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Research Article

On a Class of Anisotropic Nonlinear Elliptic Equations with Variable Exponent

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Based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solution for a class of anisotropic nonlinear elliptic equations with variable exponent $\overrightarrow{p(x)}$ growth. Furthermore, we also obtain that the weak solution is locally bounded and regular; that is, the weak solution is $L^{\infty}_{\mathrm{loc}}(\Omega)$ and $C^{1,\alpha}(\Omega)$.

1. Introduction

Since the variable exponent spaces have reflected into a various range of applications such as non-Newtonian fluids, plasticity, image processing, and nonlinear elasticity [1–4], some authors began to study the various properties of variable exponent space and some nonlinear problems with variable exponent growth. Edmunds et al. [5], Fan and Zhao [6] obtained that the variable exponent space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are reflexive Banach spaces under suitable conditions on $p(x)$. Later, Edmunds and Rákosník [7], Fan et al. [8] proved some continuous and compact Sobolev embedding theorems for the variable exponent spaces $W^{k,p(x)}(\Omega)$. For the anisotropic variable exponents spaces, in 2008, Mihăilescu et al. [9] studied the eigenvalue problems for a class of anisotropic quasilinear elliptic equations with variable exponents. In 2011, Boureanu et al. [10] proved the existence of multiple solutions for a class of anisotropic elliptic equations with variable exponents. Recently, Stancu-Dumitru [11, 12] has studied the existence of nontrivial solutions for a class of nonhomogeneous anisotropic problem. In particular, Fan [13] established some embedding theorems for anisotropic variable exponent Sobolev spaces.

In this paper, we investigate the following anisotropic nonlinear elliptic equation:

$$\begin{aligned} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} T_i(x,\nabla u) &= f - \partial_{x_i} h_i, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ with Lipschitz conditions boundary, a_i and T_i $(i = 1, 2, \ldots, N)$ are Carathéodory functions, h_i and f satisfy suitable conditions, and $\partial_{x_i} a_i(x,u,\nabla u)$ and $\partial_{x_i} h_i(x)$ are Einstein Sum; that is, $\partial_{x_i} a_i(x,u,\nabla u) = \sum_{i=1}^{N}(\partial a_i(x,u,\nabla u)/\partial x_i)$, $\partial_{x_i} h_i(x) = \sum_{i=1}^{N}(\partial h_i(x)/\partial x_i)$. We usually use critical theory to obtain the existence of weak solutions. However, since the problem (1) has no variational structure, we cannot define the energy functional for the problem (1). Therefore, based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solutions for the problem (1) in $W_0^{1,\overrightarrow{p(x)}}(\Omega)$. Furthermore, we obtain that the weak solution for the problem (1) is locally bounded.

In particular, for the special case

$$\begin{aligned} -\partial_{x_i} a_i(x, \nabla u) &= f - \partial_{x_i} h_i, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

we obtain that the weak solution is $C^{1,\alpha}(\Omega)$. To our knowledge, the above two problems have not been deeply studied in the anisotropic variable exponent Sobolev spaces.

The paper is organized as follows. In Section 2, we recall some results on anisotropic variable exponent Sobolev spaces and state our main results. In Section 3, we prove the existence, uniqueness and locally bounded of weak solution for the problem (1). In Section 4, the regularity of weak solutions for the problem (2) is proved.

2. Preliminary and Main Results

This section is dedicated to a general overview on the $W^{1,\overrightarrow{p(x)}}(\Omega)$ and $L^{p(x)}(\Omega)$; for a deeper treatment on these spaces, see [5, 7, 8, 13, 14].

Let

$$C_+\left(\overline{\Omega}\right) := \left\{h : h \in C\left(\overline{\Omega}\right), h(x) > 1 \ \forall x \in \overline{\Omega}\right\}. \tag{3}$$

Set

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \text{ and } h^- = \inf_{x\in\overline{\Omega}} h(x), \tag{4}$$

for any $p(x) \in C_+(\overline{\Omega})$, and we define the variable exponent Lebesgue space

$$\begin{aligned} &L^{p(\cdot)}(\Omega) \\ &\quad = \Big\{u : \text{a measurable real} \\ &\qquad - \text{valued function such that} \int_\Omega |u(x)|^{p(x)} dx < \infty\Big\}. \end{aligned} \tag{5}$$

We define the norm of $L^{p(x)}(\Omega)$:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\mu > 0 \ ; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}. \tag{6}$$

From [6], we have the following:

(1) $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space;

(2) $L^{p(x)}(\Omega)$ is reflexive, if and only if $1 < p^- \le p^+ < \infty$;

(3) Hölder inequality.

For all $u, v \in L^{p(x)}(\Omega)$,

$$\left|\int_\Omega uv\,dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{p'(x)}(\Omega)}, \tag{7}$$

where $1/p(x) + 1/p'(x) = 1$, $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$.

