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Research Article

Predual of Q_K Spaces

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A necessary and sufficient condition is given for a positive measure μ on $\mathbb{D}$ to be a K-Carleson measure. We give the predual of Q_K spaces in terms of this condition.

1. Introduction

In study of a function space, it is of interest to study the dual and predual of function space. It is well known that Fefferman's and Sarason's theorems are $(VMOA)^* \cong H^1$ and $(H^1)^* \cong BMOA$. Anderson et al. gave the similar results on the Bloch space in [1]. The reader can refer to [2, 3] about the predual of Q_p spaces. We note that Q_p spaces are a kind of Q_K spaces. Now our question is what is the predual of Q_K spaces. But the technique that is used to prove the predual of Q_p spaces does not work for Q_K spaces. Enlightened by [4], we started from the characterizations of K-Carleson measure by an integral operator which contains the normalized nonnegative Borel measure on the unit disk. In this paper, we obtain a principal result that the predual of Q_K spaces is the analytic space $\mathscr{C}_K$, which is introduced in Section 3. We now recall a few fundamental definitions and establish some notation.

Let $g(a,z) = -\log|\varphi_a(z)|$ be the Green function on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ with logarithmic singularity at $a \in \mathbb{D}$, where $\varphi_a(z) = (a-z)(1-\bar{a}z)^{-1}$ is the Möbius transformation of $\mathbb{D}$. Denote by $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$.

Let $K : [0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The space Q_K consists of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 K(g(a,z))\, dA(z) < \infty, \quad (1)$$

where dA is an area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$.

Equipped with the norm $|f(0)| + \|f\|_{Q_K}$, the space Q_K is Banach. It is easy to check that the space Q_K is Möbius invariant in the sense that $\|f\circ\varphi_a\|_{Q_K} = \|f\|_{Q_K}$ for any $f \in Q_K$ and $a \in \mathbb{D}$. See [5, 6] for a general theory of Q_K spaces. Note that the space Q_K gives Q_p if we choose $K(t) = t^p$ for $0 < p < \infty$. See [7, 8] for a summary of recent research for Q_p spaces.

Recall that a function $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathscr{B}$, if

$$\|f\|_{\mathscr{B}} = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)\left|f'(z)\right| < \infty. \quad (2)$$

By [5], we know that

$$Q_K \subset \mathscr{B}. \quad (3)$$

The following two conditions have played a crucial role in the study of Q_K spaces during the last years:

$$\int_1^\infty \varphi_K(t)\frac{dt}{t^2} < \infty, \quad (4)$$

$$\int_0^1 \varphi_K(t)\frac{dt}{t} < \infty, \quad (5)$$

where

$$\varphi_K(t) = \sup_{0<s<1}\frac{K(st)}{K(s)}, \quad 0 < t < \infty. \quad (6)$$

Throughout the paper, K satisfies the following condition:

$$\int_0^1 K\left(\log\frac{1}{r}\right) r\, dr < \infty. \quad (7)$$

Otherwise, the space Q_K only contains constant functions (cf. [5]). By Theorem 2.1 in [5], we may assume that K is defined on $[0,1]$ and extend its domain to $[0,\infty)$ by setting $K(t) = K(1)$ for $t > 1$. As the discussion in [6], we may assume that $K(2t) \approx K(t)$.

For a subarc $I \subset \partial\mathbb{D}$, the boundary of $\mathbb{D}$, let

$$S(I) = \{r\zeta \in \mathbb{D} : 1 - |I| < r < 1, \zeta \in I\}. \tag{8}$$

If $|I| \geqslant 1$, then we set $S(I) = \mathbb{D}$. A positive measure μ on $\mathbb{D}$ is said to be a K-Carleson measure if

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \int_{S(I)} K\left(\frac{1-|z|^2}{|I|}\right) d\mu(z) < \infty. \tag{9}$$

By results in [6], we know that a function $f \in H(\mathbb{D})$ belongs to the space Q_K if and only if $|f'(z)|^2 dA(z)$ is a K-Carleson measure.

