alpha}{\partial x^\alpha}}^{k \text{ times}} f(x). \tag{17}$$

Definition 7 (see [12–14]). Let $f(x) \in C_\alpha(a, b)$. Local fractional integral of $f(x)$ of order α in the interval $[a, b]$ is deduced by

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \end{aligned} \tag{18}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \ldots, \Delta t_j, \ldots\}$, and $[t_j, t_{j+1}]$, $j = 1, 2, \ldots, N-1$, $t_0 = a$, $t_N = b$ is a partition of the interval $[a, b]$.

For convenience, we assume that

$$\begin{aligned} {}_{\alpha}I_{\alpha}^{(\alpha)} f(x) &= 0 \quad \text{if } a = b, \\ {}_{\alpha}I_{b}^{(\alpha)} f(x) &= -{}_{b}I_{\alpha}^{(\alpha)} f(x) \quad \text{if } a > b. \end{aligned} \tag{19}$$

For any $x \in (a, b)$, we can get

$$ {}_{\alpha}I_{x}^{(\alpha)} f(x) \tag{20}$$

denoted by

$$f(x) \in {}_{\alpha}I_{x}^{(\alpha)}(a, b). \tag{21}$$

Remark 8. If $f(x) \in D_x^{(\alpha)}(a, b)$, or ${}_{\alpha}I_x^{(\alpha)}(a, b)$ then we have

$$f(x) \in C_{\alpha}(a, b). \tag{22}$$

3. Some Generalizations of Hölder Inequality and Its Reverse Form

In the section, we give some generalizations of the inequality (2) and establish its reverse form.

Theorem 9 (reverse Hölder inequality). *Let $f(x)$, $g(x) \in C_{\alpha}(a, b)$, and let $0 < p < 1$, $1/p + 1/q = 1$. Then,*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|\, (dx)^{\alpha} \\ &\quad \geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^{\alpha} \right)^{1/p} \\ &\qquad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{1/q}. \end{aligned} \tag{23}$$

Proof. Set $c = 1/p$, $q = -pd$, and then we have $d = c/(c-1)$. By inequality (2), we obtain

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^{\alpha} \\ &\quad = \frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|^p |g(x)|^{-p} (dx)^{\alpha} \\ &\quad \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|^{pc} (dx)^{\alpha} \right)^{1/c} \\ &\qquad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^{-pd} (dx)^{\alpha} \right)^{1/d} \\ &\quad = \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|\, (dx)^{\alpha} \right)^{1/c} \\ &\qquad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{1-p}. \end{aligned} \tag{24}$$

In (24), multiplying both sides by $((1/\Gamma(1+\alpha)) \int_a^b |g(x)|^q (dx)^{\alpha})^{p-1}$ yields

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^{\alpha} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{p-1} \\ &\quad \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|\, (dx)^{\alpha} \right)^{p}. \end{aligned} \tag{25}$$

Inequality (25) implies

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)\, g(x)|\, (dx)^{\alpha} \\ &\quad \geq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |f(x)|^p (dx)^{\alpha} \right)^{1/p} \\ &\qquad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |g(x)|^q (dx)^{\alpha} \right)^{1/q}. \end{aligned} \tag{26}$$

□

Combining inequality (2) and Theorem 9, we can derive the following generalization.

Corollary 10. *Let $f_j(x) \in C_{\alpha}(a, b)$, let $p_j \in R$, $j = 1, 2, \ldots m$, and let $\sum_{j=1}^{m} 1/p_j = 1$. Then,*

(1) *for $p_j > 1$, one has*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m} |f_j(x)|\, (dx)^{\alpha} \\ &\quad \leq \prod_{j=1}^{m} \left(\int_a^b \frac{1}{\Gamma(1+\alpha)} |f_j(x)|^{p_j} (dx)^{\alpha} \right)^{1/p_j}, \end{aligned} \tag{27}$$

