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Research Article

On the Aleksandrov-Rassias Problems on Linear n-Normed Spaces

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This paper generalizes T. M. Rassias' results in 1993 to n-normed spaces. If X and Y are two real n-normed spaces and Y is n-strictly convex, a surjective mapping $f : X \to Y$ preserving unit distance in both directions and preserving any integer distance is an n-isometry.

1. Introduction

Let X and Y be two metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r, that is, for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$, then r is called a conservative (or preserved) distance for the mapping f. In particular, we denote DOPP as f preserving the one distance property and SDOPP as f preserving the strong one distance property and also for f^{-1}.

In 1970 [1], Aleksandrov posed the following problem. *Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.* This question is of great significance for the Mazur-Ulam Theorem [2].

In 1993, T. M. Rassias and P. Šemrl proved the following.

Theorem 1 (see [3]). *Let X and Y be two real normed linear spaces such that one of them has a dimension greater than one. Assume also that one of them is strictly convex. Suppose that $f : X \to Y$ is a surjective mapping that satisfies SDOPP. Then, f is an affine isometry (a linear isometry up to translation).*

Theorem 2 (see [3]). *Let X and Y be two real normed linear spaces such that one of them has a dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping. Assume also that f is a surjective mapping satisfying (SDOPP). Then, f is an isometry.*

Since 2004, the Aleksandrov problem in n-normed spaces ($n \geq 2$) has been discussed, and some results are obtained [4–8].

Definition 3 (see [7]). Let X be a real linear space with $\dim X \geq n$ and $\|\cdot,\dots,\cdot\| : X^n \to R$, a function, then $(X, \|\cdot,\dots,\cdot\|)$ is called a linear n-normed space if for any $\alpha \in R$ and all $x, y, x_1, \dots, x_n \in X$

nN_1: $\|x_1,\dots,x_n\| = 0 \Leftrightarrow x_1,\dots,x_n$ are linearly dependent,

nN_2: $\|x_1,\dots,x_n\| = \|x_{j1},\dots,x_{jn}\|$ for every permutation $(j_1,\dots,j_n)$ of $(1,\dots,n)$,

nN_3: $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$,

nN_4: $\|x + y, x_2,\dots,x_n\| \leq \|x, x_2,\dots,x_n\| + \|y, x_2,\dots,x_n\|$. The function $\|\cdot,\dots,\cdot\|$ is called the n-norm on X.

Definition 4 (see [8]). Let X and Y be two real linear n-normed spaces.

(i) A mapping $f : X \to Y$ is defined to be an n-isometry if for all $x_1,\dots,x_n, y_1,\dots,y_n \in X$,

$$\begin{aligned}\|f(x_1) - f(y_1),\dots,f(x_n) - f(y_n)\| \\ = \|x_1 - y_1,\dots,x_n - y_n\|.\end{aligned} \tag{1}$$

(ii) A mapping $f : X \to Y$ is called the n-distance one preserving property (n-DOPP) if for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, $\|x_1 - y_1, \ldots, x_n - y_n\| = 1$, it follows that $\|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\| = 1$.

(iii) A mapping $f : X \to Y$ is called the n-strong distance one preserving property (n-SDOPP) if for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$, $\|x_1 - y_1, \ldots, x_n - y_n\| = 1$, it follows that $\|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\| = 1$ and conversely.

(iv) A mapping $f : X \to Y$ is called an n-Lipschitz if for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$,

$$\begin{aligned} &\|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\| \\ &\quad \le \|x_1 - y_1, \ldots, x_n - y_n\|. \end{aligned} \tag{2}$$

Definition 5 (see [7]). The points $x_0, x_1, \ldots, x_n$ of X are called n-collinear if for every i, $\{x_j - x_i : 0 \le j \ne i \le n\}$ is linearly dependent.

Definition 6. X is said to be n-strictly convex normed spaces if for any $x_0, x_1, x_2, \ldots, x_n \in X$, $x_2, \ldots, x_n \notin \operatorname{span}\{x_0, x_1\}$, and $\|x_0 + x_1, x_2, \ldots, x_n\| = \|x_0 x_2, \ldots, x_n\| + \|x_1 x_2, \ldots, x_n\|$ imply that x_0 and x_1 are linearly dependent.