In the paper, we say that $K_1 \lesssim K_2$ (for two functions K_1 and K_2) if there is a constant $C > 0$ (independent of K_1 and K_2) such that $K_1 \leq CK_2$. We say $K_1 \approx K_2$ (i.e., K_1 is comparable with K_2) whenever $K_1 \lesssim K_2 \lesssim K_1$.

2. K-Carleson Measure

For any $I \subset \partial\mathbb{D}$ and the nondecreasing K, denote

$$\omega_{I,K}(z) = K\left(\frac{1-|z|^2}{|I|}\right) \int_{\partial\mathbb{D}} \chi_I(\xi) P_z(\xi) |d\xi|, \quad z \in \mathbb{D}, \tag{10}$$

where

$$P_z(\xi) = \frac{1-|z|^2}{|\xi - z|^2} \tag{11}$$

is the poisson kernel and $\chi_I(\xi) = 1$ for $\xi \in I$, $\chi_I(\xi) = 0$ for $\xi \in \partial\mathbb{D} \setminus I$.

Lemma 1. *Let μ be a nonnegative measure on $\mathbb{D}$. Let K satisfy condition* (4). *Then for any* arc $I \subset \partial\mathbb{D}$,

$$\int_{\mathbb{D}} \omega_{I,K}(z) d\mu(z) < \infty \tag{12}$$

holds if and only if μ is a K-Carleson measure.

Proof. First we assume that (12) holds. Now we show that μ is a K-Carleson measure. For any given $z \in S(I)$, we have

$$\int_{\partial\mathbb{D}} \chi_I(\xi) P_z(\xi) |d\xi| \geq \frac{1}{4}. \tag{13}$$

This gives

$$\begin{aligned} \omega_{I,K}(z) &= K\left(\frac{1-|z|^2}{|I|}\right) \int_{\partial\mathbb{D}} \chi_I(\xi) P_z(\xi) |d\xi| \\ &\geq \frac{1}{4} K\left(\frac{1-|z|^2}{|I|}\right). \end{aligned} \tag{14}$$

Then

$$\int_{S(I)} K\left(\frac{1-|z|^2}{|I|}\right) d\mu(z) \leq 4 \int_{\mathbb{D}} \omega_{I,K}(z) d\mu(z) < \infty. \tag{15}$$

The above inequality shows that μ is a K-Carleson measure.

Conversely, suppose that μ is a K-Carleson measure. For a nonnegative integer n, we use I_n for the arc in $\partial\mathbb{D}$ which has the same center as I and length $2^n|I|$. For $\xi \in I_n$ and $n \geq 1$, we have the following estimate:

$$P_z(\xi) \lesssim \frac{1}{2^n |I|}, \quad z \in S(I_{n+1}) \setminus S(I_n). \tag{16}$$

If $z \in S(I_1)$, we have

$$\begin{aligned} &\omega_{I,K}(z) \\ &\quad = K\left(\frac{1-|z|^2}{|I|}\right) \int_{\partial\mathbb{D}} \chi_I(\xi) P_z(\xi) |d\xi| \lesssim K\left(\frac{1-|z|^2}{|I|}\right). \end{aligned} \tag{17}$$

If $z \in S(I_{n+1}) \setminus S(I_n)$, we have

$$\begin{aligned} &\omega_{I,K}(z) \\ &\quad = K\left(\frac{1-|z|^2}{|I|}\right) \int_{\partial\mathbb{D}} \chi_I(\xi) P_z(\xi) |d\xi| \lesssim \frac{1}{2^n} K\left(\frac{1-|z|^2}{|I|}\right). \end{aligned} \tag{18}$$

Then

$$\begin{aligned} \int_{\mathbb{D}} \omega_{I,K}(z) d\mu(z) &= \int_{S(I_1)} \omega_{I,K}(z) d\mu(z) \\ &\quad + \sum_{n=1}^{\infty} \int_{S(I_{n+1})\setminus S(I_n)} \omega_{I,K}(z) d\mu(z) \\ &\lesssim \|\mu\|_K \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{2^n} \int_{S(I_{n+1})\setminus S(I_n)} K\left(\frac{1-|z|^2}{|I|}\right) d\mu(z) \\ &\lesssim \|\mu\|_K \\ &\quad + \sum_{n=1}^{\infty} \frac{\varphi_K(2^n)}{2^n} \int_{S(I_{n+1})} K\left(\frac{1-|z|^2}{2^{n+1}|I|}\right) d\mu(z) \\ &\lesssim \|\mu\|_K \left(1 + \sum_{n=1}^{\infty} \frac{\varphi_K(2^n)}{2^n}\right) \\ &\lesssim \|\mu\|_K \left(1 + \int_1^{\infty} \frac{\varphi_K(s)}{s^2} ds\right). \end{aligned} \tag{19}$$