(2) *for $0 < p_1 < 1$, $p_j < 0$, $j = 2, \ldots, m$, one has*

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)} \int_a^b \prod_{j=1}^{m} |f_j(x)|\, (dx)^{\alpha} \\ &\quad \geq \prod_{j=1}^{m} \left(\int_a^b \frac{1}{\Gamma(1+\alpha)} |f_j(x)|^{p_j} (dx)^{\alpha} \right)^{1/p_j}. \end{aligned} \tag{28}$$

Proof. (1) If $m = 2$, p_1, p_2 are two positive constants and $1/p_1 + 1/p_2 = 1$. In particular, setting $p_1 > 1$, $p_2 > 1$, inequality (27) becomes inequality (2). Suppose (27) holds when $m \geq 2$. Using mathematical induction, let $p_1, p_2, \ldots, p_{m+1} > 0$ be real numbers with $\sum_{j=1}^{m+1} 1/p_j = 1$ and $f_j(x) \geq 0$, $j = 1, 2, \ldots, m, m+1$; we must have $p_i > 1$ for $j = 1, 2, \ldots, m, m+1$. In particular, we have

$$p_1 > 0, \qquad \frac{p_1}{p_1 - 1} > 0, \qquad \frac{1}{p_1} + \frac{p_1 - 1}{p_1} = 1. \tag{29}$$

By using Hölder's inequality (2), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^{m+1}\left|f_j(x)\right|(dx)^\alpha \\
&\quad= \frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|\prod_{j=2}^{m+1}\left|f_j(x)\right|(dx)^\alpha \\
&\quad\le \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left(\prod_{j=2}^{m+1}\left|f_j(x)\right|\right)^{p_1/(p_1-1)}(dx)^\alpha\right)^{(p_1-1)/p_1} \\
&\quad= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\prod_{j=2}^{m+1}\left|f_j(x)\right|^{p_1/(p_1-1)}(dx)^\alpha\right)^{(p_1-1)/p_1}
\end{aligned} \tag{30}$$

since

$$\begin{aligned}
&\frac{p_j(p_1-1)}{p_1} > 0 \quad \text{for } j = 2,\ldots,m,m+1, \\
&\sum_{j=2}^{m+1}\frac{p_1}{p_j(p_1-1)} = p_1(p_1-1)\sum_{j=2}^{m+1}\frac{1}{p_j} \\
&\qquad = p_1(p_1-1)\left(1-\frac{1}{p_1}\right) = 1.
\end{aligned} \tag{31}$$

Using induction hypothesis and inequality (30), we can get

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^{m+1}\left|f_j(x)\right|(dx)^\alpha \\
&\quad\le \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\prod_{j=2}^{m+1}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f_j(x)\right|^{p_1/(p_1-1)\cdot p_j(p_1-1)/p_1}\right.\right. \\
&\qquad\qquad\left.\left.\times(dx)^\alpha\right)^{p_1/p_j(p_1-1)}\right)^{(p_1-1)/p_1} \\
&\quad= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\prod_{j=2}^{m+1}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f_j(x)\right|^{p_j}(dx)^\alpha\right)^{1/p_j}.
\end{aligned} \tag{32}$$

Hence, this completes the proof. □

(2) The Proof of (28) is similar to the proof of (27). Clearly when $m = 2$, inequality (28) becomes Hölder's inequality (23). Now, suppose that (28) holds for some integer $m \ge 2$. Let $p_1, p_2, \ldots, p_m < 0$, and let $p_{m+1} \in R$ be such that $\sum_{j=1}^{m+1} 1/p_j = 1$, and let $f_j(x) \in L_{p,\alpha}[a,b]$, $j = 1,2,\ldots,m,m+1$. Note that $0 < p_{m+1} < 1$, since

$$p_1 > 0, \qquad 0 < \frac{p_1}{p_1-1} < 1, \qquad \frac{1}{p_1}+\frac{p_1-1}{p_1} = 1. \tag{33}$$

Observing Höder's inequality (23), we have

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^{m}\left|f_j(x)\right|(dx)^\alpha \\
&\quad= \frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|\prod_{j=2}^{m}\left|f_j(x)\right|(dx)^\alpha \\
&\quad\ge \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left(\prod_{j=2}^{m}\left|f_j(x)\right|\right)^{p_1/(p_1-1)}(dx)^\alpha\right)^{(p_1-1)/p_1} \\
&\quad= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\prod_{j=2}^{m}\left|f_j(x)\right|^{p_1/(p_1-1)}(dx)^\alpha\right)^{(p_1-1)/p_1}
\end{aligned} \tag{34}$$

unless $(1/\Gamma(1+\alpha))\int_a^b |f_1(x)|^{p_1}(dx)^\alpha = 0$.