C. Park and T. M. Rassias obtained the following.

Theorem 7 (see [8]). *Let X and Y be real linear n-normed spaces. If a mapping $f : X \to Y$ satisfies the following conditions:*

(i) *f has the n-DOPP,*

(ii) *f is n-Lipschitz,*

(iii) *f preserves the 2-collinearity,*

(iv) *f preserves the n-collinearity,*

then f is an n-isometry.

In 2009, Gao [6] researched another n-isometry and gave the 2-strictly convex concept [6].

In this paper, we generalize T. M. Rassias Theorems 1 and 7 on n-strictly convex normed spaces ($n > 1$).

2. Main Results

The proof of the following lemma was presented in [9], to be published; the proof is given again for the convenience of readers.

Lemma 8. *Let X be an n-normed space such that X has dimension greater than n and $r > 0$. Suppose that $0 < \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| \le 2r$ for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$. Then, there exists $\omega \in X$ such that*

$$\begin{aligned} \|x_1 - \omega, x_2 - y_2, \ldots, x_n - y_n\| &= r, \\ \|\omega - y_1, x_2 - y_2, \ldots, x_n - y_n\| &= r. \end{aligned} \tag{3}$$

Proof. Since $x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n$ are linearly independent and $\dim X > n$, then there exists $z_0 \in X \setminus \operatorname{span}\{x_1 - y_1, \ldots, x_n - y_n\}$ with $\|z_0, x_2 - y_2, \ldots, x_n - y_n\| = r$.

Set $y_0 = y_1 - x_1$. For any $\alpha \in R$, we have

$$\|z_0 + \alpha y_0, x_2 - y_2, \ldots, x_n - y_n\| \ne 0. \tag{4}$$

Let us define $h(\alpha)$ by

$$h(\alpha) = \frac{r(z_0 + \alpha y_0)}{\|z_0 + \alpha y_0, x_2 - y_2, \ldots, x_n - y_n\|}, \tag{5}$$

then, we obtain

$$\|h(\alpha), x_2 - y_2, \ldots, x_n - y_n\| = r. \tag{6}$$

Set

$$\begin{aligned} z_1 &= \frac{-r(y_1 - x_1)}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}, \\ z_2 &= \frac{r(y_1 - x_1)}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}. \end{aligned} \tag{7}$$

Clearly, $z_0 \ne z_1, z_2$. And we have

$$\begin{aligned} \|z_1, x_2 - y_2, \ldots, x_n - y_n\| &= r, \\ \|z_2, x_2 - y_2, \ldots, x_n - y_n\| &= r. \end{aligned} \tag{8}$$

On the other hand,

$$\lim_{\alpha \to -\infty} h(\alpha) = z_1, \qquad \lim_{\alpha \to \infty} h(\alpha) = z_2. \tag{9}$$

Thus,

$$h(-\infty) = z_1, \qquad h(+\infty) = z_2. \tag{10}$$

Define $g : h(R) \to R$ by

$$g(z) = \|z - y_0, x_2 - y_2, \ldots, x_n - y_n\|. \tag{11}$$

It follows that

$$\begin{aligned} g(z_1) &= \|z_1 - y_0, x_2 - y_2, \ldots, x_n - y_n\| \\ &= \left(1 + \frac{r}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}\right) \\ &\quad \times \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| \ge r, \\ g(z_2) & \\ &= \begin{cases} \left(1 - \dfrac{r}{\|x_1 - y_1, \ldots, x_n - y_n\|}\right)\|x_1 - y_1, \ldots, x_n - y_n\|, & \\ \qquad \text{if } \|x_1 - y_1, \ldots, x_n - y_n\| > r & \\ \left(\dfrac{r}{\|x_1 - y_1, \ldots, x_n - y_n\|} - 1\right)\|x_1 - y_1, \ldots, x_n - y_n\|, & \\ \qquad \text{if } \|x_1 - y_1, \ldots, x_n - y_n\| < r. & \end{cases} \end{aligned} \tag{12}$$

Thus, $g(z_2) \le r$.

Obviously, $g(h(\alpha))$ is continuous on R. Using the mean value theorem, there exists $\alpha_0 \in R$ such that $g(h(\alpha_0)) = r$.