Now, we recall some results on anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p(x)}}(\Omega)$ [13]; set

$$\begin{aligned} &L_+^\infty(\Omega) \\ &\quad := \left\{p \in L^\infty(\Omega) \ : p(x) \ge 1 \quad \text{for a.e. } x \in \Omega\right\}. \end{aligned} \tag{8}$$

Denote $\overrightarrow{p(x)} = (p_1(x), p_2(x), \ldots, p_N(x)) \in (L_+^\infty(\Omega))^N$ and define

$$\begin{aligned} &\forall x \in \Omega, \\ &\quad p_-(x) = \min\{p_1(x), p_2(x), \ldots, p_N(x)\}, \\ &\quad p_+(x) = \max\{p_1(x), p_2(x), \ldots, p_N(x)\}. \end{aligned} \tag{9}$$

The anisotropic variable exponent Sobolev space

$$\begin{aligned} &W^{1,\overrightarrow{p(x)}}(\Omega) \\ &\quad = \left\{u \in L^{p_+(x)}(\Omega) : D_i u \in L^{p_i(x)}(\Omega) \quad \text{for } i = 1, 2, \ldots, N\right\} \end{aligned} \tag{10}$$

is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p(x)}}(\Omega)} = \|u\|_{p_+(x)} + \sum_{i=1}^N \|D_i u\|_{p_i(x)}. \tag{11}$$

If $p_i(x) \in L_+^\infty(\Omega)$, for $i = 1, 2, \ldots, N$, $p_i^- > 1$, then $W^{1,\overrightarrow{p(x)}}(\Omega)$ is reflexive. We define $W_0^{1,\overrightarrow{p(x)}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)} = \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i(x)}(\Omega)}, \tag{12}$$

and $W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a reflexive Banach space (see [9]).

Let

$$\begin{aligned} \overline{p}(x) &= \frac{N}{\sum_{i=1}^N (1/p_i(x))}, \\ \overline{p}^*(x) &= \begin{cases} \dfrac{N\overline{p}(x)}{N - \overline{p}(x)} & \text{if } \overline{p}(x) < N, \\ +\infty & \text{if } \overline{p}(x) \ge N, \end{cases} \\ p_\infty(x) &= \max\{\overline{p}^*(x), p_+(x)\}. \end{aligned} \tag{13}$$

Hence, we have the following embedding theorem for $W_0^{1,\overrightarrow{p(x)}}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p(\cdot)} = (p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$.

Theorem 1 (see [13, Theorem 2.5]).

(i) *If* $q \in C_+^0(\overline{\Omega})$ *and*

$$q(x) < p_\infty(x) \quad \forall x \in \overline{\Omega}, \tag{14}$$

then $W_0^{1,\overrightarrow{p(\cdot)}}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$. *The embedding is compact.*

(ii) *If* $\overline{p}(x) > N$ *for all* $x \in \overline{\Omega}$, *then there exists* $\beta \in (0,1)$ *such that* $W_0^{1,\overrightarrow{p(\cdot)}}(\Omega) \hookrightarrow\hookrightarrow C^{0,\beta}(\overline{\Omega})$.

The embedding is also compact.

Theorem 2 (see [13, Theorem 2.6]). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $\overrightarrow{p(\cdot)} = (p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$. *Suppose that*

$$p_+(x) < \overline{p}^*(x) \quad \forall x \in \overline{\Omega}. \tag{15}$$

Then one has

$$\|u\|_{L^{p+(\cdot)}(\Omega)} \le C\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i(\cdot)}(\Omega)} \quad \forall u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega), \tag{16}$$

where C *is a positive constant independent of* $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$.

Remark 3. From [13], we know that if $p_+(x) < \overline{p}^*(x)$, for all $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$, for every $i = 1, 2, \ldots, N$, then $\|u\|_{L^{p_i(x)}(\Omega)} \le C\|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}$, where C is a positive constant independent of $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$.

Assume that $a_i : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $T_i : \Omega\times\mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions and satisfy the following:

(A1) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, we have

$$\lambda\sum_{i=1}^{N}|\xi_i|^{p_i(x)} \le \sum_{i=1}^{N} a_i(x,s,\xi)\,\xi_i \tag{17}$$

and $|a_i(x,s,\xi)| \le \beta[|s|^{p_\infty(x)/p_i'(x)} + |\xi_i|^{p_i(x)-1}]$, where $\lambda, \beta > 0$;

(A2) for a.e. $x \in \overline{\Omega}$, for all $\xi_i \ne \xi_i'$, $\gamma > 0$ and $\varepsilon > 0$, a_i satisfies

$$\begin{aligned}&\left[a_i(x,s,\xi) - a_i\left(x,s,\xi'\right)\right]\left(\xi_i - \xi_i'\right)\\ &\quad\ge \gamma\left(\varepsilon + |\xi_i| + |\xi_i'|\right)^{p_i(x)-2}\left|\xi_i - \xi_i'\right|^2;\end{aligned} \tag{18}$$

(A3) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^N$, $i, j = 1, 2, \ldots, N$,

$$c_1|\zeta|^2 \le \sum_{i=1}^{N}\sum_{j=1}^{N} a_{\xi_j}^i(x,s,\xi)\,\zeta_i\zeta_j \le c_2\left(1 + \sum_{i=1}^{N}|\xi_i|^{p_i(x)-2}\right)|\zeta|^2, \tag{19}$$

where $a_{\xi_j}^i(x,s,\xi) = \partial a_i(x,s,\xi)/\partial\xi_j$;

(A4) for a.e. $x \in \overline{\Omega}$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, for all $\zeta \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$c_3|\zeta|^2 \le \sum_{i=1}^{N} a_{x_k}^i(x,s,\xi)\,\zeta_i \le c_4\left(|\zeta|^2 + |s|^{q(x)}\right), \tag{20}$$

where $a_{x_k(x,s,\xi)}^i = \partial a_i(x,s,\xi)/\partial x_k$;

(T1) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$\sum_{i=1}^{N}\left|T_i(x,\xi)\right| \le c_5|s|^{q(x)/2} \le \sum_{i=1}^{N} m_i|\xi_i|^{p_i(x)-1}, \tag{21}$$

where $m_i \in L^{b_i(x)}(\Omega)$ with $1/b_i(x) = 1/p_\infty'(x) - 1/p_i'(x)$, $c_1, \ldots, c_5$ are positive constants.

