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Research Article

Some Inequalities for Bounding Toader Mean

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By finding linear relations among differences between two special means, the authors establish some inequalities for bounding Toader mean in terms of the arithmetic, harmonic, centroidal, and contraharmonic means.

1. Introduction

It is well known that the quantities

$$A(a,b)=\frac{a+b}{2},\qquad G(a,b)=\sqrt{ab},$$
$$H(a,b)=\frac{2ab}{a+b},\qquad \overline{C}(a,b)=\frac{2\left(a^2+ab+b^2\right)}{3(a+b)},$$
$$C(a,b)=\frac{a^2+b^2}{a+b},\qquad S(a,b)=\sqrt{\frac{a^2+b^2}{2}},\tag{1}$$
$$M_p(a,b)=\begin{cases}\left(\dfrac{a^p+a^p}{2}\right)^{1/p}, & p\neq 0\\ \sqrt{ab}, & p=0\end{cases}$$

are, respectively, called in the literature the arithmetic, geometric, harmonic, centroidal, contraharmonic, root-square means, and the power mean of order p of two positive numbers a and b. In [1], Toader introduced a mean

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}\,d\theta$$
$$=\begin{cases}\dfrac{2a}{\pi}\mathscr{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a>b,\\ \dfrac{2b}{\pi}\mathscr{E}\left(\sqrt{1-\left(\dfrac{b}{a}\right)^2}\right), & a<b,\\ a, & a=b,\end{cases}\tag{2}$$

where

$$\mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}\sqrt{1-r^2\sin^2\theta}\,d\theta,$$
$$\mathscr{E}'=\mathscr{E}'(r)=\mathscr{E}\left(r'\right),\quad \mathscr{E}(0)=\frac{\pi}{2},\quad \mathscr{E}(1)=1,\tag{3}$$

for $r\in[0,1]$ and $r'=\sqrt{1-r^2}$ is Legendre's complete elliptic integral of the second kind; see [2] and [3, pages 40–46].

In [4], Vuorinen conjectured that

$$M_{3/2}(a,b)<T(a,b)\tag{4}$$

for all $a,b>0$ with $a\neq b$. This conjecture was verified in [5, 6], respectively. Later in [7], it was presented that

$$T(a,b)<M_{(\ln 2)/\ln(\pi/2)}(a,b)\tag{5}$$

for all $a,b>0$ with $a\neq b$. The constants $3/2$ and $\ln 2/\ln(\pi/2)=1.53\ldots$ which appeared in (4) and (5) are the best possible.

Utilizing inequalities (4) and (5) and using the fact that the power mean $M_p(a,b)$ is continuous and strictly increasing with respect to $p\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$ may conclude that

$$A(a,b)=M_1(a,b)<T(a,b)<M_2(a,b)=S(a,b)\tag{6}$$

for all $a,b>0$ with $a\neq b$. In [8, Theorem 3.1], it was demonstrated that the double inequality

$$\alpha S(a,b)+(1-\alpha)A(a,b)$$
$$<T(a,b)<\beta S(a,b)+\left(1-\beta\right)A(a,b)\tag{7}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1}{2}, \qquad \beta \geq \frac{4-\pi}{\left(\sqrt{2}-1\right)\pi}. \tag{8}$$

Recently in [9, Theorems 1.1 to 1.3], it was shown that the double inequalities

$$\begin{aligned} &\alpha_1 \overline{C}(a,b) + (1-\alpha_1) A(a,b) \\ &\quad < T(a,b) < \beta_1 \overline{C}(a,b) + (1-\beta_1) A(a,b), \end{aligned} \tag{9}$$

$$\frac{\alpha_2}{A(a,b)} + \frac{1-\alpha_2}{\overline{C}(a,b)} < \frac{1}{T(a,b)} < \frac{\beta_2}{A(a,b)} + \frac{1-\beta_2}{\overline{C}(a,b)} \tag{10}$$

hold for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha_1 \leq \frac{3}{4}, \quad \beta_1 \geq \frac{12}{\pi} - 3, \quad \alpha_2 \leq \pi - 3, \quad \beta_2 \geq \frac{1}{4}. \tag{11}$$

The equation (4.4) in [10, page 1013] reads that

$$\begin{aligned} 2[A(a,b) - H(a,b)] &= \frac{3}{2}\left[\overline{C}(a,b) - H(a,b)\right] \\ &= C(a,b) - H(a,b) = \frac{(a-b)^2}{a+b}. \end{aligned} \tag{12}$$