Since

$$\begin{aligned}
&\frac{p_j(p_1-1)}{p_1} < 0 \quad \text{for } j = 2,\ldots,m, \\
&\frac{p_{m+1}(p_1-1)}{p_1} > 0, \\
&\sum_{j=2}^{m+1}\frac{p_1}{p_j(p_1-1)} = 1.
\end{aligned} \tag{35}$$

Combining induction hypothesis and (34), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_a^b \prod_{j=1}^{m+1}\left|f_j(x)\right|(dx)^\alpha \\
&\quad\ge \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_1(x)\right|^{p_1}(dx)^\alpha\right)^{1/p_1} \\
&\qquad\times\left(\prod_{j=2}^{m+1}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f_j(x)\right|^{p_1/(p_1-1)\cdot p_j(p_1-1)/p_1}\right.\right. \\
&\qquad\qquad\left.\left.\times(dx)^\alpha\right)^{p_1/p_j(p_1-1)}\right)^{(p_1-1)/p_1}
\end{aligned}$$

$$= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_1(x)|^{p_1}(dx)^\alpha\right)^{1/p_1} \times \prod_{j=2}^{m+1}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_j(x)|^{p_j}(dx)^\alpha\right)^{1/p_j} = \prod_{j=1}^{m+1}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_j(x)|^{p_j}(dx)^\alpha\right)^{1/p_j} \tag{36}$$

unless $(1/\Gamma(1+\alpha))\int_a^b |f_j(x)|^{p_j}(dx)^\alpha = 0$ for some $j = 1, 2, \ldots, m$.

4. Some Related Results

Theorem 11 (Minkowski inequality see [12]). *Let $f(x)$, $g(x) \in C_\alpha(a,b)$, $p > 1$. Then,*

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p(dx)^\alpha\right)^{1/p} \le \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q(dx)^\alpha\right)^{1/q}. \tag{37}$$

Next, we give reverse version of inequality (37).

Theorem 12 (reverse Minkowski's inequality). *Let $f(x)$, $g(x) \in C_\alpha(a,b)$, $0 < p < 1$. Then,*

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p(dx)^\alpha\right)^{1/p} \ge \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q(dx)^\alpha\right)^{1/q}. \tag{38}$$

Proof. Let

$$\begin{aligned} M &= \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha, \\ N &= \frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q(dx)^\alpha, \\ W &= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^q(dx)^\alpha\right)^{1/q}. \end{aligned} \tag{39}$$

By the Hölder inequality, in view of $0 < p < 1$, we have

$$\begin{aligned} W &= \frac{1}{\Gamma(1+\alpha)} \times \int_a^b \left(|f(x)|^p M^{1/p-1} + |g(x)|^p N^{1/p-1}\right)(dx)^\alpha \\ &\le \frac{1}{\Gamma(1+\alpha)}\int_a^b \left(|f(x)+g(x)|^p\left(M^{1/p}+N^{1/p}\right)^{1-p}\right)(dx)^\alpha \\ &= W^{1-p}\frac{1}{\Gamma(1+\alpha)}\int_a^b \left(|f(x)+g(x)|^p\right)(dx)^\alpha. \end{aligned} \tag{40}$$

By inequality (40), reverse Minkowski's inequality and the theorem are completely proved. □

Corollary 13. *Let $f_j(x) \in C_\alpha(a,b)$, $j = 1, 2, \ldots, m$.*

(1) *For $p > 1$, one has*

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha\right)^{1/p} \le \sum_{j=1}^m\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha\right)^{1/p}. \tag{41}$$

(2) *For $0 < p < 1$, one has*

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha\right)^{1/p} \ge \sum_{j=1}^m\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p(dx)^\alpha\right)^{1/p}. \tag{42}$$