We enumerate the hypotheses concerning f and h_i,

(F1) $f \in L^{p_\infty'(x)}(\Omega)$;

(F2) for a.e. $x \in \overline{\Omega}$, $s \in \mathbb{R}$, for some $k \in \{1, 2, \ldots, N\}$, $|f_{x_k}| \le c_6|s|^{q(x)/2}$, where $f_{x_k} = \partial f/\partial x_k$;

(H1) for $i = 1, 2, \ldots, N$, $h_i \in L^{p_i'(x)}(\Omega)$;

(H2) for a.e. $x \in \overline{\Omega}$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, $\zeta \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$c_7|\zeta|^2 \le \sum_{i=1}^{N} h_{x_k}^i\zeta_i \le c_8\left(|\zeta|^2 + |s|^{q(x)}\right), \tag{22}$$

where $h_{x_k}^i = \partial h_i/\partial x_k$, c_6, c_7, c_8 are positive constants.

Remark 4. Now, we give a simple example. Let $k = 1$, $x_0 = (3,0)$, $x = (x_1, x_2) \in \Omega = B(x_0, 1) \subset \mathbb{R}^2$, $u(x) = x_1^2 + x_2^2$, $\zeta = x_1^{3/2} + x_2^{3/2}$, $h_1(x) = x_1^2 + 2x_1x_2$, $h_2(x) = (1/2)x_1^2 + x_1x_2 + x_2^2$, $T_1(x,\xi) = x_1 + x_2 + \sum_{i=1}^{2}\xi_i$, $T_2(x,\xi) = x_1x_2 + |\xi|^2$, $f(x) = (1/2)x_1^2 + x_1x_2$. By a simple calculation, we obtain that T_i satisfies (T1), f satisfies (F2), and h_i satisfies (H2), where $i = 1, 2$.

Now, we define the weak solution of the problem (1). A function $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a weak solution of the problem (1), if for all $\varphi \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$,

$$\sum_{i=1}^{N}\int_\Omega\left[a_i(x,u,\nabla u)\,\varphi_{x_i} + T_i(x,\nabla u)\,\varphi\right] = \int_\Omega\left[f\varphi + \sum_{i=1}^{N} h_i\varphi_{x_i}\right], \tag{23}$$

where $\varphi_{x_i} = \partial\varphi/\partial x_i$.

Theorem 5. *Let* $\Omega \subset \mathbb{R}^N$ $(N > 2)$ *be a bounded open subset, for every* $i = 1, 2, \ldots, N$, $x \in \Omega$, *assume* $\overline{p(x)} < N$, $q \in C_+^0(\Omega)$ *and* $q(x) < p_\infty = \overline{p}^*(x)$, $p_i(x)$ *satisfies*

$$2 \le p_i(x) \le \frac{2Nr_i(x)}{Nr_i(x) - 2r_i(x) - 2N} \tag{24}$$

and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, *where* $r_i(x) \ge \max\{N, p_\infty(x)p_i(x)/(p_\infty(x) - p_i(x))\}$. a_i *satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1) hold, and* T_i *satisfies (T1) and the following Lipschitz condition:*

$$\left|T_i(x,\xi) - T_i\left(x,\xi'\right)\right| \le k_i(x)\left(|\xi_i| + |\xi_i'|\right)^{\delta_i(x)}\left|\xi_i - \xi_i'\right|, \tag{25}$$

where $k_i(x) \in L^{r_i(x)}(\Omega)$, $0 \le \delta_i(x) \le p_i(x)/N - p_i(x)/r_i(x) + (p_i(x) - 2)/2$. *Then there exists a unique weak solution* u *for the problem* (1). *Furthermore,* $u \in L_{\mathrm{loc}}^\infty(\Omega)$.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a bounded open subset. Suppose (A3), (A4) and (F2), (H2) hold, for $i = 1, 2, \ldots, N$, $2 \le p_i(x) \le 2N/(N-2)$, $q \in C_+^0(\Omega)$ and $q(x) < \overline{p}^*(x)$. If u is a weak solution of the problem* (2)*, then $u \in C^{1,\alpha}(\Omega)$, for all $0 < \alpha < 1$.*

3. The Proof of Theorem 5

We consider the following problem:

$$\begin{aligned} -\partial_{x_i} a_i(x, v_n, \nabla v_n) + \sum_{i=1}^{N} T_n^i(x, \nabla v_n) &= f - \partial_{x_i} h_i, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{26}$$

where $T_n^i(x, \nabla v)$ is the truncation at levels $\pm n$ of T_i. Due to [15], we obtain that there exists a weak solution $v_n \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ for the problem (26).

Lemma 7. *If $\overline{p(x)} < N$, (A1), (A2), (T1), (F1), and (H1) hold. Assume $v_n \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a weak solution of the problem* (26)*; then one has*

$$\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_n\right|^{p_i(x)} \le C. \tag{27}$$

Proof. Let $K \in \mathbb{R}^+$; K will be chosen later. There exist m measurable subsets $\Omega_1, \ldots, \Omega_m$ of Ω and m functions $v_1, \ldots, v_m$ such that if $i \ne j$, $\Omega_i \bigcap \Omega_j = \emptyset$. For $t \in \{1, 2, \ldots, m-1\}$, $|\Omega_t| = K$, if $t = m$, $|\Omega_t| \le K$. Let $\{x \in \Omega : |\nabla v_t| \ne 0\} \subset \Omega_t$, $\nabla v = \nabla v_t$ a.e. in Ω_t, $\nabla(v_1 + v_2 + \cdots + v_t)v_t = (\nabla v)v_t$, $v_1 + v_2 + \cdots + v_t = v$ in Ω. For $t \in \{1, 2, \ldots, m\}$, if $v_t \ne 0$, $\operatorname{sign}(v) = \operatorname{sign}(v_t)$.

