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Research Article

The Space of Continuous Periodic Functions Is a Set of First Category in $AP(X)$

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We prove that the space of continuous periodic functions is a set of first category in the space of almost periodic functions, and we also show that the space of almost periodic functions is a set of first category in the space of almost automorphic functions.

1. Introduction

Since the last century, the study on almost periodic type functions and their applications to evolution equations has been of great interest for many mathematicians. There is a large literature on this topic. Several books are especially devoted to almost periodic type functions and their applications to differential equations and dynamical systems. For example, let us indicate the books of Amerio and Prouse [1], Bezandry and Diagana [2], Bohr [3], Corduneanu [4], Diagana [5], Fink [6], Levitan and Zhikov [7], N'Guérékata [8, 9], Pankov [10], Shen and Yi [11], Zaidman [12], and Zhang [13].

Although almost periodic functions have a very wide range of applications now, it seems that giving an example of almost periodic (not periodic) functions is more difficult than giving an example of periodic functions. Also, there is a similar problem for almost automorphic functions. In this paper, we aim to compare the "amount" of almost periodic functions (not periodic) with the "amount" of continuous periodic functions, and we also discuss the related problems for almost automorphic functions.

2. Main Results

Throughout the rest of this paper, we denote by $\mathbb{R}$ the set of real numbers, by X a Banach space, and by $C(\mathbb{R}, X)$ the set of all continuous functions $f : \mathbb{R} \to X$.

Definition 1 (see [4]). A function $f \in C(\mathbb{R}, X)$ is called almost periodic if, for every $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$\sup_{t\in\mathbb{R}} \|f(t+\tau) - f(t)\| < \varepsilon. \tag{1}$$

We denote the collection of all such functions by $AP(X)$.

Recall that $AP(X)$ is a Banach space under the supremum norm.

Definition 2. A function $f \in C(\mathbb{R}, X)$ is called periodic if there exists $l > 0$ such that

$$f(t+l) = f(t), \quad \forall t \in \mathbb{R}. \tag{2}$$

Here, l is called a period of f. We denote the collection of all such functions by $P(X)$. For $f \in P(X)$, we call l_0 the fundamental period if l_0 is the smallest period of f.

Remark 3. Similar to the proof in [4, page 1], it is not difficult to show that if $f \in P(X)$ is not constant, and then f has the fundamental period.

Definition 4 (see [8]). A function $f \in C(\mathbb{R}, X)$ is called almost automorphic if, for every real sequence (s_m), there exists a subsequence (s_n) such that

$$g(t) = \lim_{n\to\infty} f(t + s_n) \tag{3}$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_n) = f(t) \tag{4}$$

for each $t \in \mathbb{R}$. Denote by $AA(X)$ the set of all such functions.

Recall that there exists an almost automorphic function which is not almost periodic, for instance, the following function:

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t}, \quad t \in \mathbb{R}. \tag{5}$$

Before the proof of our main results, we need to recall the notion about the first category.

Definition 5 (see [14]). Let S be a topological space. A set $E \subset S$ is said to be nowhere dense if its closure has an empty interior. The sets of the first category in S are those that are countable unions of nowhere dense sets. Any subset of S that is not of the first category is said to be of the second category in S.

Theorem 6. *$P(X)$ is a set of first category in $AP(X)$.*

Proof. For $n = 1, 2, \ldots$, we denote

$$\begin{aligned} P_n = \{ & f \in C(\mathbb{R}, X) : \text{ there exists } l \in [n, n+1] \\ & \text{such that } f(t + l) = f(t) \quad \forall t \in \mathbb{R}\}. \end{aligned} \tag{6}$$

Then, it is easy to see that

$$P(X) = \bigcup_{n=1}^{\infty} P_n. \tag{7}$$

We divide the remaining proof into two steps.

Step 1. Every P_n is a closed subset of $AP(X)$.

Let $f \in AP(X) \setminus P_n$. Then, for every $l \in [n, n+1]$, there exists $t_l \in \mathbb{R}$ such that $f(t_l + l) \neq f(t_l)$. Denote

$$\varepsilon_l := \frac{1}{4} \left\| f(t_l + l) - f(t_l) \right\| > 0, \quad l \in [n, n+1]. \tag{8}$$

In addition, due to the continuity of f, for every $l \in [n, n+1]$, there exists $\delta_l > 0$ such that

$$\left\| f(t_l + s) - f(t_l) \right\| \geq 3\varepsilon_l, \quad \forall s \in (l - \delta_l, l + \delta_l). \tag{9}$$

Obviously, we have

$$[n, n+1] \subset \bigcup_{l \in [n, n+1]} (l - \delta_l, l + \delta_l). \tag{10}$$

Then, by the Heine-Borel theorem, there exists $l_1, \ldots, l_k \in [n, n+1]$ such that

$$[n, n+1] \subset \bigcup_{i=1}^{k} \left(l_i - \delta_{l_i}, l_i + \delta_{l_i}\right), \tag{11}$$

where k is a fixed positive integer. Letting $\varepsilon = \min_{1\leq i\leq k}\{\varepsilon_{l_i}\}$, and

$$N(f, \varepsilon) := \left\{ g \in AP(X) : \|g - f\|_{AP(X)} < \varepsilon \right\}, \tag{12}$$

for every $g \in N(f, \varepsilon)$, we claim that $g \notin P_n$. In fact, for every $l \in [n, n+1]$, there exists $i \in \{1, \ldots, k\}$ such that

$$l \in \left(l_i - \delta_{l_i}, l_i + \delta_{l_i}\right). \tag{13}$$

Then, by (9), we have

$$\left\| f\left(t_{l_i} + l\right) - f\left(t_{l_i}\right) \right\| \geq 3\varepsilon_{l_i} \geq 3\varepsilon, \tag{14}$$