Proof. (1) Using Theorem 11, we have

$$\begin{aligned} &\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p(dx)^\alpha \\ &\le \frac{1}{\Gamma(1+\alpha)}\int_a^b |f_1(x)|\left|\sum_{j=1}^m f_j(x)\right|^{p-1}(dx)^\alpha \\ &\quad + \cdots + \frac{1}{\Gamma(1+\alpha)}\int_a^b |f_m(x)|\left|\sum_{j=1}^m f_j(x)\right|^{p-1}(dx)^\alpha \\ &\le \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_1(x)|^p(dx)^\alpha\right)^{1/p} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^{q(p-1)}(dx)^\alpha\right)^{1/q} \\ &\quad + \cdots + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f_m(x)|^p(dx)^\alpha\right)^{1/p} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^{q(p-1)}(dx)^\alpha\right)^{1/q} \end{aligned}$$

$$= \sum_{j=1}^{m}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_j(x)\right|^p (dx)^\alpha\right)^{1/p} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^{m} f_j(x)\right|^p (dx)^\alpha\right)^{1/q}. \tag{43}$$

Multiplying $\left((1/\Gamma(1+\alpha))\int_a^b |\sum_{j=1}^m f_j(x)|^p (dx)^\alpha\right)^{1/q}$ to two sides of (43), we get that (41) holds.

(2) The proof of (42) is similar to the proof of (38), so we omit it here. □

Corollary 14. *Let $f_j(x) \in C_\alpha(a,b)$, $j = 1,2,\ldots,m$. Then,*

(1) *for $p > 1$, one has*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^{m} f_j(x)\right|^p (dx)^\alpha > \sum_{j=1}^{m}\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_j(x)\right|^p (dx)^\alpha, \tag{44}$$

(2) *for $0 < p < 1$, one has*

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|\sum_{j=1}^{m} f_j(x)\right|^p (dx)^\alpha < \sum_{j=1}^{m}\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f_j(x)\right|^p (dx)^\alpha. \tag{45}$$

Theorem 15 (Dresher's inequality). *Let $f(x)$, $g(x) \in C_\alpha(a,b)$, and let $0 < r < 1 < p$; then*

$$\left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)+g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)+g(x)|^r (dx)^\alpha}\right)^{1/(p-r)} \le \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/(p-r)} + \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/(p-r)}. \tag{46}$$

Proof. Combining inequality (2) and Theorem 11, we have

$$\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p (dx)^\alpha\right)^{1/(p-r)} \le \left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha\right)^{1/p}\right)^{p/(p-r)}$$

$$= \left(\left(\frac{\int_a^b |f(x)|^p (dx)^\alpha}{\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/p} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/p} + \left(\frac{\int_a^b |g(x)|^p (dx)^\alpha}{\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/p} \times \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/p}\right)^{p/(p-r)}$$

$$\le \left(\left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/(p-r)} + \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/(p-r)}\right) \times \left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r}\right)^{r/(p-r)}. \tag{47}$$

Using reverse Minkowski inequality (38), we have

$$\left(\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r}\right)^{r} \le \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^r (dx)^\alpha. \tag{48}$$

By (47) and (48), we deduce that (46) holds. This completes the proof of the theorem. □

Corollary 16. *Let $f_j(x) \in C_\alpha(a,b)$, $0 < r < 1 < p$, $j = 1,2,\ldots,m$. Then,*

$$\left(\frac{(1/\Gamma(1+\alpha))\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^r (dx)^\alpha}\right)^{1/(p-r)} \le \sum_{j=1}^{m}\left(\frac{(1/\Gamma(1+\alpha))\int_a^b \left|f_j(x)\right|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b \left|f_j(x)\right|^r (dx)^\alpha}\right)^{1/(p-r)}. \tag{49}$$

Theorem 17 (reverse Dresher's inequality). *Let $f(x), g(x) \in C_\alpha(a,b)$, $p \le 0 \le r < 1$. Then,*

$$\left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)+g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)+g(x)|^r (dx)^\alpha}\right)^{1/(p-r)} \ge \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/(p-r)} + \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/(p-r)}. \tag{50}$$