Choose v_t as test function of the problem (26) and fix $t \in \{1, 2, \ldots, m\}$. Using Young inequality, Hölder inequality, the embedding theorem of the $W_0^{1,\overrightarrow{p(x)}}(\Omega)$, and the hypotheses (A1), (F1), and (H1), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} \\ &\quad \le c_1 \left(\|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega \left|T^i(x, \nabla v)\right| |v_t| \right. \\ &\qquad \left. + \sum_{i=1}^{N} \int_\Omega |h_i|^{p_i'(x)} \right), \end{aligned} \tag{28}$$

where $c_1 > 0$ is a constant, $A_t = \prod_{i=1}^{N} \|\partial_{x_i} v_t\|_{L^{p_i(x)}(\Omega)}$. By (T1), Young inequality and Hölder inequality, we obtain, for $c_2 > 0$,

$$\begin{aligned} &\left| \sum_{i=1}^{N} \int_\Omega T^i(x, \nabla v) v_t \right| \\ &\quad \le \sum_{i=1}^{N} m_i \int_\Omega \left|\partial_{x_i} v\right|^{p_i(x)-1} |v_t| \\ &\quad \le \sum_{i=1}^{N} m_i \sum_{s=1}^{t} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)-1} |v_t| \\ &\quad \le c_2 \sum_{i=1}^{N} \sum_{s=1}^{t} \\ &\qquad \times \left[\int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} |\Omega_s|^{1/p_i^- - 1/p_\infty^+} \right. \\ &\qquad \left. + A_t^{p_+^+/N} |\Omega_s|^{1/p_i^- - 1/p_\infty^+} \right] \\ &\quad \le c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \\ &\qquad \times \left[\int_{\Omega_t} \left|\partial_{x_i} v_t\right|^{p_i(x)} \right. \\ &\qquad \left. + \sum_{s=1}^{t-1} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} + A_t^{p_+^+/N} \right], \end{aligned} \tag{29}$$

where $p_i^- = \inf_{x \in \overline{\Omega}} p_i(x)$, $p_+^+ = \sup_{x \in \overline{\Omega}} p_+(x)$, $p_\infty^+ = \sup_{x \in \overline{\Omega}} p_\infty(x)$.

Combining (28) with (29), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} \\ &\quad \le c_1 \left\{ \|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p_i'(x)} \right. \\ &\qquad + c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \\ &\qquad \times \left[\int_{\Omega_t} \left|\partial_{x_i} v_t\right|^{p_i(x)} \right. \\ &\qquad \left. \left. + \sum_{s=1}^{t-1} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} + A_t^{p_+^+/N} \right] \right\}. \end{aligned} \tag{30}$$

Let K satisfy $c_1 c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} < 1$; then we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} \\ &\quad \le c_3 \left\{ \|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p_i'(x)} \right. \\ &\qquad + \sum_{s=1}^{t-1} \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \left(\sum_{j=1}^{N} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_j(x)} \right) \\ &\qquad \left. + \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} A_t^{p_+^+/N} \right\}, \end{aligned} \tag{31}$$

where $c_3 > 0$ is a constant. Let $t = 1$; we obtain

$$\begin{aligned}\int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)} &\le \sum_{i=1}^N \int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)} \\ &\le c_3 \left[\|f\|_{L^{p'_\infty(x)}} A_1^{1/N} + \sum_{i=1}^N \int_\Omega |h_i|^{p'_i(x)} \right. \\ &\quad \left. + \sum_{i=1}^N K^{1/p_i^- - 1/p_\infty^+} A_1^{p_+^+/N} \right].\end{aligned} \tag{32}$$

Now, we recall the following classical inequality, for $a_1, a_2, \ldots, a_N$ are positive numbers,

$$\prod_{i=1}^N a_i^{1/N} \le \frac{1}{N} \sum_{i=1}^N a_i. \tag{33}$$

From (33), we have

$$A_1 = \prod_{i=1}^N \left\|\partial_{x_i} v_1\right\|_{L^{p_i(x)}(\Omega)} \le \left(\frac{1}{N} \sum_{i=1}^N \left\|\partial_{x_i} v_1\right\|_{L^{p_i(x)}(\Omega)} \right)^N. \tag{34}$$

Combining (32) with (34), there exists a constant c_4 such that

$$\sum_{i=1}^N \int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)} \le c_4. \tag{35}$$

Furthermore, put (35) in (31) and iterate on t; we have

$$\begin{aligned}\sum_{i=1}^N \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} &\le c_5 \left[\|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^N \int_\Omega |h_i|^{p'_i(x)} \right. \\ &\quad \left. + 1 + \sum_{i=1}^N K^{1/p_i^- - 1/p_\infty^+} A_1^{p_+^+/N} \right].\end{aligned} \tag{36}$$

Therefore, we obtain

$$\sum_{i=1}^N \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} \le c_6, \tag{37}$$

for some constant $c_6 > 0$. □

Lemma 8. *If $\overline{p(x)} < N$, and a_i satisfies the hypotheses (A1), (A2), the hypotheses (T1), (F1), and (H1) hold. Then there exists at least a weak solution for the problem* (1).