Motivated by (12), we further find that

$$\begin{aligned} 6\left[\overline{C}(a,b) - A(a,b)\right] &= 3\left[C(a,b) - \overline{C}(a,b)\right] \\ &= 2[C(a,b) - A(a,b)] = \frac{(a-b)^2}{a+b}. \end{aligned} \tag{13}$$

It is not difficult to see that the double inequality (9) can be rearranged as

$$\alpha_1 < \frac{T(a,b) - A(a,b)}{\overline{C}(a,b) - A(a,b)} < \beta_1. \tag{14}$$

Therefore, replacing the denominator in (14) by one of differences in (12) and (13) yields

$$\begin{aligned} &\frac{1}{2}\alpha_1 C(a,b) - \frac{1}{2}\alpha_1 \overline{C}(a,b) + A(a,b) \\ &\quad < T(a,b) < \frac{1}{2}\beta_1 C(a,b) - \frac{1}{2}\beta_1 \overline{C}(a,b) + A(a,b), \end{aligned} \tag{15}$$

$$\begin{aligned} &\frac{1}{3}\alpha_1 C(a,b) + \left(1 - \frac{1}{3}\alpha_1\right) A(a,b) \\ &\quad < T(a,b) < \frac{1}{3}\beta_1 C(a,b) + \left(1 - \frac{1}{3}\beta_1\right) A(a,b), \end{aligned} \tag{16}$$

$$\begin{aligned} &\left(1 + \frac{1}{3}\alpha_1\right) A(a,b) - \frac{1}{3}\alpha_1 H(a,b) \\ &\quad < T(a,b) < \left(1 + \frac{1}{3}\beta_1\right) A(a,b) - \frac{1}{3}\beta_1 H(a,b), \end{aligned} \tag{17}$$

$$\begin{aligned} &\frac{1}{4}\alpha_1 \overline{C}(a,b) - \frac{1}{4}\alpha_1 H(a,b) + A(a,b) \\ &\quad < T(a,b) < \frac{1}{4}\beta_1 \overline{C}(a,b) - \frac{1}{4}\beta_1 H(a,b) + A(a,b), \end{aligned} \tag{18}$$

$$\begin{aligned} &\frac{1}{6}\alpha_1 C(a,b) - \frac{1}{6}\alpha_1 H(a,b) + A(a,b) \\ &\quad < T(a,b) < \frac{1}{6}\beta_1 C(a,b) - \frac{1}{6}\beta_1 H(a,b) + A(a,b), \end{aligned} \tag{19}$$

where α_1 and β_1 satisfy (11). On the other hand, the arithmetic mean $A(a,b)$ in the numerator of (14) can also be replaced by the harmonic, contraharmonic, or centroidal means.

For our own convenience, we denote the difference of means in (12) and (13) by

$$M_{CH}(a,b) = \frac{(a-b)^2}{a+b}. \tag{20}$$

The quantity $M_{CH}(a,b)$ is nonnegative and convex on $(0,\infty) \times (0,\infty)$. See [11, Theorem 2.1].

Now we naturally pose the following problem.

Problem 1. What are the best constants α and β such that the double inequality

$$\alpha < \frac{T(a,b) - H(a,b)}{M_{CH}(a,b)} < \beta \tag{21}$$

holds for all positive numbers a and b with $a \neq b$?

The main purposes of this paper are to answer the previous problem, to provide an alternative proof for inequalities (14) to (19), and, finally, to remark the connection between Toader mean and the complete elliptic integral of the second kind.

2. Lemmas

To attain our main purposes, we need the following lemmas.

For $0 < r < 1$, denote $r' = \sqrt{1-r^2}$. Legendre's complete elliptic integrals of the first kind may be defined in [12, 13] by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2\theta}}\,d\theta,$$
$$\mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \quad \mathcal{K}(0) = \frac{\pi}{2}, \quad \mathcal{K}(1) = \infty. \tag{22}$$

Lemma 2 (see [14, Appendix E, pages 474-475]). *For $0 < r < 1$ and $r' = \sqrt{1-r^2}$, one has*

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - (r')^2\mathcal{K}}{r(r')^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d\left(\mathcal{E} - (r')^2\mathcal{K}\right)}{dr} = r\mathcal{K}, \qquad \mathcal{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathcal{E} - (r')^2\mathcal{K}}{1+r}. \tag{23}$$

Lemma 3 (see [14, Theorem 1.25]). *For $-\infty < a < b < \infty$, let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$, differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is (strictly) increasing (or (strictly) decreasing, resp.) on (a,b), so are the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)}, \qquad \frac{f(x) - f(b)}{g(x) - g(b)}. \tag{24}$$

Lemma 4 (see [14, Theorem 3.21]). *The function*

$$h(r) = \frac{\mathcal{E} - (r')^2\mathcal{K}}{r^2} \tag{25}$$

is strictly increasing and convex from $(0,1)$ onto $(\pi/4, 1)$.