Set $\omega_0 = h(\alpha_0)$, $\omega = \omega_0 + x_1$, we have

$$\|\omega_0 - y_0, x_2 - y_2, \dots, x_n - y_n\| = r. \tag{13}$$

And from $\|h(\alpha), x_2 - y_2, \dots, x_n - y_n\| = r$, we have

$$\begin{aligned} &\|\omega - x_1, x_2 - y_2, \dots, x_n - y_n\| = r, \\ &\|\omega - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad = \|\omega_0 + x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad = \|\omega_0 - y_0, x_2 - y_2, \dots, x_n - y_n\| = r. \end{aligned} \tag{14}$$

□

Lemma 9. *Let X and Y be two real linear n-normed spaces whose dimensions are greater than n, and let Y be n-strictly convex normed space. Suppose that $f : X \to Y$ is a surjective mapping satisfying (n-SDOPP) with preserving distance k for any $k \in N$. Then, f preserves distance $1/k$ for any $k \in N$.*

Proof. Firstly, f is injective. Suppose, on the contrary, that there are $x_0, x_1 \in X$, $x_0 \neq x_1$, such that $f(x_0) = f(x_1)$. As $\dim X > n$, it follows that there exist vectors $x_2, \dots, x_n \in X$ such that $x_1 - x_0, \dots, x_n - x_0$ are linearly independent. Then, $\|x_1 - x_0, \dots, x_n - x_0\| \neq 0$.

Set

$$z_2 := x_0 + \frac{x_2 - x_0}{\|x_1 - x_0, \dots, x_n - x_0\|}. \tag{15}$$

Clearly,

$$\|x_1 - x_0, z_2 - x_0, x_3 - x_0, \dots, x_n - x_0\| = 1. \tag{16}$$

Then

$$\begin{aligned} &\|f(x_1) - f(x_0), f(z_2) - f(x_0), \\ &\quad f(x_3) - f(x_0), \dots, f(x_n) - f(x_0)\| = 1. \end{aligned} \tag{17}$$

This implies that $f(x_0) \neq f(x_1)$, which is a contradiction. Therefore, f is a bijective mapping.

Let $x_1, \dots, x_n, y_1, \dots, y_n \in X$ and ($k \in N \setminus \{1\}$) satisfying

$$\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{1}{k}. \tag{18}$$

By Lemma 8, we can find $w_1 \in X$ with

$$\begin{aligned} &\|x_1 - w_1, x_2 - y_2, \dots, x_n - y_n\| = 1, \\ &\|w_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = 1. \end{aligned} \tag{19}$$

Set

$$u_1 = w_1 + k(y_1 - w_1), \qquad v_1 = w_1 + k(x_1 - w_1). \tag{20}$$

Clearly, we have

$$\begin{aligned} &\|x_1 - v_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad = \|(k-1)(x_1 - w_1), x_2 - y_2, \dots, x_n - y_n\| \\ &\quad = k - 1 \\ &\quad\quad \|w_1 - v_1, x_2 - y_2, \dots, x_n - y_n\| = k. \end{aligned} \tag{21}$$

It follows from the hypothesis of f preserving any integer k; then,

$$\begin{aligned} &\|f(x_1) - f(w_1), f(x_2 - y_2), \dots, f(x_n) - f(y_n)\| = 1, \\ &\|f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = k - 1, \\ &\|f(w_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = k. \end{aligned} \tag{22}$$

Clearly, we have

$$\begin{aligned} &\|f(w_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\quad = \|f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\quad\quad + \|f(x_1) - f(w_1), f(x_2 - y_2), \dots, f(x_n) - f(y_n)\|. \end{aligned} \tag{23}$$

We conclude that

$$\begin{aligned} &f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \\ &\quad \notin \operatorname{span}\{f(x_1) - f(v_1), f(x_1) - f(w_1)\}. \end{aligned} \tag{24}$$

Otherwise, if for some $f(x_i) - f(y_i)$, we have $\mu_i, \lambda_i \in R$ with $\mu_i \neq 0$ or $\lambda_i \neq 0$ such that

$$f(x_i) - f(y_i) = \mu_i (f(x_1) - f(v_1)) + \lambda_i (f(x_1) - f(w_1)). \tag{25}$$

Suppose that $\lambda_i \neq 0$. Then,

$$\begin{aligned} k - 1 = {} & \|f(x_1) - f(v_1), \dots, f(x_i) \\ & -f(y_i), \dots, f(x_n) - f(y_n)\| \\ = {} & |\lambda_i| \, \|(x_1) - f(v_1), \dots, f(x_1) \\ & -f(w_1), \dots, f(x_n) - f(y_n)\|. \end{aligned} \tag{26}$$