Proof. By (27), we obtain that there exists a sequence $\partial_{x_i} u_n$ which is bounded in $L^{p_i(x)}(\Omega)$; we have for $i = 1, 2, \ldots, N$, $\partial_{x_i} u_n \rightharpoonup \partial_{x_i} u$ weakly in $L^{p_i(x)}(\Omega)$ and $u_n \to u$ strongly in $L^{p_i(x)}(\Omega)$. We argue as in [16] to prove

$$\partial_{x_i} u_n \longrightarrow \partial_{x_i} u \quad \text{a.e. in } \Omega \text{ for } i = 1, 2, \ldots, N. \tag{38}$$

From (38), we obtain

$$\begin{aligned} a_i(x, u_n, \nabla u_n) &\longrightarrow a_i(x, u, \nabla u) \quad \text{a.e. in } \Omega, \\ T_n^i(x, \nabla u_n) &\longrightarrow T_n^i(x, \nabla u) \quad \text{a.e. in } \Omega. \end{aligned} \tag{39}$$

By the hypotheses (A1) and (T1), for $\kappa_i(x) \in [1, p_i'(x)]$ and $c > 0$, we have

$$\begin{aligned}\int_\Omega \left|a_i(x, u_n, \nabla u_n)\right|^{\kappa_i(x)} &\le c \left[\left(\int_\Omega |u|^{p_\infty(x)} \right)^{\kappa_i(x)/p_i'(x)} \right. \\ &\quad \left. + \left(\int_\Omega \left|\partial_{x_i} u\right|^{p_i(x)} \right)^{\kappa_i(x)/p_i'(x)} \right] |\Omega|^{(p_i'^+ - \kappa_i^-)/p_i'^-}, \\ \int_\Omega \left|T_n^i(x, \nabla u_n)\right|^{\kappa_i(x)} &\le c \left[\int_\Omega \left|\partial_{x_i} u\right|^{p_i(x)} \right]^{\kappa_i(x)/p_i'(x)} |\Omega|^{(p_i'^+ - \kappa_i^-)/p_i'^-}.\end{aligned} \tag{40}$$

By Vitali Theorem, we have

$$\begin{aligned} a_i(x, u_n, \nabla u_n) &\longrightarrow a_i(x, u, \nabla u) \quad \text{strongly in } L^{\kappa_i(x)}(\Omega), \\ T_n^i(x, \nabla u_n) &\longrightarrow T_i(x, \nabla u) \quad \text{strongly in } L^{\kappa_i(x)}(\Omega), \end{aligned} \tag{41}$$

for any $\kappa_i(x) \in [1, p_i'(x)]$. Hence, we complete the proof of Lemma 8. □

Lemma 9. *If $\overline{p(x)} < N$, $p_\infty(x) = \overline{p}^*(x)$, a_i satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1) hold, for $i = 1, 2, \ldots, N$, $x \in \Omega$, $p_i(x)$ satisfies*

$$2 \le p_i(x) \le \frac{2N r_i(x)}{N r_i(x) - 2 r_i(x) - 2N}, \tag{42}$$

where $r_i(x) \ge \max\{N, p_\infty(x) p_i(x)/(p_\infty(x) - p_i(x))\}$, T_i satisfies (T1) and (25). *Then there exists a unique weak solution for the problem* (1).

Proof. Suppose that there exist two distinct weak solutions u and v for the problem (1). Denote $\varphi = \max\{u - v, 0\}$, and for $t \in [1, \sup \varphi]$, let $\Omega_t = \{x \in \Omega : t < \varphi < \sup \varphi\}$. Otherwise, if $t \in [0, 1)$, we choose $m = 1/t$ and let $\Omega_m = \{x \in \Omega : m < \varphi < \sup \varphi\}$. We choose

$$\varphi_t = \begin{cases} \varphi - t, & \varphi > t, \\ 0, & \text{otherwise}, \end{cases} \tag{43}$$

as test function. From (A2) and (25), we have

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_t} \left|\partial_{x_i} \varphi_t\right|^2 \left(\varepsilon + \left|\partial_{x_i} u\right| + \left|\partial_{x_i} v\right|\right)^{p_i(x)-2} &\le \frac{1}{\gamma} \sum_{i=1}^N \int_{\Omega_t} k_i(x) \left(\left|\partial_{x_i} u\right| + \left|\partial_{x_i} v\right|\right)^{\delta_i(x)} \left|\partial_{x_i} \varphi_t\right| \varphi_t\end{aligned} \tag{44}$$

when $\delta_+(x) \geq (p_i(x)-2)/2$. By Young inequality, for c independent on t, we obtain

$$\sum_{i=1}^{N}\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\left(\varepsilon+\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{p_i(x)-2} \leq c\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)-p_i(x)+2}\varphi_t^2. \tag{45}$$

Then we have

$$\begin{aligned}\frac{1}{C}\left(\int_{\Omega_t}\varphi_t^{2^*}\right)^{2/2^*} &\leq \prod_{i=1}^{N}\left(\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\right)^{1/N}\\ &\leq c_1\prod_{i=1}^{N}\left(\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\left(\varepsilon+\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{p_i(x)-2}\right)^{1/N}\\ &\leq c_1\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)-p_i(x)+2}\varphi_t^2\\ &\leq c_1\left(\int_{\Omega_t}\varphi_t^{2^*}\right)^{2/2^*}\\ &\quad\times\sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)}\right)^{2/N},\end{aligned} \tag{46}$$

where c_1 is a constant independent on t. Hence, we have

$$\frac{1}{C}\leq\sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)}\right)^{2/N}. \tag{47}$$

Since $N/r_i(x)+((2\delta_i(x)-p_i(x)+2)/p_i(x))(N/2)\leq 1$, we have

$$\lim_{t\to\sup\varphi}\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)}=0. \tag{48}$$

Therefore, (48) leads to a contradiction.