We have the desired result by condition (4). □

Let $\mathbb{M}$ be the set of all nonnegative measure σ on $\mathbb{D}$ with the normalized condition $\sigma(\mathbb{D}) = 1$. For $\xi \in \partial\mathbb{D}$, let

$$\Gamma(\xi) = \{z \in \mathbb{D}, |z - \xi| < 2(1 - |z|)\}. \tag{20}$$

For any $z \in \mathbb{D}$, denote

$$\omega_{\sigma,K,z}(\xi) = \int_{\Gamma(\xi)} K\left(\frac{1-|z|^2}{1-|w|}\right) d\sigma(w), \tag{21}$$

$$\omega_{\sigma,K}(z) = \int_{\partial\mathbb{D}} \omega_{\sigma,K,z}(\xi) P_z(\xi) |d\xi|. \tag{22}$$

The following estimate can be found in [9]:

$$\begin{aligned}\int_{\partial\mathbb{D}} \chi_{\Gamma(\xi)}(w) |d\xi| \\ = 4 \arcsin\left(\min\left(1, (1-|w|)\sqrt{\frac{4}{|w|}}\right)\right) \approx 1-|w|^2.\end{aligned} \tag{23}$$

Then we have

$$\begin{aligned}\int_{\partial\mathbb{D}} \omega_{\sigma,K,z}(\xi) |d\xi| \\ &= \int_{\partial\mathbb{D}} \int_{\Gamma(\xi)} K\left(\frac{1-|z|^2}{1-|w|}\right) d\sigma(w) |d\xi| \\ &= \int_{\partial\mathbb{D}} \int_{\mathbb{D}} K\left(\frac{1-|z|^2}{1-|w|}\right) \chi_{\Gamma(\xi)}(w) d\sigma(w) |d\xi| \\ &= \int_{\mathbb{D}} K\left(\frac{1-|z|^2}{1-|w|}\right) \left(\int_{\partial\mathbb{D}} \chi_{\Gamma(\xi)}(w) |d\xi|\right) d\sigma(w) \\ &\lesssim K(1) \int_{\mathbb{D}} \left(1-|w|^2\right) d\sigma(w) \\ &\leq K(1) \sigma(\mathbb{D}).\end{aligned} \tag{24}$$

This shows that $\omega_{\sigma,K,z} \in L^1(\partial\mathbb{D})$. Hence, the definition of $\omega_{\sigma,K}$ is logical.

Theorem 2. *Let K satisfy condition* (4). *For all $\sigma \in \mathbb{M}$,*

$$\int_{\mathbb{D}} \omega_{\sigma,K}(z) d\mu(z) < \infty, \tag{25}$$

if and only if μ is a K-Carleson measure.

Proof. Suppose that μ is a K-Carleson measure. For any $w \in \mathbb{D}$, denote $I_w = \{\xi \in \partial\mathbb{D} : w \in \Gamma(\xi)\}$. Clearly, I_w is an arc on $\partial\mathbb{D}$ with the midpoint $w/|w|$. We obtain the following estimate by (23):

$$|I_w| = \int_{\partial\mathbb{D}} \chi_{I_w}(\xi) |d\xi| \approx 1-|w|. \tag{26}$$