Proof. Let $\alpha_1 \ge 0$, $\alpha_2 \ge 0$, $\beta_1 > 0$, $\beta_2 > 0$, and $-1 < \lambda < 0$, using Radon's inequality (see [3])

$$\sum_{k=1}^n \frac{a_k^p}{b_k^{p-1}} < \frac{\left(\sum_{k=1}^n a_k\right)^p}{\left(\sum_{k=1}^n b_k\right)^{p-1}}, \quad a_k \ge 0,\ b_k > 0,\ 0 < p < 1. \tag{51}$$

We have

$$\frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda} + \frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda} \le \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^\lambda}, \tag{52}$$

if and only if sequence (α) and sequence (β) are proportional. Let

$$\begin{aligned}
\alpha_1 &= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p},\\
\beta_1 &= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r},\\
\alpha_2 &= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha\right)^{1/p},\\
\beta_2 &= \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r},
\end{aligned} \tag{53}$$

and set $\lambda = r/(p-r)$. Observing (52)–(53), we have

$$\begin{aligned}
&\frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda} + \frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda}\\
&= \frac{\left((1/\Gamma(1+\alpha))\int_a^b |f(x)|^p (dx)^\alpha\right)^{(\lambda+1)/p}}{\left((1/\Gamma(1+\alpha))\int_a^b |f(x)|^r (dx)^\alpha\right)^{\lambda/r}}\\
&\quad + \frac{\left((1/\Gamma(1+\alpha))\int_a^b |g(x)|^p (dx)^\alpha\right)^{(\lambda+1)/p}}{\left((1/\Gamma(1+\alpha))\int_a^b |g(x)|^r (dx)^\alpha\right)^{\lambda/r}}\\
&= \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |f(x)|^r (dx)^\alpha}\right)^{1/(p-r)}\\
&\quad + \left(\frac{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b |g(x)|^r (dx)^\alpha}\right)^{1/(p-r)}\\
&\le \frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^\lambda}\\
&= \left(\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha\right)^{1/p}\right]^{p/(p-r)}\right)\\
&\quad \times \left(\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r}\right]^{r/(p-r)}\right)^{-1}.
\end{aligned} \tag{54}$$

Since $-1 < \lambda = r/(p-r) < 0$, let $p < 0 < r$, and let $0 < r \le 1$, and by Theorem 12, we obtain, respectively,

$$\begin{aligned}
&\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^p (dx)^\alpha\right)^{1/p} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^p (dx)^\alpha\right)^{1/p}\right]^p\\
&\ge \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^p (dx)^\alpha,\\
&\left[\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)|^r (dx)^\alpha\right)^{1/r} + \left(\frac{1}{\Gamma(1+\alpha)}\int_a^b |g(x)|^r (dx)^\alpha\right)^{1/r}\right]^r\\
&\le \frac{1}{\Gamma(1+\alpha)}\int_a^b |f(x)+g(x)|^r (dx)^\alpha.
\end{aligned} \tag{55}$$

Observing (54)-(55), we obtain the desired results, and the theorem is completely proved. □

Corollary 18. *Let $f_j(x) \in C_\alpha(a,b)$, and let $p \le 0 \le r < 1$, $j = 1, 2, \ldots, m$. Then,*

$$\left(\frac{(1/\Gamma(1+\alpha))\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b \left|\sum_{j=1}^m f_j(x)\right|^r (dx)^\alpha}\right)^{1/(p-r)} \ge \sum_{j=1}^m \left(\frac{(1/\Gamma(1+\alpha))\int_a^b \left|f_j(x)\right|^p (dx)^\alpha}{(1/\Gamma(1+\alpha))\int_a^b \left|f_j(x)\right|^r (dx)^\alpha}\right)^{1/(p-r)}. \tag{56}$$

Acknowledgments

The authors would like to thank the anonymous referees for their valuable comments on the original version of this paper. This work was supported by the NNSFC (no. 11201433) and Scientific Research Project of Guangxi Education Department (no. 201204LX672).

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