On the other hand, if $\delta_+(x) < (p_i(x)-2)/2$, from (44), and Young inequality, for some constant c_2 (independent on t), we have

$$\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\leq c_2\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)}\varphi_t^2. \tag{49}$$

Argue as in (46), for some constant c_3 (independent on t), we have

$$\frac{1}{C}\leq c_3\sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{N\delta_+(x)}\right)^{2/N}. \tag{50}$$

Since $N/r_i(x)+N\delta_i(x)/p_i(x)\leq 1$, we obtain a contradiction. □

Lemma 10. *If (A1), (T1), (F1), and (H1) hold, for $i=1,2,\ldots,N$, $2\leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x))<2$, $q\in C_+^0(\Omega)$ and $q(x)<p_\infty=\overline{p}^*(x)$, and u is a weak solution of the problem* (1)*, then $u\in L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. We denote $E=W_0^{1,\overrightarrow{p(x)}}(\Omega)$. Let $t>0$, $\varphi=\max\{u-t,0\}$, and choose φ as test function to the problem (1). We have

$$\sum_{i=1}^{N}\int_{\Omega}\left[a_i(x,u,\nabla u)\varphi_{x_i}+T_i(x,\nabla u)\varphi\right]=\int_{\Omega}\left[f\varphi+\sum_{i=1}^{N}h_i\varphi_{x_i}\right]. \tag{51}$$

Let $x_0\in\Omega$, fix $k>1$, $B(x_0,R)$ be a ball, and let $\delta\in(0,1)$, $\delta R<s<t<R$, and $A_{k,t}=\{x\in B(x_0,t),\ u(x)>k\}$. Using (51), (A1), (F1), (H1), and Hölder inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_{k,s}}\left|\partial_{x_i}u\right|^{p_i(x)} &\leq \sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^{p_i(x)}\\ &\leq c\left[\|f\|_{L^{p'_\infty}(A_{k,t})}\|u-k\|_{L^{p_\infty}(A_{k,t})}\right.\\ &\quad+\sum_{i=1}^{N}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right||u-k|\\ &\quad\left.+\sum_{i=1}^{N}\int_{A_{k,t}}h_i\partial_{x_i}u\right].\end{aligned} \tag{52}$$

For $2\leq p_i(x)$ and (T1), by Young inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right||u-k| &\leq\sum_{i=1}^{N}\left(\frac{1}{2}\int_{A_{k,t}}|u-k|^2+\frac{1}{2}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right|^2\right)\\ &\leq\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u-k|^{p_i(x)}+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u|^{q(x)}\\ &\leq c\int_{A_{k,t}}\left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*}+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u|^{q(x)},\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_{k,t}}h_i\partial_{x_i}u &\leq\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}h_i^2+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^2\\ &\leq\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}h_i^2+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^{p_i(x)}.\end{aligned} \tag{53}$$

As $2\leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x))<2$, if $\overline{p(x)}<N$, there exists $\varepsilon=1/2-\overline{p(x)}/\overline{p(x)}^*>0$ such that

$$c\left(\int_{A_{k,t}}h_i^2\right)^{1/2}\left|A_{k,t}\right|^{1/2}\leq c\left|A_{k,t}\right|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \tag{54}$$

If $\overline{p(x)} \geq N$, a similar estimate is true; just only choose a suitable $\overline{p(x)}^*$.

Due to $q \in C_+^0(\Omega)$ and $q(x) < p_\infty(x) = \overline{p(x)}^*$, we have

$$\|f\|_{L^{p'_\infty}} \|u-k\|_{L^{p_\infty}} \leq c \int_{A_{k,t}} |u|^{q(x)}. \tag{55}$$

From

$$\begin{aligned} \int_{A_{k,t}} |u|^{q(x)} &= \int_{A_{k,t}} |(u-k)+k|^{q(x)} \\ &\leq c \int_{A_{k,t}} |u-k|^{q(x)} + ck^{q^+} |A_{k,t}| \\ &\leq c \int_{A_{k,t}} \left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*} + ck^{q^+} |A_{k,t}|, \end{aligned} \tag{56}$$

and (52)–(55), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_{A_{k,s}} \left|\partial_{x_i} u\right|^{p_i(x)} \\ &\leq \frac{1}{2} \sum_{i=1}^N \int_{A_{k,t}} \left|\partial_{x_i} u\right|^{p_i(x)} + c \int_{A_{k,t}} \left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*} \\ &\quad + ck^{q^+} |A_{k,t}| + c|A_{k,t}|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \end{aligned} \tag{57}$$

Using Lemma 3.1 in [17], and (57), we have

$$\begin{aligned} &\sum_{i=1}^N \int_{A_{k,\delta R}} \left|\partial_{x_i} u\right|^{p_i(x)} \\ &\leq c \int_{A_{k,R}} \left|\frac{u-k}{R-\delta R}\right|^{\overline{p(x)}^*} + ck^{q^+} |A_{k,R}| \\ &\quad + c|A_{k,R}|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \end{aligned} \tag{58}$$

By Lemma 2.4 in [18], (58), we obtain that u is bounded from above on $B(x_0, R/2)$. Note that $-u$ is also a weak solution of the problem (1), where $\widetilde{a}_i(x,u,\nabla u) = a_i(x,-u,-\nabla u)$ and $\widetilde{T}_i(x,\nabla u) = T_i(x,-\nabla u)$. Hence, $-u$ is also bounded from above on $B(x_0, R/2)$, and $u \in L^\infty(B(x_0, R/2))$. This implies that $u \in L^\infty_{\text{loc}}(\Omega)$. □

Proof of Theorem 5. From Lemmas 8, 9, and 10, we obtain that there exists a uniqueness weak solution $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ for the problem (1), and what is more, $u \in L^\infty_{\text{loc}}(\Omega)$. Hence we complete the proof of Theorem 5. □

4. The Proof of Theorem 6

In this section, we prove the regularity of weak solutions for the problem (2).