Note that $K(t) \approx K(2t)$ for any $0 < t < \infty$. Then

$$\begin{aligned}\omega_{\sigma,K}(z) &= \int_{\partial\mathbb{D}} \omega_{\sigma,K,z}(\xi) P_z(\xi) |d\xi| \\ &= \int_{\partial\mathbb{D}} \int_{\Gamma(\xi)} K\left(\frac{1-|z|^2}{1-|w|}\right) P_z(\xi) d\sigma(w) |d\xi| \\ &= \int_{\partial\mathbb{D}} \int_{\mathbb{D}} K\left(\frac{1-|z|^2}{1-|w|}\right) \chi_{\Gamma(\xi)}(w) P_z(\xi) d\sigma(w) |d\xi| \\ &\approx \int_{\mathbb{D}} K\left(\frac{1-|z|^2}{1-|w|}\right) \left(\int_{\partial\mathbb{D}} \chi_{I_w}(\xi) P_z(\xi) |d\xi|\right) d\sigma(w) \\ &= \int_{\mathbb{D}} \omega_{I_w,K}(z) d\sigma(w).\end{aligned} \tag{27}$$

Therefore, by Lemma 1 and Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{D}} \omega_{\sigma,K}(z) d\mu(z) &\approx \int_{\mathbb{D}} \int_{\mathbb{D}} \omega_{I_w,K}(z) d\sigma(w) d\mu(z) \\ &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \omega_{I_w,K}(z) d\mu(z)\right) d\sigma(w) \\ &< \infty.\end{aligned} \tag{28}$$

Conversely, we assume that (25) holds. For any arc I, let w be the point in $\mathbb{D}$ such that $I_w = I$. Let σ be the point mass at w. Then

$$\omega_{\sigma,K}(z) = K\left(\frac{1-|z|^2}{1-|w|}\right) \int_{\partial\mathbb{D}} \chi_{I_w}(\xi) P_z(\xi) |d\xi| = \omega_{I_w,K}(z). \tag{29}$$

This and Lemma 1 give that

$$\int_{\mathbb{D}} \omega_{\sigma,K}(z) d\mu(z) = \int_{\mathbb{D}} \omega_{I_w,K}(z) d\mu(z) < \infty. \tag{30}$$

The proof is complete. □

3. Predual of Q_K Spaces

In this section, we will apply Theorem 2 to get the predual of Q_K spaces.

Definition 3. Let K be a right-continuous and nondecreasing function. Denote by M_K the set of all measurable functions G on $\mathbb{D}$ such that the measure $|G(z)|^2 dA(z)$ is a K-Carleson measure.

By Theorem 2, we can define

$$\|G\|_{M_K} = \sup_{\sigma \in \mathbb{M}} \left(\int_{\mathbb{D}} |G(z)|^2 \omega_{\sigma,K}(z) dA(z)\right)^{1/2}. \tag{31}$$

It is easy to check that $\|\cdot\|_{M_K}$ is a norm.

Denote by M^K the set of all measurable functions F on $\mathbb{D}$ such that

$$|||F|||_{M^K} = \inf_{\sigma\in\mathbb{M}}\left(\int_{\mathbb{D}} |F(z)|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z)\right)^{1/2} < \infty. \tag{32}$$

Lemma 4. *For the space M^K, define*

$$\|F\|_{M^K} = \sup_{G\in M_K} \frac{|\langle F,G\rangle|}{\|G\|_{M_K}}, \tag{33}$$

where

$$\langle F,G\rangle = \int_{\mathbb{D}} F(z)\overline{G(z)}\left(1-|z|^2\right) dA(z). \tag{34}$$

Then $\|\cdot\|_{M^K}$ is a norm.

Proof. It is obvious that $F = 0$ a.e., then $\|F\|_{M^K} = 0$. Conversely, if we set

$$G(z) = \frac{F(z)}{|F(z)|}, \quad F\in M^K \tag{35}$$

then

$$\begin{aligned}\langle F,G\rangle &= \int F(z)\overline{G(z)}\left(1-|z|^2\right) dA(z)\\ &= \int_{\mathbb{D}} |F(z)|\left(1-|z|^2\right) dA(z) = 0.\end{aligned} \tag{36}$$

This implies $F = 0$ a.e. We have

$$\|F\|_{M^K} = 0 \Longleftrightarrow F = 0 \text{ a.e.} \tag{37}$$

For any given $a\in\mathbb{C}$, it is easy to see that

$$\|aF\|_{M^K} = \sup_{G\in M_K}\frac{|\langle aF,G\rangle|}{\|G\|_{M_K}} = |a| \sup_{G\in M_K}\frac{|\langle F,G\rangle|}{\|G\|_{M_K}} = |a|\,\|F\|_{M^K}. \tag{38}$$