Lemma 5. *The function*

$$f(r) = \frac{2\mathcal{E} - (r')^2\mathcal{K}}{r^2} \tag{26}$$

is strictly decreasing on $(0,1)$ and satisfies

$$\lim_{r\to 0^+} f(r) = \infty, \qquad \lim_{r\to 1^-} f(r) = 2. \tag{27}$$

Proof. By the first three formulas in Lemma 2, simple computations lead to

$$f'(r) = \frac{(r')^2\mathcal{K} - 3\mathcal{E}}{r^3} \triangleq \frac{f_1(r)}{r^3},$$
$$f_1'(r) = \frac{(r')^2\mathcal{K} + \mathcal{K} - 2\mathcal{E}}{r} \triangleq \frac{f_2(r)}{r}, \tag{28}$$
$$f_2'(r) = \frac{r}{(r')^2}\left[\mathcal{E} - (r')^2\mathcal{K}\right] \triangleq \frac{r^3}{(r')^2}h(r),$$

where the function $h(r)$ is defined by (25) in Lemma 4.

From

$$f_1'(0) = f_2(0) = f_2'(0) = 0, \quad f_1(1) = -3, \tag{29}$$

it follows that

$$f_2'(r) > 0, \quad f_2(r) > 0, \quad f_1'(r) > 0,$$
$$f_1(r) < 0, \quad f'(r) < 0. \tag{30}$$

Hence, the function $f(r)$ is strictly decreasing on $(0,1)$. Further, by easily obtained limits in (27), the proof of Lemma 5 is complete. □

3. Some Inequalities for Bounding Toader Mean

Now, we are in a position to give an affirmative solution to Problem 1 and to provide an alternative proof for inequalities (14) to (19).

Theorem 6. *The double inequality*

$$\alpha M_{CH}(a,b) + H(a,b) < T(a,b) < \beta M_{CH}(a,b) + H(a,b) \tag{31}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha \le \frac{5}{8} = 0.625, \qquad \beta \ge \frac{2}{\pi} = 0.636\ldots. \tag{32}$$

Proof. Without loss of generality, assume that $a > b > 0$. Let $t = b/a$. Then, $t \in (0,1)$ and

$$\frac{T(a,b) - H(a,b)}{M_{CH}(a,b)} = \frac{(2/\pi)\,\mathcal{E}\left(\sqrt{1-t^2}\right) - (2t/(1+t))}{(1-t)^2/(1+t)}. \tag{33}$$

Let $r = (1-t)/(1+t)$. Then, $r \in (0,1)$, and by the last formula in Lemma 2,

$$\frac{T(a,b) - H(a,b)}{M_{CH}(a,b)} = \frac{(2/\pi)\,\mathcal{E}\left(2\sqrt{r}/(1+r)\right) + r - 1}{2r^2/(1+r)}$$
$$= \frac{f_1(r)}{f_2(r)} + \frac{1}{2}, \tag{34}$$

where

$$f_1(r) = \frac{2}{\pi}\left[2\mathcal{E} - (r')^2\mathcal{K}\right] - 1, \qquad f_2(r) = 2r^2. \tag{35}$$

By the middle two formulas in Lemma 2, a straightforward calculation leads to

$$f_1(0) = f_2(0) = 0, \quad f_1'(r) = \frac{2}{\pi}\frac{\mathcal{E} - (r')^2\mathcal{K}}{r}, \quad f_2'(r) = 4r. \tag{36}$$

So, we have

$$\frac{f_1'(r)}{f_2'(r)} = \frac{1}{2\pi}\frac{\mathcal{E} - (r')^2\mathcal{K}}{r^2} = \frac{h(r)}{2\pi}. \tag{37}$$