Assume that

$$\begin{aligned} &\|f(x_1) - f(v_1), \dots, f(x_1) - f(w_1), \dots, f(x_n) - f(y_n)\| \\ &\quad \neq 0. \end{aligned} \tag{27}$$

Set

$$\begin{aligned} s_j = {} & f(x_j) \\ & + (f(x_j) - f(y_j)) \\ & \times (\|f(x_1) - f(v_1), \dots, f(x_1) - f(w_1), \dots, \\ & \quad f(x_n) - f(y_n)\|)^{-1}, \quad (j \geq 2). \end{aligned} \tag{28}$$

Then, for $j \neq i$,

$$\begin{aligned} &\|f(x_1) - f(v_1), \dots, s_j - f(x_j), \dots, f(x_1) \\ &\quad -f(w_1), \dots, f(x_n) - f(y_n)\| = 1. \end{aligned} \tag{29}$$

Since f is bijective and preserves n-SDOPP on both directions. Then, there exists $t_j \in X$ with $f(t_j) = s_j$ which satisfies that

$$\left\|x_1 - v_1, t_j - x_j, \dots, x_1 - w_1, \dots, x_n - y_n\right\| = 1. \quad (30)$$

However, by (20), $x_1 - v_1 = (1-k)(x_1 - w_1)$, and thus $x_1 - v_1$, $x_1 - w_1$ are linear dependent. Then,

$$\left\|x_1 - v_1, t_j - x_j, \dots, x_1 - w_1, \dots, x_n - y_n\right\| = 0. \quad (31)$$

This contradiction implies that

$$\begin{aligned}&\|f(x_1) - f(v_1), \dots, f(x_1) - f(w_1), \dots, f(x_n) - f(y_n)\| \\ &\quad = 0.\end{aligned} \quad (32)$$

This also contradicts with (26). Since Y is n-strictly convex, then there exists $\alpha > 0$ such that

$$f(x_1) - f(v_1) = \alpha(f(x_1) - f(w_1)). \quad (33)$$

Then,

$$f(x_1) = \frac{1}{1+\alpha} f(v_1) + \frac{\alpha}{1+\alpha} f(w_1). \quad (34)$$

Since

$$\begin{aligned}\|f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| &= k-1, \\ \|f(x_1) - f(w_1), f(x_2 - y_2), \dots, f(x_n) - f(y_n)\| &= 1,\end{aligned} \quad (35)$$

then $\alpha = k-1$. Thus,

$$f(x_1) = \frac{1}{k} f(v_1) + \frac{k-1}{k} f(w_1), \quad (36)$$

Similarly,

$$f(y_1) = \frac{1}{k} f(u_1) + \frac{k-1}{k} f(w_1). \quad (37)$$

Hence,

$$\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| = \frac{1}{k}. \quad (38)$$

□

Lemma 10. *Let X and Y be real n-normed spaces such that $\dim X \geq n$. If a mapping $f : X \to Y$ preserves the distance $1/k$ for each $k \in \mathbb{N}$, then f preserves the distance zero.*

Proof. Choose $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\|x_1 - y_1, \dots, x_n - y_n\| = 0$; that is, $x_1 - y_1, \dots, x_n - y_n$ are linearly dependent. Assume that $\{x_{m+1} - y_{m+1}, \dots, x_n - y_n\}$ is a maximum linearly independent group of $\{x_1 - y_1, \dots, x_n - y_n\}$ $(m < n)$. As $\dim X \geq n$, we can find a finite sequence of vectors $\omega_1, \omega_2, \dots, \omega_m \in X$ such that $x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n$ are linearly independent. Hence, it holds that

$$\|x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| \neq 0. \quad (39)$$

We will prove that

$$\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \leq \frac{1}{k}, \quad (40)$$

for every $k \in \mathbb{N}$. Let $m = 1$. We can find a vector $\omega_1 \in X$ such that $x_1 - \omega_1, x_2 - y_2, \dots, x_n - y_n$ are linearly independent. Set

$$v_1 = x_1 + \frac{x_1 - \omega_1}{2k\|x_1 - \omega_1, x_2 - y_2, \dots, x_n - y_n\|}, \quad (41)$$

for arbitrarily fixed $k \in \mathbb{N}$. Then,

$$\begin{aligned}&\|x_1 - v_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{1}{2k}, \\ &\|v_1 - x_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad - \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\leq \|(v_1 - x_1) + (x_1 - y_1), x_2 - y_2, \dots, x_n - y_n\| \\ &\leq \|v_1 - x_1, x_2 - y_2, \dots, x_n - y_n\| \\ &\quad + \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|.\end{aligned} \quad (42)$$