Given $F_1, F_2 \in M^K$, we have

$$\begin{aligned}\left\|F_1+F_2\right\|_{M^K} &= \sup_{G\in M_K}\frac{\left|\langle F_1+F_2,G\rangle\right|}{\|G\|_{M_K}}\\ &\le \sup_{G\in M_K}\frac{\left|\langle F_1,G\rangle\right|}{\|G\|_{M_K}} + \sup_{G\in M_K}\frac{\left|\langle F_2,G\rangle\right|}{\|G\|_{M_K}}\\ &= \left\|F_1\right\|_{M^K} + \left\|F_2\right\|_{M^K}.\end{aligned} \tag{39}$$

The proof is complete. □

Remark 5. Note that $\|F\|_{M^K} \le |||F|||_{M^K}$. In fact, for any $\sigma\in\mathbb{M}$, $F\in M^K$, and $G\in M_K$, we have

$$\begin{aligned}|\langle F,G\rangle| &= \left|\int_{\mathbb{D}} F(z)\overline{G(z)}\left(1-|z|^2\right) dA(z)\right|\\ &\le \left(\int_D |F(z)|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z)\right)^{1/2}\\ &\quad\times\left(\int_D |G(z)|^2\omega_{\sigma,K}(z)\,dA(z)\right)^{1/2}\\ &\le \left(\int_D |F(z)|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z)\right)^{1/2}\\ &\quad\times\|G\|_{M_K}.\end{aligned} \tag{40}$$

This shows that $|\langle F,G\rangle| \le |||F|||_{M^K}\|G\|_{M_K}$. Hence, $\|F\|_{M^K} \le |||f|||_{M^K}$.

Theorem 6. *Let K satisfy condition* (4). *If M^K is equipped with the norm*

$$\|F\|_{M^K} = \sup_{G\in M_K}\frac{|\langle F,G\rangle|}{\|G\|_{M_K}}, \tag{41}$$

then $(M^K)^* \cong M_K$ under the pairing

$$\langle F,G\rangle = \int_{\mathbb{D}} F(z)\overline{G(z)}\left(1-|z|^2\right) dA(z). \tag{42}$$

Proof. For any given $G \in M_K$, it is easy to see that $|G(z)|^2 dA(z)$ is a K-Carleson measure. By Theorem 2, for any $\sigma\in\mathbb{M}$,

$$\int_{\mathbb{D}} |G(z)|^2\omega_{\sigma,K}(z)\,dA(z) < \infty. \tag{43}$$

By Lemma 4, we have

$$|\langle F,G\rangle| \le \|F\|_{M^K}\|G\|_{M_K}. \tag{44}$$

This shows that $L(F) = \langle F,G\rangle$ is a bounded functional on M^K for $F\in M^K$. We have $\|L\| = \|G\|_{M_K}$ by the elementary knowledge of functional analysis, where $\|\cdot\|$ is norm of L. This gives $G\in (M^K)^*$.

Conversely, let L be a bounded linear functional on M^K. For any given $F\in M^K$, we have

$$\begin{aligned}|L(F)| &\le \|L\|\,\|F\|_{M^K} \le \|L\|\cdot|||F|||_{M^K}\\ &= \|L\| \inf_{\sigma\in M}\int_{\mathbb{D}} |F(z)|^2 \mid \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z),\end{aligned} \tag{45}$$

where $\|\cdot\|$ is norm of L. So for any fixed $\sigma\in\mathbb{M}$, we have

$$|L(F)| \le \|L\|\cdot\|F\|_{L^2(\sigma,K,2)}, \tag{46}$$

where the space $L^2(\sigma, K, 2)$ consists of all Lebesgue measure functions F on $\mathbb{D}$ such that

$$\|F\|_{L^2(\sigma,K,2)} = \left(\int_{\mathbb{D}} |F(z)|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z) \right)^{1/2} < \infty. \tag{47}$$