Combining this with Lemmas 3 and 4 reveals that the function $f_1(r)/f_2(r)$ is strictly increasing on $(0,1)$. Moreover, using L'Hôpital's rule, we obtain

$$\lim_{r\to 0^+}\frac{f_1(r)}{f_2(r)}+\frac{1}{2}=\lim_{r\to 0^+}\frac{1}{2\pi}\frac{\mathscr{E}-(r')^2\mathscr{K}}{r^2}+\frac{1}{2}=\frac{5}{8},$$

$$\lim_{r\to 1^-}\frac{f_1(r)}{f_2(r)}+\frac{1}{2}=\lim_{r\to 1^-}\frac{(2/\pi)\left[2\mathscr{E}-(r')^2\mathscr{K}\right]-1}{2r^2}+\frac{1}{2}$$
$$=\lim_{r\to 1^-}\frac{\mathscr{E}}{\pi r^2}+\lim_{r\to 1^-}\frac{h(r)}{\pi}=\frac{2}{\pi}. \tag{38}$$

The proof of Theorem 6 is thus complete. □

Theorem 7. *For all $a,b>0$ with $a\neq b$,*

(1) *the double inequality*

$$\alpha_1 M_{CH}(a,b)+A(a,b)<T(a,b)<\beta_1 M_{CH}(a,b)+A(a,b) \tag{39}$$

holds if and only if

$$\alpha_1\le\frac{1}{8}=0.125,\qquad \beta_1\ge\frac{2}{\pi}-\frac{1}{2}=0.136\ldots, \tag{40}$$

(2) *the double inequality*

$$\alpha_2 M_{CH}(a,b)+C(a,b)<T(a,b)<\beta_2 M_{CH}(a,b)+C(a,b) \tag{41}$$

holds if and only if

$$\alpha_2\le-\frac{3}{8}=-0.375,\qquad \beta_2\ge\frac{2}{\pi}-1=-0.363\ldots, \tag{42}$$

(3) *the double inequality*

$$\alpha_3 M_{CH}(a,b)+\overline{C}(a,b)<T(a,b)<\beta_3 M_{CH}(a,b)+\overline{C}(a,b) \tag{43}$$

holds if and only if

$$\alpha_3\le-\frac{1}{24}=-0.041\ldots,\qquad \beta_3\ge\frac{2}{\pi}-\frac{2}{3}=-0.030\ldots. \tag{44}$$

Proof. From the identities in (12), it follows that

$$H(a,b)=A(a,b)-\frac{1}{2}M_{CH}(a,b)$$
$$=C(a,b)-M_{CH}(a,b)=\overline{C}(a,b)-\frac{2}{3}M_{CH}(a,b). \tag{45}$$

Substituting these into the inequality (31) in Theorem 6 acquires inequalities (39) to (43) in Theorem 7. □

Theorem 8. *The inequality*

$$T(a,b)>\lambda M_{CH}(a,b) \tag{46}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\le 2/\pi$.

Proof. Without loss of generality, assume that $a>b>0$. Let $t=b/a$. Then, $t\in(0,1)$ and

$$\frac{T(a,b)}{M_{CH}(a,b)}=\frac{(2/\pi)\,\mathscr{E}\left(\sqrt{1-t^2}\right)}{(1-t)^2/(1+t)}. \tag{47}$$

Let $r=(1-t)/(1+t)$. Then, $r\in(0,1)$, and by the last formula in Lemma 2,

$$\frac{T(a,b)}{M_{CH}(a,b)}=\frac{(2/\pi)\,\mathscr{E}\left(2\sqrt{r}/(1+r)\right)}{2r^2/(1+r)}=\frac{1}{\pi}\frac{2\mathscr{E}-(r')^2\mathscr{K}}{r^2}. \tag{48}$$

Therefore, from Lemma 5, it follows that function $T(a,b)/M_{CH}(a,b)$ is strictly decreasing and

$$\lim_{r\to 1^-}\frac{T(a,b)}{M_{CH}(a,b)}=\lim_{r\to 1^-}\frac{1}{\pi}\frac{2\mathscr{E}-(r')^2\mathscr{K}}{r^2}=\frac{2}{\pi}. \tag{49}$$

Theorem 8 is thus proved. □

4. Remarks

Finally, we would like to remark several things, including the connection between Toader mean and the complete elliptic integral of the second kind.

Remark 9. The double inequality (39) is equivalent to (9). Consequently, inequalities (14) to (19) are recovered once again.

Remark 10. The coefficient 3/2 in (12) corrects an error which appeared at the corresponding position in (4.4) in [10, page 1013]. Luckily, this error does not influence the correctness of any other conclusions in [10].