Since $\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = 0$, we get

$$\|v_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = \frac{1}{2k}. \quad (43)$$

Since f preserves the distance $1/(2k)$, we see that

$$\begin{aligned}&\|f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\leq \|f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &\quad + \|f(v_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n)\| \\ &= \frac{1}{2k} \cdot 2 = \frac{1}{k}.\end{aligned} \quad (44)$$

For $m \geq 2$, we set

$$\begin{aligned}v_1 &= x_1 + (x_1 - \omega_1) \\ &\quad \times (2^m k \|x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} \\ &\quad - y_{m+1}, \dots, x_n - y_n\|)^{-1},\end{aligned} \quad (45)$$

$$v_i = 2x_i - \omega_i, \quad (46)$$

for any $i \in \{2, 3, \dots, m\}$. Then, we have

$$x_i - v_i = \omega_i - x_i, \qquad v_i - y_i = (x_i - \omega_i) + (x_i - y_i), \quad (47)$$

for each $i \in \{2, 3, \dots, m\}$. Since $x_i - y_i, x_{m+1} - y_{m+1}, \dots, x_n - y_n$ are linearly dependent, we get

$$\|\dots, x_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n\| = 0, \quad (48)$$

and hence,

$$\begin{aligned}
&\|\ldots, x_i - \omega_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad - \|\ldots, x_i - y_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad \le \|\ldots, (x_i - \omega_i) + (x_i - y_i), \ldots, x_{m+1} \\
&\qquad - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad \le \|\ldots, x_i - \omega_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\qquad + \|\ldots, x_i - y_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\|,
\end{aligned} \tag{49}$$

which together with (48) implies that

$$\begin{aligned}
&\|\ldots, v_i - y_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad = \|\ldots, x_i - \omega_i, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\|,
\end{aligned} \tag{50}$$

for all $i \in \{2, 3, \ldots, m\}$. By a similar argument, we further obtain that

$$\begin{aligned}
&\|v_1 - y_1, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad = \|v_1 - x_1, \ldots, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\|.
\end{aligned} \tag{51}$$

In view of (45), (50), and (51), we conclude that

$$\begin{aligned}
&\|v_1 - y_1, \mu_2, \ldots, \mu_m, x_{m+1} - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad = \|x_1 - v_1, x_2 - \omega_2, \ldots, x_m - \omega_m, x_{m+1} \\
&\qquad - y_{m+1}, \ldots, x_n - y_n\| \\
&\quad = \frac{1}{2^m k},
\end{aligned} \tag{52}$$

where μ_i denotes either $v_i - y_i$ or $x_i - v_i$ for $i \in \{2, 3, \ldots, m\}$.

Since f preserves the distance $1/(2^m k)$ for any $k \in \mathbb{N}$, it follows from (52) that

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), f(x_3) \\
&\quad - f(y_3), \ldots, f(x_n) - f(y_n)\| \\
&\quad \le \|f(x_1) - f(v_1), \\
&\qquad f(x_2) - f(v_2), \ldots, f(x_{m-1}) - f(v_{m-1}), \\
&\qquad f(x_m) - f(v_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \ldots, f(x_n) - f(y_n)\| \\
&\quad + \|f(x_1) - f(v_1), \\
&\qquad f(x_2) - f(v_2), \ldots, f(x_{m-1}) - f(v_{m-1}), \\
&\qquad f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \ldots, f(x_n) - f(y_n)\| \\
&\quad + \|f(x_1) - f(v_1), \\
&\qquad f(x_2) - f(v_2), \ldots, f(v_{m-1}) - f(y_{m-1}), \\
&\qquad f(x_m) - f(v_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \ldots, f(x_n) - f(y_n)\| \\
&\quad + \|f(x_1) - f(v_1), \\
&\qquad f(x_2) - f(v_2), \ldots, f(v_{m-1}) - f(y_{m-1}), \\
&\qquad f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \ldots, f(x_n) - f(y_n)\| \\
&\quad + \cdots + \\
&\quad + \|f(v_1) - f(y_1), \\
&\qquad f(v_2) - f(y_2), \ldots, f(v_{m-1}) - f(y_{m-1}), \\
&\qquad f(v_m) - f(y_m), \\
&\qquad f(x_{m+1}) - f(y_{m+1}), \ldots, f(x_n) - f(y_n)\| \\
&\quad = \frac{1}{2^m k} \cdot 2^m = \frac{1}{k},
\end{aligned} \tag{53}$$