Hence, L can be extended to L_σ as a bounded linear functional on $L^2(\sigma, K, 2)$ such that $L(F) = L_\sigma(F)$ for any $F \in M^K$ and $\|L_\sigma\| \le \|L\|$. By the Hölder inequality, we obtain that the dual of $L^2(\sigma, K, 2)$ is $L^2(\omega_{\sigma,K}(z)dA(z))$ under the pair

$$\langle F, G\rangle = \int_{\mathbb{D}} F(z)\overline{G(z)}\left(1-|z|^2\right) dA(z), \tag{48}$$

where the space $L^2(\omega_{\sigma,K}(z)dA(z))$ consists of all Lebesgue measurable functions F on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |F(z)|^2 \omega_{\sigma,K}(z)\, dA(z) < \infty. \tag{49}$$

Then there exists a $G_\sigma \in L^2(\omega_{\sigma,K}(z)dA(z))$ such that

$$L_\sigma(h) = \left\langle h, G_\sigma \right\rangle, \quad h \in L^2(\sigma, K, 2). \tag{50}$$

Note that the function G_σ is independent of σ. In fact, for any given $\tau \in \mathbb{M}$ which is different from σ, we have

$$L_\tau(h) = \left\langle h, G_\tau \right\rangle, \quad h \in L^2(\tau, K, 2). \tag{51}$$

Given any $z_0 \in \mathbb{D}$, consider the Bergman disk $B(z_0, r) = \{z \in \mathbb{D}, |\varphi_{z_0}(z)| < r\}$. Define

$$h(z) = e^{i \arg\{G_\sigma(z) - G_\tau(z)\}} \chi_{B(z_0,r)}(z), \tag{52}$$

to be the test function. It is easy to see that $h \in M^K \subset L^2(\sigma, K, 2) \cap L^2(\tau, K, 2)$. Then we have

$$\left\langle h, G_\sigma \right\rangle = L_\sigma(h) = L_\tau(h) = \left\langle h, G_\tau \right\rangle. \tag{53}$$

The above equalities show that

$$\begin{aligned} 0 &= \left\langle h, G_\sigma - G_\tau \right\rangle \\ &= \int_{B(z_0,r)} h(z)\overline{\left(G_\sigma(z) - G_\tau(z)\right)}\left(1-|z|^2\right) dA(z) \\ &= \int_{B(z_0,r)} \left|G_\sigma(z) - G_\tau(z)\right|\left(1-|z|^2\right) dA(z). \end{aligned} \tag{54}$$

Hence, $G_\sigma = G_\tau$ a.e. on $B(z_0, r)$ for any given $z_0 \in \mathbb{D}$. This implies that $G_\sigma = G_\tau$ *a.e.* on $\mathbb{D}$. We now have a $G \in L^2(\omega_{\sigma,K}(z)dA(z))$ so that, for any $\sigma \in \mathbb{M}$,

$$L(F) = \langle F, G\rangle, \quad F \in L^2(\sigma, K, 2), \tag{55}$$

$$\int_{\mathbb{D}} |G(z)|^2 \omega_{\sigma,K}(z)\, dA(z) = \left\|L_\sigma\right\| \le \|L\| < \infty. \tag{56}$$

Theorem 2 shows that $|G(z)|^2 dA(z)$ is a K-Carleson measure. Hence, $G \in M_K$. □

Definition 7. Let K be a right-continuous and nondecreasing function. Let $\mathbb{M}$ be the set of all nonnegative measure σ on $\mathbb{D}$ with the normalized condition $\sigma(\mathbb{D}) = 1$. Denote by $\mathscr{C}_K$ the set of all analytic functions $f \in H(\mathbb{D})$ such that

$$|||f|||_{C_K} = \inf_{\sigma\in\mathbb{M}} \int_{\mathbb{D}} \left|f'(z)\right|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z) < \infty, \tag{57}$$

where $\omega_{\sigma,K}$ is defined as in (22).

Remark 8. In fact, $f \in C_K$ if and only if $f' \in L^2(\sigma, K, 2)$ for any $\sigma \in \mathbb{M}$. Obviously, we have $|||f|||_{\mathscr{C}_K} = \inf_{\sigma\in\mathbb{M}} \|f'\|_{L^2(\sigma,K,2)}$. See (47) about the definition of $\|\cdot\|_{L^2(\sigma,K,2)}$.