Proof of Theorem 6. We denote $E = W_0^{1,\overrightarrow{p(x)}}(\Omega)$. For understanding, we write $a^i(u,Du) = a_i(u,\nabla u)$, and $h^i = h_i$. If g is any function with compact support in Ω, for $1 \leq k \leq N$, for small enough m, we define

$$\Delta_m^k g(x) = \frac{g(x+me_k) - g(x)}{m}, \tag{59}$$

where $e_k = (0,0,\ldots,\underset{k}{1},\ldots,0)$. Hence, if for any $\phi \in E$, ϕ has compact support in Ω and $|m| < (1/2)\,\text{dist}(\text{supp}\,\phi, \partial\Omega)$; from the definition of weak solution for the problem (1), we have

$$\sum_{i=1}^N \int_\Omega \Delta_m^k a^i(x,Du)\phi_{x_i} - \int_\Omega \Delta_m^k f\phi - \sum_{i=1}^N \int_\Omega \Delta_m^k h^i \phi_{x_i} = 0. \tag{60}$$

For the definition of Δ_m^k and [19], we have

$$\begin{aligned} &\Delta_m^k a^i(x,Du) \\ &= \frac{1}{m}\int_0^1 \frac{d}{dt}\left[a^i\left(x+tme_k, Du+tm\Delta_m^k Du\right)\right] dt \\ &= \int_0^1 \left[a^i_{x_k} + \sum_{j=1}^N a^i_{\xi_j} D_j\left(\Delta_m^k u\right)\right] dt \\ &\doteq e^i + g_{ij} D_j\left(\Delta_m^k u\right), \end{aligned} \tag{61}$$

where $e^i = \int_0^1 a^i_{x_k} dt$, $g_{ij} = \int_0^1 \sum_{j=1}^N a^i_{\xi_j} dt$, and

$$\begin{aligned} \Delta_m^k f &= \frac{1}{m}\int_0^1 \frac{d}{dt} f(x+tme_k)\, dt \\ &= \int_0^1 f_{x_k} dt \doteq F, \\ \Delta_m^k h^i &= \frac{1}{m}\int_0^1 \frac{d}{dt} h^i(x+tme_k)\, dt \\ &= \int_0^1 h^i_{x_k} dt \doteq H^i. \end{aligned} \tag{62}$$

Then, combining (60) with (61)-(62), we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega e^i D_i\phi + \sum_{i,j=1}^N \int_\Omega g_{ij} D_j\left(\Delta_m^k u\right) D_i\phi \\ &\quad - \int_\Omega F\phi - \sum_{i=1}^N \int_\Omega H^i D_i\phi = 0. \end{aligned} \tag{63}$$

Due to (A4), (F2), (H2), (63), Young inequality, we have

$$\begin{aligned} &\sum_{i,j=1}^N \int_\Omega g_{ij} D_j\left(\Delta_m^k u\right) D_i\phi \\ &= \int_\Omega F\phi + \sum_{i=1}^N \int_\Omega H^i D_i\phi - \sum_{i=1}^N \int_\Omega e^i D_i\phi \\ &\leq c\int_\Omega |D\phi|^2 + c\int_\Omega |u|^{q(x)} + c\int_\Omega |u|^{q(x)/2}\phi \\ &\leq c\int_\Omega |D\phi|^2 + c\int_\Omega |u|^{q(x)} + c\int_\Omega \phi^2; \end{aligned} \tag{64}$$

that is,

$$\sum_{i,j=1}^{N}\int_{\Omega} g_{ij}D_j\left(\Delta_m^k u\right)D_i\phi \le c\int_{\Omega}\left|D\phi\right|^2 + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}\phi^2. \tag{65}$$

Choose $\phi = \psi^2\Delta_m^k u$, where $\psi \in C_0^1(\overline{\Omega})$, $|\psi| \le 1$, and (65), we obtain

$$\begin{aligned}\sum_{i,j=1}^{N}\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_j\left(\Delta_m^k u\right) &\le -2\sum_{i,j=1}^{N}\int_{\Omega}\psi u g_{ij}D_i\psi D_j\left(\Delta_m^k u\right) + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 \\ &\quad + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}\left(\psi^2\Delta_m^k u\right)^2.\end{aligned} \tag{66}$$

From (66), (A3), Cauchy inequality and Young inequality, we have

$$\begin{aligned}&\sum_{i,j=1}^{N}\int_{\Omega}\left|g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_j\left(\Delta_m^k u\right)\right| \\ &\le 2\sum_{i,j=1}^{N}\int_{\Omega}\left|\psi u g_{ij}D_i\psi D_j\left(\Delta_m^k u\right)\right| \\ &\qquad + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}|\psi|^4\left|\Delta_m^k u\right|^2 \\ &\le 2\sum_{i,j=1}^{N}\left[\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_i\left(\Delta_m^k u\right)\right]^{1/2} \\ &\qquad \times\left[\int_{\Omega} g_{ij}\left(\Delta_m^k u\right)^2 D_i\psi D_j\psi\right]^{1/2} + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 \\ &\qquad + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}|\psi|^4\left|\Delta_m^k u\right|^2 \\ &\le 2\sum_{i,j=1}^{N}\left[\frac{\varepsilon}{2}\left(\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_i\left(\Delta_m^k u\right)\right) + \frac{1}{2\varepsilon}\right. \\ &\qquad \left.\times\left(\int_{\Omega} g_{ij}\left(\Delta_m^k u\right)^2 D_i\psi D_j\psi\right)\right] + c\int_{\Omega}|u|^{q(x)} \\ &\quad + c\int_{\Omega}\psi^2\left|D\left(\Delta_m^k u\right)\right|^2 \\ &\quad + c\int_{\Omega}\left|\Delta_m^k u\right|^2|D\psi|^2 + c\int_{\Omega}\left|\Delta_m^k u\right|^2.\end{aligned} \tag{67}$$