where k is an arbitrary positive integer. Hence, we conclude that

$$\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = 0, \tag{54}$$

which implies that f preserves the distance zero. □

Remark 11. In ([9], Lemma 2.2 to be published), we give the same method under the condition of f preserving 2-colinear.

Theorem 12. *Let X and Y be real n-normed spaces such that* $\dim X > n$ *and Y is n-strictly convex. If a surjective mapping $f : X \to Y$ has the n-SDOPP and preserves the distance k for any $k \in \mathbb{N}$, then f is an affine n-isometry.*

Proof. Assume that $\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| > 0$ for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.

Take positive integers k, m such that

$$\frac{m-1}{k} \le \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| \le \frac{m}{k}. \tag{55}$$

Set

$$p_i = x_1 + \frac{i}{k} \cdot \frac{y_1 - x_1}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}, \tag{56}$$

for $i = 0, 1, \ldots, m-2$, and

$$p_m = y_1. \tag{57}$$

Clearly, for $i = 1, \ldots, m-2$,

$$\begin{aligned}
&\|p_i - p_{i-1}, x_2 - y_2, \ldots, x_n - y_n\| = \frac{1}{k},\\
0 &< \|p_m - p_{m-2}, x_2 - y_2, \ldots, x_n - y_n\|\\
&= \left\| y_1 - x_1 - \frac{m-2}{k} \cdot \frac{y_1 - x_1}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}, x_2 - y_2, \ldots, x_n - y_n \right\|\\
&= \left(1 - \frac{m-2}{k} \cdot \frac{1}{\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|}\right) \cdot \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|\\
&= \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| - \frac{m-2}{k}\\
&\le \frac{m}{k} - \frac{m-2}{k} = \frac{2}{k}.
\end{aligned} \tag{58}$$

According to Lemma 8, there exists $p_{m-1} \in X$ such that

$$\begin{aligned}
\|p_{m-1} - p_{m-2}, x_2 - y_2, \ldots, x_n - y_n\| &= \frac{1}{k},\\
\|p_{m-1} - y_1, x_2 - y_2, \ldots, x_n - y_n\| &= \frac{1}{k}.
\end{aligned} \tag{59}$$

It follows from Lemma 9 that we have

$$\|f(p_i) - f(p_{i-1}), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = \frac{1}{k}, \tag{60}$$

for $i = 0, 1, 2, \ldots, m$.

On the other hand,

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2) \ldots, f(x_n) - f(y_n)\|\\
&\le \sum_{i=1}^{m} \|f(p_i) - f(p_{i-1}), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = \frac{m}{k}.
\end{aligned} \tag{61}$$

Hence

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&\le \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|.
\end{aligned} \tag{62}$$

Suppose that

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&< \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|.
\end{aligned} \tag{63}$$

For any $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$, with

$$\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| \ne 0, \tag{64}$$

find a positive integer k_0 satisfying $\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| < k_0$.

Set $z_1 = x_1 + k_0(y_1 - x_1)/\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|$. Clearly, $\|z_1 - x_1, x_2 - y_2, \ldots, x_n - y_n\| = k_0$, and $\|z_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| = k_0 - \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|$.

It follows that $\|f(z_1) - f(x_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = k_0$ and

$$\begin{aligned}
k_0 &= \|f(z_1) - f(x_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&\le \|f(z_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&\quad + \|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&< k_0 - \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|\\
&\quad + \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| = k_0.
\end{aligned} \tag{65}$$

Then (63) is not valid. Hence,

$$\begin{aligned}
&\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\|\\
&= \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|.
\end{aligned} \tag{66}$$

□

Corollary 13. *Let X and Y be two real linear n-normed spaces. Suppose that mapping $f : X \to Y$ preserves any positive integer k-distance and Lipschitz condition. Then, f is an n-isometry.*

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