We need the following result to proof the main theorem (cf. [10]). Let $\psi \in H(\mathbb{D})$. Define an operator on $H(\mathbb{D})$ as

$$T\psi(z) = \int_{\mathbb{D}} |\psi(w)| \frac{\left(1-|w|^2\right)^{b-1}}{|1-\overline{w}z|^{b+1}} dA(w), \quad b \ge 1. \tag{58}$$

Lemma 9. *Suppose K satisfies conditions* (4) *and* (5). *If $d\mu(z) = |\psi(z)|^2 dA(z)$ is a K-Carleson measure, then $|T\psi(z)|^2 dA(z)$ is a K-Carleson measure.*

Theorem 10. *Let K satisfy conditions* (4) *and* (5). *The dual of $\mathscr{C}_K$ is Q_K under the pairing*

$$\left\langle f, g\right\rangle_C = \int_{\mathbb{D}} f'(z)\overline{g'(z)}\left(1-|z|^2\right) dA(z). \tag{59}$$

Proof. Choose $g \in Q_K$. Then $|g'(z)|^2 dA(z)$ is a K-Carleson measure. Theorem 2 gives that, for any $\sigma \in \mathbb{M}$,

$$\int_{\mathbb{D}} \left|g'(z)\right|^2 \omega_{\sigma,K}(z)\, dA(z) < \infty. \tag{60}$$

By the Hölder inequality, we have

$$\begin{aligned} \left|\left\langle f, g\right\rangle_C\right| &= \left|\int_{\mathbb{D}} f'(z)\overline{g'(z)}\left(1-|z|^2\right) dA(z)\right| \\ &\le \left(\int_{\mathbb{D}} \left|f'(z)\right|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z) \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{D}} \left|g'(z)\right|^2 \omega_{\sigma,K}(z)\, dA(z) \right)^{1/2}. \end{aligned} \tag{61}$$

This and Theorem 2 give that

$$\left|\left\langle f, g\right\rangle_C\right| \lesssim \inf_{\sigma\in M} \left(\int_{\mathbb{D}} \left|f'(z)\right|^2 \frac{\left(1-|z|^2\right)^2}{\omega_{\sigma,K}(z)} dA(z) \right)^{1/2} \left\|g'\right\|_{M_K}. \tag{62}$$

Therefore, $g \in (\mathscr{C}^K)^*$.

Conversely, let L be a bounded linear function on $\mathscr{C}_K$. Since $f \in \mathscr{C}_K \Leftrightarrow f' \in H(\mathbb{D}) \cap M^K$, L can be viewed as a bounded linear functional $\widetilde{L}$ on M^K; that is, $L(f) = \widetilde{L}(f')$. By

Theorem 6, there exists a $G \in M_K$ such that $\widetilde{L}(F) = \langle F, G\rangle$ for any $F \in M^K$.

Consider the Bergman projection P from $L^2(\mathbb{D})$ to the Bergman space A^2:

$$P(G)(z) = \int_{\mathbb{D}} \frac{G(w)}{(1-\overline{w}z)^2} dA(w), \quad G \in L^2(\mathbb{D}). \tag{63}$$

It is easy to see that

$$|P(G)(z)| \le \int_{\mathbb{D}} \frac{|G(w)|}{|1-\overline{w}z|^2} dA(w). \tag{64}$$

Hence, $|P(G)(z)|^2 dA(z)$ is a K-Carleson measure by Lemma 9. This shows that $P(G)$ is analytic and in M_K. Let g be the function satisfying $g' = P(G)$. Then $g \in Q_K$.

For $f \in \mathscr{C}_K$, we have $f' \in H(\mathbb{D}) \cap M^K$. The Bergman projection P is self-adjoint. Hence,

$$\begin{aligned} L(f) &= \widetilde{L}(f') = \langle f', G\rangle = \langle f', P(G)\rangle \\ &= \langle f', g'\rangle = \langle f, g\rangle_C. \end{aligned} \tag{65}$$

We obtain

$$L(f) = \int_{\mathbb{D}} f'(z)\overline{g'(z)}\left(1-|z|^2\right) dA(z), \quad f \in C_K. \tag{66}$$

We complete the proof of the predual Theorem 10. □

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