which yields that

$$\begin{aligned} & \left\| g\left(t_{l_i} + l\right) - g\left(t_{l_i}\right) \right\| \\ & \quad \geq \left\| f\left(t_{l_i} + l\right) - f\left(t_{l_i}\right) \right\| - \left\| f\left(t_{l_i} + l\right) - g\left(t_{l_i} + l\right) \right\| \\ & \qquad - \left\| f\left(t_{l_i}\right) - g\left(t_{l_i}\right) \right\| \\ & \quad \geq 3\varepsilon - \varepsilon - \varepsilon = \varepsilon > 0, \end{aligned} \tag{15}$$

where $\|g - f\|_{AP(\mathbb{R})} < \varepsilon$ was used. So, we know that $N(f, \varepsilon) \subset AP(X) \setminus P_n$, which means that P_n is a closed subset of $AP(X)$.

Step 2. Every P_n has an empty interior.

It suffices to prove that, for every $f \in P_n$ and $\delta > 0$, $N(f, \delta) \bigcap (AP(X) \setminus P_n) \neq \emptyset$. Now let $f \in P_n$ and $\delta > 0$. In the following, we discuss two cases.

Case I. f is constant.

We denote

$$f_\delta(t) = \frac{\cos t + \cos\left(\sqrt{2}t\right)}{3} \cdot \delta \cdot x_0 + f(t), \quad t \in \mathbb{R}, \tag{16}$$

where $x_0 \in X$ is some constant with $\|x_0\| = 1$. Then $f_\delta \in N(f, \delta)$ and $f_\delta \notin P_n$ since f_δ is not periodic.

Case II. f is not constant.

By Remark 3, f has a fundamental period l_0. We denote

$$f_\delta(t) = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R}, \tag{17}$$

where $M_f = \sup_{t\in\mathbb{R}} \|f(t)\|$. Obviously, $f_\delta \in N(f, \delta)$. Also, we claim that $f_\delta \notin P_n$. In fact, if this is not true, then there exists $T \in [n, n+1]$ such that

$$f_\delta(t + T) = f_\delta(t), \quad t \in \mathbb{R}, \tag{18}$$

that is,

$$f(t+T) + f\left(\frac{t+T}{\pi}\right) \cdot \frac{\delta}{M_f} = f(t) + f\left(\frac{t}{\pi}\right) \cdot \frac{\delta}{M_f}, \quad t \in \mathbb{R}. \tag{19}$$

Let

$$\begin{aligned} F_1(t) &= f(t + T) - f(t), \\ F_2(t) &= \frac{\delta}{M_f}\left[f\left(\frac{t}{\pi}\right) - f\left(\frac{t+T}{\pi}\right) \right], \quad t \in \mathbb{R}. \end{aligned} \tag{20}$$

Then $F_1(t) \equiv F_2(t)$. If $F_1(t) \equiv F_2(t) \equiv C$, where C is a fixed constant, then

$$f(t+T) = f(t) + C, \quad t \in \mathbb{R}, \tag{21}$$

which yields

$$C = \frac{f(kT) - f(0)}{k} \longrightarrow 0, \quad k \longrightarrow \infty, \tag{22}$$

since f is bounded. Thus, we have

$$f(t+T) = f(t), \qquad f\left(\frac{t}{\pi}\right) = f\left(\frac{t+T}{\pi}\right), \quad t \in \mathbb{R}. \tag{23}$$

Noting that l_0 is the fundamental period of f and πl_0 is the fundamental period of $f(\cdot/\pi)$, there exist two positive integers p and q such that

$$pl_0 = T = q\pi l_0, \tag{24}$$

that is, $\pi = p/q$, which is a contradiction. If $F_1 = F_2$ is not constant, then, by Remark 3, we can assume that T_0 is the fundamental period of F_1 and F_2. Noting that l_0 is a period of F_1 and πl_0 is a period of F_2, similar to the above proof, we can also show that π is a rational number, which is a contradiction.

In conclusion, $P(X)$ is countable unions of closed subsets with empty interior. So $P(X)$ is a set of first category. □

Remark 7. Since $AP(X)$ is a set of second category, it follows from Theorem 6 that $AP(X) \setminus P(X)$ is a set of second category, which means that, to some extent, the "amount" of almost periodic functions (not periodic) is far more than the "amount" of continuous periodic functions.

Theorem 8. *$AP(X)$ is a set of first category in $AA(X)$.*

Proof. Firstly, $AP(X)$ is a closed subset of $AA(X)$. Secondly, $AP(X)$ has an empty interior in $AA(X)$. In fact, letting

$$\varphi(t) = \sin \frac{1}{2 + \cos t + \cos\left(\sqrt{2}t\right)}, \quad t \in \mathbb{R}, \tag{25}$$

for every $f \in AP(X)$ and $\delta > 0$, we have $f_\delta \notin AP(X)$ and $f_\delta \in N(f, \delta)$, where

$$f_\delta(t) = f(t) + \frac{\delta x_0}{M_\varphi}\varphi(t), \quad t \in \mathbb{R}, \quad M_\varphi = \sup_{t\in\mathbb{R}} \|\varphi(t)\|, \tag{26}$$

and $x_0 \in X$ is some constant with $\|x_0\| = 1$. This completes the proof. □

Remark 9. By Theorem 8, $AA(X) \setminus AP(X)$ is a set of second category in $AA(X)$, which means that, to some extent, the "amount" of almost automorphic functions (not almost periodic) is far more than the "amount" of almost periodic functions.

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