Remark 11. We point out that Toader mean $T(a,b)$ satisfies

$$T(a,b)=R_E\left(a^2,b^2\right), \tag{50}$$

where

$$R_E(x,y)=\frac{1}{\pi}\int_0^\infty\left(\frac{x}{t+x}+\frac{y}{t+y}\right)\frac{t}{\sqrt{(t+x)(t+y)}}dt \tag{51}$$

is the complete symmetric elliptic integral of the second kind and is a symmetric and homogeneous function; see [15, equation (9.2-3)] and [16, page 250, equation (1.6)]. Numerous inequalities involving R_E, $\mathscr{E}$, and $\mathscr{K}$ are known in the mathematical literature; see [2, 16–19] and [3, pages 40–46] and closely related references therein. In the past years, the fact that Toader mean T and the elliptic integral R_E are the same has been overlooked by several researchers.

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References

[1] Gh. Toader, "Some mean values related to the arithmetic-geometric mean," *Journal of Mathematical Analysis and Applications*, vol. 218, no. 2, pp. 358–368, 1998.

[2] B.-N. Guo and F. Qi, "Some bounds for the complete elliptic integrals of the first and second kinds," *Mathematical Inequalities & Applications*, vol. 14, no. 2, pp. 323–334, 2011.

[3] F. Qi, D.-W. Niu, and B.-N. Guo, "Refinements, generalizations, and applications of Jordan's inequality and related problems," *Journal of Inequalities and Applications*, vol. 2009, Article ID 271923, 52 pages, 2009.

[4] M. Vuorinen, "Hypergeometric functions in geometric function theory," in *Special Functions and Differential Equations (Madras, 1997)*, pp. 119–126, Allied Publ., New Delhi, India, 1998.

[5] R. W. Barnard, K. Pearce, and K. C. Richards, "An inequality involving the generalized hypergeometric function and the arc length of an ellipse," *SIAM Journal on Mathematical Analysis*, vol. 31, no. 3, pp. 693–699, 2000.

[6] S.-L. Qiu and J.-M. Shen, "On two problems concerning means," *Journal of Hangzhou Insitute Electronic Engineering*, vol. 17, no. 3, pp. 1–7, 1997 (Chinese).

[7] H. Alzer and S.-L. Qiu, "Monotonicity theorems and inequalities for the complete elliptic integrals," *Journal of Computational and Applied Mathematics*, vol. 172, no. 2, pp. 289–312, 2004.

[8] Y.-M. Chu, M.-K. Wang, and S.-L. Qiu, "Optimal combinations bounds of root-square and arithmetic means for Toader mean," *Proceedings of the Indian Academy of Sciences. Mathematical Sciences*, vol. 122, no. 1, pp. 41–51, 2012.

[9] Y. Hua and F. Qi, "The best bounds for Toader mean in terms of the centroidal and arithmetic means," http://arxiv.org/abs/1303.2451.

[10] W.-D. Jiang and F. Qi, "Some sharp inequalities involving Seiffert and other means and their concise proofs," *Mathematical Inequalities & Applications*, vol. 15, no. 4, pp. 1007–1017, 2012.

[11] I. J. Taneja, "Refinement of inequalities among means," *Journal of Combinatorics, Information & System Sciences*, vol. 31, no. 1-4, pp. 343–364, 2006.

[12] F. Bowman, *Introduction to Elliptic Functions with Applications*, Dover, New York, NY, USA, 1961.

[13] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, vol. 67 of *Die Grundlehren der mathematischen Wissenschaften*, Springer-Verlag, New York, NY, USA, 2nd edition, 1971.

[14] G. D. Anderson, M. K. Vamanamurthy, and M. K. Vuorinen, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, NY, USA, 1997.

[15] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, NY, USA, 1977.

[16] E. Neuman, "Bounds for symmetric elliptic integrals," *Journal of Approximation Theory*, vol. 122, no. 2, pp. 249–259, 2003.

[17] H. Kazi and E. Neuman, "Inequalities and bounds for elliptic integrals," *Journal of Approximation Theory*, vol. 146, no. 2, pp. 212–226, 2007.

[18] H. Kazi and E. Neuman, "Inequalities and bounds for elliptic integrals. II," in *Special Functions and Orthogonal Polynomials*, vol. 471 of *Contemp. Math.*, pp. 127–138, American Mathematical Society, Providence, RI, USA, 2008.

[19] L. Yin and F. Qi, "Some inequalities for complete ellipticintegrals," http://arxiv.org/abs/1301.4385.