By (A3), we obtain

$$\begin{aligned}&\int_{\Omega}\psi^2\left|D\left(\Delta_m^k u\right)\right|^2 \\ &\le c\int_{\Omega}\left[1 + \sum_{i=1}^{N}\left(\left|D_i u(x)\right|^{p_i(x)-2}\right.\right. \\ &\qquad \left.\left. + \left|D_i u(x + me_k)\right|^{p_i(x)-2}\right)\right] \\ &\qquad \times\left|\Delta_m^k u\right|^2|D\psi|^2 + c\int_{\Omega}|u|^{q(x)}.\end{aligned} \tag{68}$$

Assume $p_N = p_+^+ = \sup_{x\in\overline{\Omega}} p_+(x)$; we have for $i = 1, 2, \ldots, N$, $p_i(x) \le p_N$. For $q(x) < \overline{p}^*(x)$, let $k = N$ and take $m \to 0$; then we have

$$\int_{\Omega}\psi^2\left|D\left(D_N u\right)\right|^2 \le c\int_{\Omega}\left(1 + \sum_{i=1}^{N}\left|D_i u\right|^{p_i(x)-2}\right)|D\psi|^2\left|D_N u\right|^2. \tag{69}$$

Hence, $D_N u \in W_{\mathrm{loc}}^{1,2}(\Omega)$. If we take $\phi = \psi^2(\Delta_m^N u - r)^+$, where $r > 0$, $|\psi| < 1$, $(\Delta_m^N u - r)^+ = \max\{\Delta_m^N u - r, 0\}$. Since $D_N u \in W_{\mathrm{loc}}^{1,2}(\Omega)$, $D_N u \in L_{\mathrm{loc}}^{2N/(N-2)}(\Omega)$, we have

$$\begin{aligned}&\int_{\Omega}\psi^2\left|D(D_N u - r)^+\right|^2 \\ &\le c\int_{\Omega}\left(1 + \sum_{i=1}^{N}\left|D_i u\right|^{p_i(x)-2}\right)|D\psi|^2\left|(D_N u - r)^+\right|^2 \\ &\le c\int_{\Omega}\left(1 + \sum_{i=1}^{N-1}\left|D_i u\right|^{p_i(x)-2}\right)|D\psi|^2\left|(D_N u - r)^+\right|^2 \\ &\quad + c\int_{\Omega}\left|D_N u\right|^{p_N-2}|D\psi|^2\left|(D_N u - r)^+\right|^2 \\ &\le c\int_{\Omega}\left|(D_N u - r)^+\right|^{p_N}|D\psi|^2 + cr^{p_N-2}\int_{\Omega}\left|(D_N u - r)^+\right|^2 \\ &\quad + c\sum_{i=1}^{N}\int_{\Omega}\left(\left|(D_N u - r)^+\right|^{2N/(N-2)}|D\psi|^{2N/(N-2)}\right)^{(N-2)/N} \\ &\qquad \times\left(\int_{\{D_N u>k\}\bigcap \operatorname{supp}\psi}\left(1 + \left|D_i u\right|^{p_i(x)-2}\right)^{N/2}\right)^{2/N}.\end{aligned} \tag{70}$$

For $2 \le p_i(x) \le 2N/(N-2)$, we have

$$\begin{aligned}&\left(\int_{\{D_N u>k\}\bigcap \operatorname{supp}\psi}\left(1 + \left|D_i u\right|^{p_i(x)-2}\right)^{N/2}\right)^{2/N} \\ &\quad \le c\left|\{D_N u > k\}\bigcap \operatorname{supp}\psi\right|^{2/N-2/p_-+1}.\end{aligned} \tag{71}$$

If we fix $B(x_0, R)$, $\operatorname{supp}\psi \subset B(x_0, R)$, where $R < 1$; let $r \geq 1$, $0 < \delta < 1$, and take $\psi \in C_0^1(B(x_0, R))$, $\psi \equiv 1$ on $B(x_0, \delta R)$; $0 \leq \psi \leq 1$, $|D\psi| \leq c/(R - \delta R)$, then we have

$$\begin{aligned}
&\int_{A_{r,\delta R}} \left|D(D_N u - r)^+\right|^2 \\
&\quad \leq c \int_{A_{r,R}} \left(\frac{|D_N u - r|}{R - \delta R}\right)^{2N/(N-2)} \\
&\qquad + c\left(\int_{A_{r,R}} \left(\frac{|D_N u - r|}{R - \delta R}\right)^{2N/(N-2)}\right)^{(N-2)/N} \\
&\qquad \times |A_{r,R}|^{2/N-2/p_-^- +1} + cr^{2N/(N-2)} |A_{r,R}|.
\end{aligned} \tag{72}$$

Hence, (72) implies that $D_N u$ is in $L^\infty_{\mathrm{loc}}(\Omega)$. So we obtain $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \bigcap W^{2,2}_{\mathrm{loc}}(\Omega)$. By De Giorgi-Moser regularity theorem, for any $\omega \subset\subset \Omega$ and $0 < \alpha < 1$, we have $D_N u \in C^{0,\alpha}(\omega)$; then $u \in C^{1,\alpha}(\Omega)$, for $0 < \alpha < 1$. □

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